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Abstract. The first aim of this paper is to contribute for investigating on soft $S-$ metric space which is based on soft points of soft sets and to prove some important theorems on sequential compact and totally bounded in soft $S-$ metric space. Moreover, we introduce soft uniformly continuous mapping and examine some of its properties.

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Keywords: Soft set, Soft S-metric, Soft sequential compact metric space, Soft uniformly continuous mapping.

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1. INTRODUCTION

Metric space is one of the most useful and important notions in mathematics and applied sciences. Some authors have tried to give generalizations of metric spaces in different ways. Sedghi et al. [15] modified the concept of $D-$metric space and gave the concept of D^*-metric space. Later, they initiated the notion of $S-$metric space which is different from other space as generalization of a metric space [16]. Some authors have proved fixed point type theorems in these spaces.

Metric spaces provide a powerful tool to the study of optimization and approximation theory, variational inequalities and so many. After Molodtsov [13] initiated concept of soft set theory as a new mathematical tool for dealing with uncertainties. Applications of soft set theory in other disciplines and real life problems were progressing rapidly. The study of soft metric space was initiated by Das and Samanta [6]. Gunduz et al. [8] initiated the notion of soft $S-$metric space which is based on soft points of soft sets and gave some of its properties. Moreover, they [9] proved some fixed point theorems of soft contractive mappings on soft $S-$metric space.

Topological structures of soft sets have been studied by some authors. Shabir and Naz [17] initiated the study of soft topological spaces which are defined over an initial universe with a fixed set of parameters. They showed that a soft topological space gives a parameterized family of topological spaces. Theoretical studies of soft topological spaces have also been researched by some authors in [3, 4, 5, 7, 10, 12, 14, 18, 19, 20, 21], etc.

The purpose of this paper is to contribute for investigating on soft $S-$metric space which is based on soft points of soft sets. Firstly, we focus on compact sets in soft $S-$metric space and explore the differences and similarities between the point set topology and soft topology. In addition to, we define the concepts of sequential compact and totally bounded in soft $S-$metric space and prove some important theorems on this space. Finally, we give Lebesque number for soft open cover in soft $S-$metric space. We show that soft compact $S-$metric space and soft sequentially compact $S-$metric space are equivalent structures. Moreover, we introduce soft uniformly continuous mapping and examine some of its properties.

2. Preliminaries

We briefly give some basic definitions of concepts which serve a background to this work. Throughout this paper, X denotes initial universe, E denotes the set of all parameters, $P(X)$ denotes the power set of X and $SP(\widetilde{X})$ be the collection of all soft points of $\widetilde{X}$, $\mathbb{R}(E)^*$ denotes the set of all non-negative soft real numbers.

Definition 2.1 ([13]). A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$.

Definition 2.2 ([1]). For two soft sets (F, E) and (G, E) over X, (F, E) is called a soft subset of (G, E), denoted by $(F, E)\widetilde{\subseteq}(G, E)$, if $\forall a \in E$, $F(a) \subseteq G(a)$.

Similarly, (F, E) is called a soft superset of (G, E), denoted by $(F, E)\widetilde{\supseteq}(G, E)$, if (G, E) is a soft subset of (F, E).

Two soft sets (F, E) and (G, E) over X are called soft equal, if (F, E) is a soft subset of (G, E) and (G, E) is a soft subset of (F, E).

Definition 2.3 ([1]). (i) The intersection of two soft sets (F, E) and (G, E) over X, denoted by $(F, E)\widetilde{\cap}(G, E) = (H, E)$, is the soft set (H, E), where $\forall a \in E$, $H(a) = F(a) \cap G(a)$.

(ii) The union of two soft sets (F, E) and (G, E) over X, denoted by $(F, E)\widetilde{\cup}(G, E) = (H, E)$, is the soft set (H, E), where $\forall a \in E$, $H(a) = F(a) \cup G(a)$.

Definition 2.4 ([11]). (i) A soft set (F, E) over X is said to be a null soft set denoted by Φ, if for all $a \in E$, $F(a) = \varnothing$.

(ii) A soft set (F, E) over X is said to be an absolute soft set denoted by $\widetilde{X}$ if for all $a \in E$, $F(a) = X$.

Definition 2.5 ([17]). (i) The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E)\widetilde{\setminus}(G, E)$, is defined as $H(a) = F(a)\setminus G(a)$ for all $a \in E$.

