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ABSTRACT. In this paper, we investigate rough $\Delta\mathcal{I}_2$-convergence as an extension of rough convergence. We define the set of rough $\Delta\mathcal{I}_2$-limit points of a sequence and analyze the results with proofs.

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Corresponding Author: Ömer KİŞİ (okisi@bartin.edu.tr)

1. Introduction

Statistical convergence of a real number sequence was firstly originated by Fast [18]. Theory of $\mathcal{I}$-convergence has become an important working area after the study of Kostyrko et al. [29]. Nuray and Ruckle [37] introduced the same concept as generalized statistical convergence. A lot of improvement has been made in the area of $\mathcal{I}$-convergence after the studies of [5, 6, 10, 11, 17, 20, 30, 34, 35, 36, 42].

Rough convergence was firstly given by Phu [38, 39, 40] in finite dimensional normed spaces. He denoted that the set LIM_x^r is bounded, convex and closed and he investigated the concept of rough Cauchy sequence. Aytar [4] examined of rough statistical convergence and obtained the set of rough statistical limit points of a sequence and showed two statistical convergence criteria related with this set. Also, Aytar [3] obtained the rough limit set and the core of a real sequence. In [33], rough statistical convergence of double sequences in finite dimensional normed linear spaces was studied. Rough convergence of double sequences has been originated by Malik and Maity [32] and some properties of this convergence were given. Dündar and Çakan [15, 16] investigated the rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and also obtained the notion of rough convergence and the set of rough limit points of a double sequence. Also, we refer ([1, 2, 12, 13, 14, 19, 21, 22, 23, 24, 25, 41, 43]) for details in the area of rough convergence.

The concept of rough sets, fuzzy sets and intuitionistic fuzzy sets are closely related. We benefit from some important studies to prepear our study (for details, see [7, 8, 9, 26, 27, 28, 31]).

In view of the recent studies of rough convergence, it looks like very natural to extend the interesting concept of rough $\Delta\mathcal{I}_2$-convergence further by using ideals which we mainly do here.

2. Definitions and notations

We recall some notations and basic definitions used in this paper.

Definition 2.1 ([29])**.** A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal, if it satisfies the following conditions:

(i) $emptyset \in \mathcal{I}$,

(ii) for each $\mathcal{P}, \mathcal{R} \in \mathcal{I}$, $\mathcal{P} \cup \mathcal{R} \in \mathcal{I}$,

(iii) for each $\mathcal{P} \in \mathcal{I}$ and each $\mathcal{R} \subseteq \mathcal{P}$, $\mathcal{R} \in \mathcal{I}$.

An ideal is called non-trivial, if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible, if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 2.2 ([29])**.** A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$, if it satisfies the following conditions:

(i) $\varnothing \notin \mathcal{F}$,

(ii) for each $\mathcal{P}, \mathcal{R} \in \mathcal{F}$, have $\mathcal{P} \cap \mathcal{R} \in \mathcal{F}$,

(iii) for each $\mathcal{P} \in \mathcal{F}$ and each $\mathcal{R} \supseteq \mathcal{P}$, $\mathcal{R} \in \mathcal{F}$.

Lemma 2.3 ([29])**.** *If $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is a nontrivial ideal, then the family of sets*

$$\mathcal{F}(\mathcal{I}) = \{\mathcal{M} \subset \mathbb{N} : \exists \mathcal{A} \in \mathcal{I} : \mathcal{M} = \mathbb{N} \setminus \mathcal{A}\}$$

is a filter of $\mathbb{N}$.

In this case, $\mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.4 ([20])**.** A real sequence $x = (x_k)$ is said to be Δ-ideal convergent to $x \in \mathbb{R}$, provided for each $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |\Delta x_k - x| \geq \varepsilon\} \in \mathcal{I}.$$

Definition 2.5 ([6])**.** A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible, if $\{i\} \times \mathbb{N},\ \mathbb{N} \times \{i\} \in \mathcal{I}_2$, for each $i \in \mathbb{N}$.