(ii) The complement of a soft set (F, E), denoted by $(F, E)^c$, is defined $(F, E)^c = (F^c, E)$, where $F^c : E \to P(X)$ is a mapping given by $F^c(a) = X\setminus F(a)$, $\forall a \in E$ and F^c is called the soft complement function of F.

Definition 2.6 ([17]). Let $\widetilde{\tau}$ be the collection of soft sets over X, then $\widetilde{\tau}$ is said to be a soft topology on X, if

(i) $\Phi, \widetilde{X}$ belong to $\widetilde{\tau}$,
(ii) the union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$,
(iii) the intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.
The triplet $(X, \widetilde{\tau}, E)$ is called a soft topological space over X. Then each member of $\widetilde{\tau}$ are said to be a soft open set in X.

Definition 2.7 ([17])**.** Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. A soft set (F, E) over X is said to be a soft closed set in X, if its complement $(F, E)^c$ belongs to $\widetilde{\tau}$.

Proposition 2.8 ([17])**.** *Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. Then the collection $\widetilde{\tau}_a = \{F(a) : (F, E) \in \widetilde{\tau}\}$ for each $a \in E$, defines a topology on X.*

Definition 2.9 ([17])**.** Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X and (F, E) be a soft set over X. Then the soft closure of (F, E), denoted by $\overline{(F, E)}$, is the intersection of all soft closed super sets of (F, E). The soft interior of (F, E), denoted by $(F, E)^0$, is the union of all soft open subsets of (F, E).

Definition 2.10 ([2])**.** Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X and (F, E) be a soft set over X. Then the soft bounded of (F, E), denoted by $\partial(F, E)$, is defined $\partial(F, E) = \overline{(F, E)} \widetilde{\cap} \overline{(F, E)^c}$.

Definition 2.11 ([2, 6])**.** Let (F, E) be a soft set over X. The soft set (F, E) is called a soft point, denoted by (x_a, E), if for the element $a \in E$, $F(a) = \{x\}$ and $F(a') = \varnothing$ for all $a' \in E - \{a\}$ (briefly denoted by x_a).
It is obvious that each soft set can be expressed as a union of soft points. For this reason, to give the family of all soft sets on X it is sufficient to give only soft points on X.

Definition 2.12 ([2])**.** Two soft points x_a and $y_{a'}$ over a common universe X, we say that the soft points are different, if $x \neq y$ or $a \neq a'$.

Definition 2.13 ([2])**.** The soft point x_a is said to be belonging to the soft set (F, E), denoted by $x_a \widetilde{\in} (F, E)$, if $x_a(a) \in F(a)$,i.e., $\{x\} \subseteq F(a)$, for $a \in E$.

Definition 2.14 ([2])**.** Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. A soft set $(F, E) \widetilde{\subseteq} (X, E)$ is called a soft neighborhood of the soft point $x_a \widetilde{\in} (F, E)$, if there exists a soft open set (G, E) such that $x_a \widetilde{\in} (G, E) \widetilde{\subset} (F, E)$.

Definition 2.15 ([6])**.** Let $\mathbb{R}$ be the set of all real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set and denoted by (F, E). If (F, E) is a singleton soft set, then it will be called a soft real number and denoted by $\widetilde{r}, \widetilde{s}, \widetilde{t}$ etc. Here $\widetilde{r}, \widetilde{s}, \widetilde{t}$ will denote a particular type of soft real numbers such that $\widetilde{r}(a) = r$, for all $a \in E$. $\widetilde{0}$ and $\widetilde{1}$ are the soft real numbers, where $\widetilde{0}(a) = 0$, $\widetilde{1}(a) = 1$, for all $a \in E$, respectively.

Definition 2.16 ([6])**.** Let $\widetilde{r}, \widetilde{s}$ be two soft real numbers. Then the following statements hold:
(i) $\widetilde{r} \widetilde{\leq} \widetilde{s}$, if $\widetilde{r}(a) \leq \widetilde{s}(a)$, for all $a \in E$,
(ii) $\widetilde{r} \widetilde{\geq} \widetilde{s}$, if $\widetilde{r}(a) \geq \widetilde{s}(a)$, for all $a \in E$,

(iii) $\widetilde{r}\widetilde{<}\widetilde{s}$, if $\widetilde{r}(a) < \widetilde{s}(a)$, for all $a \in E$,
(iv) $\widetilde{r}\widetilde{>}\widetilde{s}$, if $\widetilde{r}(a) > \widetilde{s}(a)$, for all $a \in E$.

Definition 2.17 ([16])**.** Let X be a non-empty set and $S : X^3 \to [0, \infty)$ be a function satisfying the following conditions: for all $x, y, z, t \in X$,
(i) $S(x, y, z) = 0$ if and only if $x = y = z$,
(ii) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$.
Then S is called $S-$metric on X and the pair (X, S) is called $S-$metric space.

3. Compact sets in soft S-metric spaces

Definition 3.1 ([8])**.** A soft S-metric on $SP(\widetilde{X})$ is a mapping $S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$ that satisfies the following conditions, for each soft points $x_a, y_b, z_c, u_d \in SP(\widetilde{X})$,
(S1) $S(x_a, y_b, z_c) \geq \widetilde{0}$,
(S2) $S(x_a, y_b, z_c) = \widetilde{0}$ if and only if $x_a = y_b = z_c$,
(S3) $S(x_a, y_b, z_c) \leq S(x_a, x_a, u_d) + S(y_b, y_b, u_d) + S(z_c, z_c, u_d)$.
Then the soft set $\widetilde{X}$ with a soft S-metric is called a soft S-metric space and denoted by $\left(\widetilde{X}, S, E\right)$.

Definition 3.2 ([8])**.** Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space and $\widetilde{r}$ be a non-negative soft real number. For $\widetilde{r} > \widetilde{0}$ and $x_a \in SP(\widetilde{X})$, we define the soft open ball $B_S(x_a, \widetilde{r})$ and soft closed ball $\mathbf{B}_S(x_a, \widetilde{r})$ with center x_a and a radius $\widetilde{r}$ as follows:

$$\begin{aligned} B_S(x_a, \widetilde{r}) &= \left\{ y_b \in SP(\widetilde{X}) : S(y_b, y_b, x_a) < \widetilde{r} \right\}, \\ \mathbf{B}_S(x_a, \widetilde{r}) &= \left\{ y_b \in SP(\widetilde{X}) : S(y_b, y_b, x_a) \leq \widetilde{r} \right\}. \end{aligned}$$

Definition 3.3 ([8])**.** Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space and (F, E) be a soft set.

(i) If for every $x_a \in (F, E)$, there exists $\widetilde{r} > \widetilde{0}$ such that $B_S(x_a, \widetilde{r}) \subset (F, E)$, then the soft set (F, E) is called a soft open set in $\left(\widetilde{X}, S, E\right)$.

(ii) The soft set (F, E) is said to be soft S-bounded, if there exists $\widetilde{r} > \widetilde{0}$ such that $S(x_a, x_a, y_b) < \widetilde{r}$, for all $x_a, y_b \in (F, E)$.

(iii) A soft sequence $\{x^n_{a_n}\}$ in $\left(\widetilde{X}, S, E\right)$ said to converges to x_b, denoted by $\lim\limits_{n\to\infty} x^n_{a_n} = x_b$, if $S\left(x^n_{a_n}, x^n_{a_n}, x_b\right) \to \widetilde{0}$ as $n \to \infty$.

(iv) A soft sequence $\{x^n_{a_n}\}$ in $\left(\widetilde{X}, S, E\right)$ is called a Cauchy sequence, if for $\widetilde{\varepsilon} > \widetilde{0}$ there exists $n_0 \in \mathbb{N}$ such that $S\left(x^n_{a_n}, x^n_{a_n}, x^m_{a_m}\right) < \widetilde{\varepsilon}$, for each $n, m \geq n_0$.

(v) The soft S-metric space $\left(\widetilde{X}, S, E\right)$ is said to be complete, if every Cauchy sequence is convergent.

Definition 3.4. Let $\{x^n_{a_n}\}$ be a sequence in a soft S-metric space $\left(\widetilde{X}, S, E\right)$. A soft point $x_a \in SP(\widetilde{X})$ is a cluster soft point of the sequence $\{x^n_{a_n}\}$, if for every soft open ball $B_S(x_a, \widetilde{\varepsilon})$ and for every $n \in \mathbb{N}$, there exists a $m \geq n$ such that $x^m_{a_m} \in B_S(x_a, \widetilde{\varepsilon})$.

Definition 3.5. Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space. $\left(\widetilde{X}, S, E\right)$ is called a soft sequential compact S-metric space, if every soft sequence has a soft subsequence that converges in $\left(\widetilde{X}, S, E\right)$.