It is obvious that a strongly admissible ideal is admissible also. We will take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.6 ([6])**.** A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$, we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write

$$\mathcal{I}_2\text{-}\lim_{m,n\to\infty} x_{mn} = L.$$

Definition 2.7 ([38])**.** A sequence $x = (x_k)$ is said to be r-convergent to x_*, denoted by $x_k \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \; \exists i_\varepsilon \in \mathbb{N}: \;\; k \geq i_\varepsilon \Rightarrow \|x_k - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_k \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_k)$.

During the paper, let $r > 0$ and $\mathbb{R}^n$ demonstrate the real n-dimensional space with the norm $\|.\|$. Think $x = (x_k) \subset X = \mathbb{R}^n$.

Definition 2.8 ([15])**.** A sequence $x = (x_k)$ is said to be rough $\mathcal{I}$-convergent (r-$\mathcal{I}$-convergent) to x_* with the roughness degree r, denoted by $x_k \xrightarrow{r-\mathcal{I}} x_*$, provided that $\{k \in \mathbb{N} : \|x_k - x_*\| \geq r + \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$ or equivalently, if the condition

$$\mathcal{I}\text{-}\limsup \|x_k - x_*\| \leq r$$

holds. We can write $x_k \xrightarrow{r-\mathcal{I}} x_*$ iff the inequality $\|x_k - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all k.

Definition 2.9 ([15])**.** A double sequence $x = (x_{mn})$ is called to be rough convergent to x_*, provided that

$$\forall \varepsilon > 0 \;\; \exists k_\varepsilon \in \mathbb{N} : m, n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon,$$

with the roughness degree r or equivalently, if

$$\limsup \; \|x_{mn} - x_*\| \leq r.$$

In this case, we write $x_{mn} \xrightarrow{r} x_*$.

Definition 2.10 ([16])**.** A double sequence $x = (x_{mn})$ is called to be r-$\mathcal{I}_2$-convergent to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*$, provided that

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I}_2\text{-}\limsup \|x_{mn} - x_*\| \leq r$$

is satisfied. In addition, we can write $x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*$ iff the inequality $\|x_{mn} - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all (m, n).

3. Main results

Definition 3.1. For some given real number $r \geq 0$, a double sequence (Δx_{kl}) is called to be rough $\mathcal{I}_2$-convergent to x_* with the roughness degree r, provided that

$$\{(k, l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2, \tag{3.1}$$

for every $\varepsilon > 0$ or similarly, if the statement

$$\mathcal{I}_2 - \limsup \|\Delta x_{kl} - x_*\| \leq r \tag{3.2}$$

is satisfied. In addition, we can write $\Delta x \xrightarrow{r-\mathcal{I}_2} x_*$ if the inequality $\|\Delta x_{kl} - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and for almost all (k, l).

We obtain the concept of $\mathcal{I}_2$-cluster point of (Δx_{kl}) and of $\mathcal{I}_2$-boundedness for a double sequence (Δx_{kl}).

$c \in \mathbb{R}^n$ is called a $\mathcal{I}_2$-cluster point of a double sequence (Δx_{kl}) stated that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - c\| < \varepsilon\} \notin \mathcal{I}_2$$

for every $\varepsilon > 0$. We demonstrate the set of all $\mathcal{I}_2$-cluster point of a double sequence (Δx_{kl}) by $\mathcal{I}_2 (\Gamma_{\Delta x})$.

A double sequence (Δx_{kl}) is called to be $\mathcal{I}_2$-bounded ,if there exists $M > 0$ such that

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl}\| \geq M\} \in \mathcal{I}_2.$$

Remark 3.2. r-convergence implies r-$\Delta\mathcal{I}_2$-convergence according as the roughness degree.

If we take $r = 0$, then we get the ordinary $\Delta\mathcal{I}_2$-convergence of a double sequence.

Overall, the r-$\mathcal{I}_2$-limit of a double sequence might not be unique for the roughness degree r. Hence, we should think the so-called rough $\mathcal{I}_2$-limit set of (Δx_{kl}), which is shown by

$$\mathcal{I}_2\text{-LIM}^r \Delta x := \left\{x^* \in \mathbb{R}^n : \Delta x_{kl} \overset{r-\mathcal{I}_2}{\to} x_*\right\}.$$

A double sequence (Δx_{kl}) is called to be r-$\mathcal{I}_2$-convergent if $\mathcal{I}_2$-$LIM^r \Delta x \neq \varnothing$.