Proposition 3.6. *Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space. If $\left(\widetilde{X}, S, E\right)$ is a soft sequential compact S-metric space, then (X, S_a) is sequential compact S-metric space, for each $a \in E$.*

Proof. Let $\{x^n\}$ be any sequence in (X, S_a), for each $a \in E$. Then the sequence $\{x^n\}$ is written as $\{x_a^n\}$ in soft S-metric space. The sequence $\{x_a^n\}$ has soft convergent subsequence $\{x_a^{n_k}\}$. This means that the subsequence $\{x^{n_k}\}$ converges in (X, S_a). □

The converse of Proposition 3.6 may not be true in general.

Example 3.7. Let $E = \mathbb{N}, X = [0, 1]$ and soft S-metric defined as follows

$$S\left(x_a, y_b, z_c\right) = |a - c| + |b - c| + |x - z| + |y - z|,$$

where $\mathbb{N}$ is a natural number set. It is clear that (X, S_a) is a sequential compact S-metric space, for each $a \in E$. However, the soft sequence $\left\{\left(\frac{1}{2^k}\right)_{a_n}^n\right\}_{k,n}$ does not have a convergent soft subsequence in $\left(\widetilde{X}, S, E\right)$.

Proposition 3.8. *Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space. Then $\left(\widetilde{X}, S, E\right)$ is a soft sequential compact S-metric space if and only if every infinite soft set has a cluster soft point.*

Proof. Let (F, E) be an infinite soft set and $\left\{x_{a_n}^n\right\}$ be a soft sequence in (F, E). Then $\left\{x_{a_n}^n\right\}$ has a convergent soft subsequence $\left\{x_{a_{n_k}}^{n_k}\right\}$. Assume that $\left\{x_{a_{n_k}}^{n_k}\right\}$ converges to a soft point x_a. Since

$$\Phi \neq B_S\left(x_a, \widetilde{\varepsilon}\right) \cap \left\{x_{a_{n_k}}^{n_k}\right\} \subset B_S\left(x_a, \widetilde{\varepsilon}\right) \cap (F, E),$$

x_a is a cluster soft point in (F, E).

Conversely, let $\left\{x_{a_n}^n\right\}$ be an arbitrary soft sequence in $\left(\widetilde{X}, S, E\right)$. If $\left\{x_{a_n}^n\right\}$ is finite, then there exists a fixed convergent subsequence of the soft sequence. If we take $\left\{x_{a_n}^n\right\} = (F, E)$, then $\left\{x_{a_n}^n\right\}$ is a soft set. According to condition, this soft set has a cluster soft point as x_a. Since

$$B_S\left(x_a, \widetilde{\varepsilon}\right) \cap (F, E) \neq \Phi,$$

we choose $\left\{x_{a_{n_k}}^{n_k}\right\} \in B_S\left(x_a, \widetilde{\varepsilon}\right) \cap (F, E)$, for each $\widetilde{\varepsilon} > \widetilde{0}$. Thus the subsequence $\left\{x_{a_{n_k}}^{n_k}\right\}$ converges to x_a. □

Definition 3.9. Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space and (D,E) be a soft set of soft points. If $\widetilde{X} \subset \bigcup_{x_a \in (D,E)} B_S\left(x_a, \widetilde{\varepsilon}\right)$ is satisfied, then (D,E) is said to be a soft $\widetilde{\varepsilon}$-net in soft S-metric space $\left(\widetilde{X}, S, E\right)$.

Definition 3.10. Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space. If for each $\widetilde{\varepsilon} > \widetilde{0}$, there exists a finite soft $\widetilde{\varepsilon}$-net of $\left(\widetilde{X}, S, E\right)$, then $\left(\widetilde{X}, S, E\right)$ is said to be totally bounded soft S- metric space.

Definition 3.11. Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space and (D,E) be a soft set of soft points. The diameter of a non-empty soft set in $\left(\widetilde{X}, S, E\right)$ is defined by:

$$S\left((D,E)\right) = \sup\left\{S\left(x_a, x_a, y_b\right) : x_a, y_b \in (D,E)\right\}.$$

Lemma 3.12. *Let $\left(\widetilde{X}, S, E\right)$ be a totally bounded soft S-metric space and (F,E) be an infinite soft set. Then for each $\widetilde{\varepsilon} > \widetilde{0}$, there exists an infinite soft set $(G,E) \subset (F,E)$ such that $S\left((G,E)\right) < \widetilde{\varepsilon}$.*

Proof. Let $\widetilde{\varepsilon} > \widetilde{0}$ be arbitrary. Since $\left(\widetilde{X}, S, E\right)$ is a totally bounded soft S-metric space, there exists a finite soft $\widetilde{\varepsilon}-$net as $\widetilde{H} = \left\{x^1_{a_1}, x^2_{a_2}, ..., x^n_{a_n}\right\}$. Then we have

$$\widetilde{X} = \bigcup_{i=1}^{n} B_S\left(x^i_{a_i}, \widetilde{\varepsilon}\right) \text{ and } \widetilde{F} = \bigcup_{i=1}^{n} B_S\left(x^i_{a_i}, \widetilde{\varepsilon}\right) \cap (F,E).$$

For $1 \leq i \leq n$, at least one of the soft sets $\widetilde{F}$ must have infinite elements. If we denote this soft set as (G,E), it is clear that $S\left((G,E)\right) < \widetilde{\varepsilon}$. ☐

Theorem 3.13. *Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space. $\left(\widetilde{X}, S, E\right)$ is a totally bounded soft S-metric space if and only if every soft sequence has a Cauchy subsequence in $\widetilde{X}$.*

Proof. Let $\left(\widetilde{X}, S, E\right)$ be a totally bounded soft S-metric space and $\left\{x^n_{a_n}\right\}$ be any soft sequence. If $(F,E) = \left\{x^n_{a_n}\right\}$ is finite, the proof is completed. Assume that $(F,E) = \left\{x^n_{a_n}\right\}$ is infinite. Then from Lemma 3.12, there exists an infinite soft set $(F_1,E) \subset (F,E)$ such that $S\left((F_1,E)\right) < \widetilde{1}$. We choose the number $n_1 \in \mathbb{N}$ such that $x^{n_1}_{a_{n_1}} \in (F_1,E)$. If we apply to the Lemma 3.12 to (F_1,E), we obtain an infinite $(F_2,E) \subset (F_1,E)$ such that $S\left((F_2,E)\right) < \widetilde{\frac{1}{2}}$. Here we take the number $n_2 > n_1$ such that $x^{n_2}_{a_{n_2}} \in (F_2,E)$. Thus we get the soft subsequence $\left\{x^{n_k}_{a_{n_k}}\right\}$.

Now let us show that the sequence $\left\{x^{n_k}_{a_{n_k}}\right\}$ is a Cauchy subsequence. For arbitrary $\widetilde{\varepsilon} > \widetilde{0}$, we choose the number k_o such that $\widetilde{\frac{1}{k_o}} < \widetilde{\varepsilon}$. Then since $x^{n_k}_{a_{n_k}}, x^{n_m}_{a_{n_m}} \in (F_{k_o},E)$ for each $k, m \geq k_o$,

$$S\left(x^{n_k}_{a_{n_k}}, x^{n_k}_{a_{n_k}}, x^{n_m}_{a_{n_m}}\right) < \widetilde{\frac{1}{k_o}} < \widetilde{\varepsilon}$$

is satisfied. This means that $\left\{x_{a_{n_k}}^{n_k}\right\}$ is a Cauchy subsequence.

Conversely, assume that $\left(\widetilde{X}, S, E\right)$ is not totally bounded soft S-metric space. In this case, $\left(\widetilde{X}, S, E\right)$ has not a finite soft $\widetilde{\varepsilon}_0$-net, for some $\widetilde{\varepsilon}_0 > \widetilde{0}$. Let $x_{a_1}^1 \in \widetilde{X}$ be an arbitraray soft point. Then there can be found a soft point $x_{a_2}^2 \in \widetilde{X}$ such that $S\left(x_{a_1}^1, x_{a_1}^1, x_{a_2}^2\right) \geq \widetilde{\varepsilon}_0$. Since the soft set $\left\{x_{a_1}^1, x_{a_2}^2\right\}$ is not a $\widetilde{\varepsilon}_0$-net, there is a soft point $x_{a_3}^3 \in \widetilde{X}$ such that $S\left(x_{a_1}^1, x_{a_1}^1, x_{a_3}^3\right) \geq \widetilde{\varepsilon}_0$, $S\left(x_{a_2}^2, x_{a_2}^2, x_{a_3}^3\right) \geq \widetilde{\varepsilon}_0$. Thus we form a soft sequence $\left\{x_{a_k}^k\right\}$ such that $S\left(x_{a_i}^i, x_{a_i}^i, x_{a_j}^j\right) \geq \widetilde{\varepsilon}_0$, for all i, j. It is clear that $\left\{x_{a_k}^k\right\}$ has not a Cauchy subsequence which contradicts the fact that every soft sequence has a Cauchy subsequence given by assumption. □