As indicated above, we can't say that the r-$\mathcal{I}_2$-limit of a double sequence is unique for the roughness degree $r > 0$. The following theorem is related to this claim.

Theorem 3.3. *For a sequence* (Δx_{kl}),

$$diam\,(\mathcal{I}_2\text{-}LIM^r\,(\Delta x_{kl})) \leq 2r.$$

In general, $diam\,(\mathcal{I}_2\text{-}LIM^r\,(\Delta x_{kl}))$ *has no smaller bound.*

Proof. Assume that

$$diam\,(\mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})) = \sup\{\|y - z\|\,; y, z \in \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})\} > 2r.$$

Then, there exist $y, z \in \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$ such that $\|y - z\| > 2r$. Now, we select $\varepsilon > 0$ so that $\varepsilon < \frac{\|y-z\|}{2} - r$. Since $y, z \in \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$, for every $\varepsilon > 0$, we have

$$A_1\,(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y\| \geq r + \varepsilon\} \in \mathcal{I}_2$$

and

$$A_2\,(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - z\| \geq r + \varepsilon\} \in \mathcal{I}_2.$$

In that case, we have

$$A_1^c\,(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y\| < r + \varepsilon\} \in \mathcal{F}\,(\mathcal{I}_2)$$

and

$$A_2^c\,(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - z\| < r + \varepsilon\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

Using the properties of $\mathcal{F}\,(\mathcal{I}_2)$, $A_1^c\,(\varepsilon) \cap A_2^c\,(\varepsilon) \neq \varnothing$ and we get$A_1^c\,(\varepsilon) \cap A_2^c\,(\varepsilon) \in \mathcal{F}\,(\mathcal{I}_2)\,.$

Thus, we can write

$$\|y - z\| \leq \|\Delta x_{kl} - y\| + \|\Delta x_{kl} - z\| < 2\,(r + \varepsilon) < 2\left(r + \frac{\|y - z\|}{2} - r\right) = \|y - z\|\,,$$

for all $(k,l) \in A_1^c(\varepsilon) \cap A_2^c(\varepsilon)$, which is contradiction. So

$$diam\,(\mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})) \leq 2r.$$

For proof the second part of the theorem, take a double sequence (Δx_{kl}) such that $\mathcal{I}_2\text{-}\lim \Delta x_{kl} = x_*$. Let $\varepsilon > 0$. Then, we can write

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| \geq \varepsilon\} \in \mathcal{I}_2.$$

Thus, we have

$$\|\Delta x_{kl} - y\| \leq \|\Delta x_{kl} - x_*\| + \|x_* - y\| \leq \|\Delta x_{kl} - x_*\| + r < r + \varepsilon$$

for each

$$y \in \overline{B}_r\,(x_*) = \{y \in \mathbb{R}^n : \|y - x_*\| \leq r\}$$

So, we get

$$\|\Delta x_{kl} - y\| < r + \varepsilon,$$

for each $(k,l) \in \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| < \varepsilon\}$. Since the double sequence (Δx_{kl}) is $\mathcal{I}_2$-convergent to x_*, we have

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - x_*\| < \varepsilon\} \in \mathcal{F}\,(\mathcal{I}_2)\,.$$

Hence, we have $y \in \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$ and as a result,

$$\mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl}) = \overline{B}_r\,(x_*)\,. \tag{3.3}$$

This shows that in general upper bound $2r$ of the diameter of the set $\mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$ can not be decreased any more. □

Now we give some geometrical and topological properties of the r-$\mathcal{I}_2$-limit set of (Δx_{kl}).

Theorem 3.4. *The set $\mathcal{I}_2$-LIM$^r\,(\Delta x_{kl})$ of a sequence (Δx_{kl}) is a closed set.*

Proof. If $\mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl}) = \varnothing$, then there is nothing to prove.