Theorem 3.14. *Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space. $\left(\widetilde{X}, S, E\right)$ is a soft sequential compact S-metric space if and only if $\left(\widetilde{X}, S, E\right)$ is both complete and totally bounded soft S-metric space.*

Proof. Let $\left(\widetilde{X}, S, E\right)$ be a soft sequential compact S- metric space. Then every soft sequence $\left\{x_{a_n}^n\right\}$ has a soft subsequence that converges in $\left(\widetilde{X}, S, E\right)$. Since the soft subsequence is a Cauchy sequence in $\left(\widetilde{X}, S, E\right)$, by Theorem 3.13, $\left(\widetilde{X}, S, E\right)$ is totally bounded soft S-metric space. If $\left\{x_{a_n}^n\right\}$ is a Cauchy sequence in $\left(\widetilde{X}, S, E\right)$ which has a convergent soft subsequence, then it is also convergent, i.e., $\left(\widetilde{X}, S, E\right)$ is complete soft S-metric space.

Conversely, let $\left(\widetilde{X}, S, E\right)$ be a complete and totally bounded soft S-metric space and $\left\{x_{a_n}^n\right\}$ be an arbitrary soft sequence. Since $\left(\widetilde{X}, S, E\right)$ is totally bounded, $\left\{x_{a_n}^n\right\}$ has a Cauchy subsequence. Since $\left(\widetilde{X}, S, E\right)$ is complete soft S-metric space, the Cauchy subsequence converges. Then $\left(\widetilde{X}, S, E\right)$ is a soft sequential compact S-metric space. □

Definition 3.15. Let $\left(\widetilde{X}, S, E\right)$ be a soft S-metric space and a family $\mathcal{U}$ be a soft open cover of $\left(\widetilde{X}, S, E\right)$. A number $\widetilde{\varepsilon} > \widetilde{0}$ is called a Lebesque number of $\mathcal{U}$, if for each soft point $x_a \in SP(\widetilde{X})$, there exists $(F, E) \in \mathcal{U}$ such that $B_S\left(x_a, \widetilde{\varepsilon}\right) \subset (F, E)$, for all soft points $x_a \in SP(\widetilde{X})$.

Proposition 3.16. *If $\left(\widetilde{X}, S, E\right)$ is a soft sequentially compact $S-$metric space, then every soft open cover in $\left(\widetilde{X}, S, E\right)$ has a Lebesque number.*

Proof. Assume that soft open cover $\mathcal{U}$ has not a Lebesque number. Then for any n and for each $(F, E) \in \mathcal{U}$ there exists a soft sequence $\left\{x_{a_n}^n\right\}$, where $B_S\left(x_{a_n}^n, \widetilde{\frac{1}{n}}\right) \subset (F, E)$ is not satisfied. Thus we obtain a soft sequence $\left\{x_{a_n}^n\right\}$ satisfying the above condition. Since $\left(\widetilde{X},S,E\right)$ is a soft sequentially compact S-metric space, $\left\{x_{a_n}^{n}\right\}$ has a soft subsequence $\left\{x_{a_{n_k}}^{n_k}\right\}$ converging to x_a. Let $x_a\in(F,E)\,\widetilde{\in}\,\mathcal{U}$. Since (F,E) is a soft open set, there is a soft open ball $B_S\left(x_a,\widetilde{\frac{2}{m}}\right)$ such that $B_S\left(x_a,\widetilde{\frac{2}{m}}\right)\subset(F,E)$. Also since $\left\{x_{a_{n_k}}^{n_k}\right\}$ converges to x_a, there exists a number k_o such that $x_{a_{n_k}}^{n_k}\in B_S\left(x_a,\widetilde{\frac{2}{m}}\right)$, whenever $k\geq k_o$. We take the number $k\geq k_o$ as $n_k\geq m$. Then

$$B_S\left(x_{a_{n_k}}^{n_k},\widetilde{\frac{1}{n_k}}\right)\subset B_S\left(x_a,\widetilde{\frac{2}{m}}\right)\subset(F,E)\in\mathcal{U}.$$

This contradicts with the choice of the soft point $x_{a_{n_k}}^{n_k}$. □

Theorem 3.17. *Let $\left(\widetilde{X},S,E\right)$ be a soft S-metric space. Then the following statements are equivalent:*

(1) $\left(\widetilde{X},S,E\right)$ is a soft compact S-metric space,

(2) $\left(\widetilde{X},S,E\right)$ is a soft sequentially compact S- metric space.

Proof. (1)⇒ (2): Let $\left(\widetilde{X},S,E\right)$ be a soft compact S-metric space but not a soft sequential compact S-metric space. Then there is an infinite soft set (F,E) which does not have a cluster point in $\left(\widetilde{X},S,E\right)$. Thus there is a soft number $\widetilde{r}_{x_a}$ such that $B_S\left(x_a,\widetilde{r}_{x_a}\right)\cap(F,E)=\{x_a\}$, for all $x_a\in(F,E)$. A family $\{B_S\left(x_a,\widetilde{r}_{x_a}\right)\}_{x_a\in(F,E)}\cup(F,E)^c$ is a soft open cover in $\left(\widetilde{X},S,E\right)$ and this soft open cover has not a finite soft subcover. But this implies that $\left(\widetilde{X},S,E\right)$ is not soft compact S-metric space, which is a contradiction.

(2)⇒ (1): Let $\left(\widetilde{X},S,E\right)$ be a soft sequential compact S-metric space and $\mathcal{U}$ be any soft open cover in $\left(\widetilde{X},S,E\right)$. Then by Proposition 3.16, $\mathcal{U}$ has a Lebesque number $\widetilde{\varepsilon}>\widetilde{0}$. Since $\left(\widetilde{X},S,E\right)$ is totally bounded soft S-metric space, $\left(\widetilde{X},S,E\right)$ has finite $\widetilde{\frac{\varepsilon}{3}}$-net as $\left\{x_{a_1}^{1},x_{a_2}^{2},...,x_{a_n}^{n}\right\}$. For each $k=1,2,...,n$,

$$S\left(B_S\left(x_{a_k}^{k},\widetilde{\frac{\varepsilon}{3}}\right)\right)\leq\widetilde{\frac{2\varepsilon}{3}}\leq\widetilde{\varepsilon}.$$

Thus we obtain a soft set $B_S\left(x_{a_k}^{k},\widetilde{\frac{\varepsilon}{3}}\right)\subset(F_k,E)\,\widetilde{\in}\,\mathcal{U}$. Since

$$\widetilde{X}=\bigcup_{k=1}^{n}B_S\left(x_{a_k}^{k},\widetilde{\frac{\varepsilon}{3}}\right)\subset\bigcup_{k=1}^{n}(F_k,E),$$

$\left(\widetilde{X},S,E\right)$ is a soft compact S-metric space. □

Definition 3.18. Let $\left(\widetilde{X}, S, E\right)$ and $\left(\widetilde{Y}, S', E'\right)$ be two soft S-metric spaces. Then the mapping $(f,\varphi) : \left(\widetilde{X}, S, E\right) \to \left(\widetilde{Y}, S', E'\right)$ is called soft uniformly continuous mapping, if for every $\widetilde{\varepsilon} > \widetilde{0}$, there exists a $\widetilde{\delta} > \widetilde{0}$ such that

$$S'\left((f,\varphi)(x_a), (f,\varphi)(x_a), (f,\varphi)(y_b)\right) < \widetilde{\varepsilon},$$

whenever $x_a, y_b \in SP(\widetilde{X})$ satisfy $S(x_a, x_a, y_b) < \widetilde{\delta}$.

Proposition 3.19. *If $(f,\varphi) : \left(\widetilde{X}, S, E\right) \to \left(\widetilde{Y}, S', E'\right)$ is a soft uniformly continuous mapping, then $f_a : (X, S_a) \to \left(Y, S'_{\varphi(a)}\right)$ is also uniformly continuous mapping for each $a \in E$.*

Proof. The proof is straightforward. □

The converse of the the Proposition 3.19 may not be true in general. This is shown by the following example.