Assume that $\mathcal{I}_2\text{-LIM}^r \neq \varnothing$. Now, consider a double sequence (Δy_{kl}) in $\mathcal{I}_2$-LIM$^r\,(\Delta x_{kl})$ with $\lim\limits_{k,l\to\infty} \Delta y_{kl} = y_*$. We must show that $y_* \in \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$. Choose $\varepsilon > 0$. Since $\Delta y_{kl} \to y_*$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|\Delta y_{kl} - y_*\| < \varepsilon,$$

for all $k,l \geq n_\varepsilon$.

Now select an $(k_0,l_0) \in \mathbb{N} \times \mathbb{N}$ such that $k \geq k_0$, $l \geq l_0$. Then, we obtain

$$\|\Delta y_{k_0 l_0} - y_*\| < \varepsilon.$$

In other words, since $(\Delta y_{kl}) \subseteq \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$, we have $(\Delta y_{k_0 l_0}) \in \mathcal{I}_2\text{-LIM}^r\,(\Delta x_{kl})$, that is,

$$A\,(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - \Delta y_{k_0 l_0}\| \geq r + \varepsilon\} \in \mathcal{I}_2. \tag{3.4}$$

Now, let us denote that the inclusion

$$A^c\,(\varepsilon) \subseteq A^c\,(2\varepsilon) \tag{3.5}$$

holds, where

$$A\,(2\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_*\| \geq r + 2\varepsilon\}$$

Put $(p,q) \in A^c(\varepsilon)$. Then we obtain

$$\|\Delta x_{pq} - \Delta y_{k_0 l_0}\| < r + \varepsilon.$$

Thus

$$\|\Delta x_{pq} - y_*\| \leq \|\Delta x_{pq} - \Delta y_{k_0 l_0}\| + \|\Delta y_{k_0 l_0} - y_*\| < r + 2\varepsilon,$$

that is, $(p,q) \in A^c(2\varepsilon)$, which gives (3.5). So, we prove $A(2\varepsilon) \subseteq A(\varepsilon)$. Because of $A(\varepsilon) \in \mathcal{I}_2$ by (3.4) (i.e., $y_* \in \mathcal{I}_2$-LIM$^r(\Delta x_{kl})$), which completes the proof. □

Theorem 3.5. *The set $\mathcal{I}_2$-LIM$^r(\Delta x_{kl})$ of a double sequence (Δx_{kl}) is convex.*

Proof. Let $y_0, y_1 \in \mathcal{I}_2$-LIM$^r(\Delta x_{kl})$ and $\varepsilon > 0$ be given. Let

$$A_0(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_0\| \geq r + \varepsilon\} \in \mathcal{I}_2$$

and

$$A_1(\varepsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - y_1\| \geq r + \varepsilon\} \in \mathcal{I}_2.$$

Then, we have

$$\|\Delta x_{kl} - [(1-\lambda) y_0 + \lambda y_1]\| = \|(1-\lambda)(\Delta x_{kl} - y_0) + \lambda(\Delta x_{kl} - y_1)\| < r + \varepsilon,$$

for each $(k,l) \in A_0^c(\varepsilon) \cap A_1^c(\varepsilon)$ and each $\lambda \in [0,1]$. Because of $(A_0^c(\varepsilon) \cap A_1^c(\varepsilon)) \in \mathcal{F}(\mathcal{I}_2)$, by the property of $\mathcal{F}(\mathcal{I}_2)$, we get

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : \|\Delta x_{kl} - [(1-\lambda) y_0 + \lambda y_1]\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

which shows that $(1-\lambda) y_0 + \lambda y_1 \in \mathcal{I}_2$-LIM$^r(\Delta x_{kl})$ for any $\lambda \in [0,1]$. Thus the set $\mathcal{I}_2$-LIM$^r(\Delta x_{kl})$ is convex. □