Example 3.20. Let $E = \mathbb{R}$ be a set of parameters and $X = \mathbb{R}^2$. Consider usual metrics on this sets and define soft $S-$metric by $S(x_a, y_b, z_c) = |a-c| + |b-c| + d(x,z) + d(y,z)$. Then if we define the soft mapping $(f,\varphi) : \left(\widetilde{X}, S, E\right) \to \left(\widetilde{X}, S, E\right)$ as follows $(f,\varphi)(x_a) = \left(\frac{1}{4}x\right)_{3a}$, then

$$\begin{aligned} S\left((f,\varphi)(0,1)_2, (f,\varphi)(0,1)_2, (f,\varphi)(1,0)_1\right) &= S\left(\left(0,\frac{1}{4}\right)_6, \left(0,\frac{1}{4}\right)_6, \left(\frac{1}{4},0\right)_3\right) \\ &= 6 + \frac{\sqrt{2}}{2}, \end{aligned}$$

$$S\left((0,1)_2, (0,1)_2, (1,0)_1\right) = 2 + 2\sqrt{2}.$$

Since $6 + \frac{\sqrt{2}}{2} > 2 + 2\sqrt{2}$, we see that the soft mapping (f,φ) is not a soft contraction mapping and thus it is not soft uniformly continuous. But the mapping $f_a : (X, S_a) \to (X, S_{3a})$ is uniformly continuous mapping, for each $a \in E$.

Theorem 3.21. *If $(f,\varphi) : \left(\widetilde{X}, S, E\right) \to \left(\widetilde{Y}, S', E'\right)$ is a soft continuous mapping and $\left(\widetilde{X}, S, E\right)$ is a soft sequentially compact $S-$metric space, then (f,φ) is a soft uniformly continuous mapping.*

Proof. Since $\left(\widetilde{X}, S, E\right)$ is a soft sequentially compact S- metric space, it is also soft compact metric space. Since (f,φ) is a soft continuous mapping, for any $\widetilde{\varepsilon} > \widetilde{0}$ and for any soft point x_a, there exists a soft number $\widetilde{\delta} > \widetilde{0}$ such that

$$\text{for every soft point } y_b,\ S(x_a, x_a, y_b) < 2\widetilde{\delta}(x_a),$$

where we have $S'\left((f,\varphi)(x_a), (f,\varphi)(x_a), (f,\varphi)(y_b)\right) < \frac{\widetilde{\varepsilon}}{2}$.
Then the family $\mathcal{U}=\left\{B_S\left(x_a, \widetilde{\delta}(x_a)\right)\right\}_{x_a \in SP(\widetilde{X})}$ is a soft open cover in $\left(\widetilde{X}, S, E\right)$.

Since $\left(\widetilde{X}, S, E\right)$ is a soft compact S-metric space, this soft open cover has a finite soft subcover such as

$$\left\{B_S\left(x^1_{a_1}, \widetilde{\delta}\left(x^1_{a_1}\right)\right), ..., B_S\left(x^n_{a_n}, \widetilde{\delta}\left(x^n_{a_n}\right)\right)\right\}.$$

Let us take

$$\widetilde{\delta} = \min\left\{\widetilde{\delta}\left(x^1_{a_1}\right), ..., \widetilde{\delta}\left(x^n_{a_n}\right)\right\}.$$

Now consider only two soft points $x_a, y_b \in SP(\widetilde{X})$ such that $S\left(x_a, x_a, y_b\right) < \widetilde{\delta}$. Assume that $z_c \in B_S\left(x^i_{a_i}, \widetilde{\delta}\left(x^i_{a_i}\right)\right)$, $1 \leq i \leq n$. Then $S\left(z_c, z_c, x^i_{a_i}\right) < \widetilde{\delta}\left(x^i_{a_i}\right)$ and $S\left(y_b, y_b, x^i_{a_i}\right) \leq 2\widetilde{\delta}\left(x^i_{a_i}\right)$ is satisfied. Since (f, φ) is a soft continuous mapping at the soft point $x^i_{a_i}$, $S'\left((f,\varphi)\left(z_c\right), (f,\varphi)\left(z_c\right), (f,\varphi)\left(x^i_{a_i}\right)\right) < \frac{\widetilde{\varepsilon}}{2}$. So (f, φ) is a soft uniformly continuous mapping. □

4. Conclusion

The purpose of this paper is to contribute for investigating on soft S- metric space. Firstly, we focus on compact sets in soft S-metric space and explore the differences and similarities between the point set topology and soft topology. In addition to, we define the concepts of sequential compact and totally bounded in soft S-metric space and prove some important theorems on this space. Finally, we give Lebesque number for soft open cover in soft S-metric space. We show that soft compact S-metric space and soft sequentially compact S-metric space are equivalent structures. Moreover, we introduce soft uniformly continuous mapping and examine some of its properties.

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