Theorem 3.6. *Let $r > 0$. Then, (Δx_{kl}) is rough $\mathcal{I}_2$-convergent to x_* iff there exists (Δy_{kl}) such that*

$$\mathcal{I}_2 - \lim \Delta y_{kl} = x_* \textit{ and } \|\Delta x_{kl} - \Delta y_{kl}\| \leq r, \textit{ for all } (k,l) \in \mathbb{N} \times \mathbb{N}. \tag{3.6}$$

Proof. Necessity: Let $\Delta x \xrightarrow{r-\mathcal{I}_2} x_*$. Then, we have

$$\mathcal{I}_2 - \limsup \|\Delta x_{kl} - x_*\| \leq r. \tag{3.7}$$

Now, we define

$$\Delta y_{kl} := \begin{cases} x_* & \text{if } \|\Delta x_{kl} - x_*\| \leq r \\ \Delta x_{kl} + r\frac{x_* - \Delta x_{kl}}{\|\Delta x_{kl} - x_*\|} & \text{otherwise.} \end{cases}$$

Then, we have

$$\|\Delta y_{kl} - x_*\| = \begin{cases} 0 & \text{if } \|\Delta x_{kl} - x_*\| \leq r \\ \\ \|\Delta x_{kl} - x_*\| - r & \text{otherwise,} \end{cases}$$

by the definition of Δy_{kl},

$$\|\Delta x_{kl} - \Delta y_{kl}\| \leq r \tag{3.8}$$

for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. By (3.7) and the definition of Δy_{kl}, we get

$$\mathcal{I}_2 - \limsup \|\Delta y_{kl} - x_*\| = 0,$$

which implies that $\mathcal{I}_2 - \lim \Delta y_{kl} = x_*$.

Sufficiently: Assume that (3.6) holds. Because of $\mathcal{I}_2-\lim \Delta y_{kl}=x_*$, we have

$$A(\varepsilon)=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta y_{kl}-x_*\|\geq r+\varepsilon\}\in\mathcal{I}_2,$$

for each $\varepsilon>0$. Now we define the set

$$B(\varepsilon)=\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-x_*\|\geq r+\varepsilon\}.$$

It is easy to understand that the inclusion

$$B(\varepsilon)\subseteq A(\varepsilon)$$

holds. Because of $A(\varepsilon)\in\mathcal{I}_2$, we get $B(\varepsilon)\in\mathcal{I}_2$. Hence (Δx_{kl}) is rough $\mathcal{I}_2$-convergent to x_*. □

Lemma 3.7. *For an arbitrary $c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$ of a double sequence (Δx_{kl}), we have*

$$\|x_*-c\|\leq r \text{ for all } x_*\in\mathcal{I}_2\text{-}LIM^r(\Delta x_{kl}).$$

Proof. If possible suppose that there exists $c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$ and $x_*\in\mathcal{I}_2\text{-LIM}^r(\Delta x_{kl})$ such that $\|x_*-c\|>r$. Let $\varepsilon=\frac{\|x_*-c\|-r}{2}$. Then, we have

(3.9)
$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-c\|\geq r+\varepsilon\}\subseteq\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-x_*\|\geq r+\varepsilon\}.$$

Because of $c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$, we get

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-c\|\geq r+\varepsilon\}\notin\mathcal{I}_2.$$

By the definition of $\mathcal{I}_2$-convergence, since

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-x_*\|\geq r+\varepsilon\}\in\mathcal{I}_2,$$

so by (3.9), we have

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-c\|<\varepsilon\}\in\mathcal{I}_2,$$

which contradicts with the fact $c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$. □

Theorem 3.8. (1) *If $c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$, then*

(3.10)
$$\mathcal{I}_2\text{-}LIM^r(\Delta x_{kl})\subseteq\overline{B}_r(c).$$

(2)

(3.11)
$$\mathcal{I}_2\text{-}LIM^r(\Delta x_{kl})=\bigcap_{c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)}\overline{B}_r(c)=\left\{x_*\in\mathbb{R}^n:\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)\subseteq\overline{B}_r(x_*)\right\}.$$

Proof. (1) If $c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$, then by Lemma 3.7, we have $\|x_*-c\|\leq r$, for all

$$x_*\in\mathcal{I}_2-LIM^r(\Delta x_{kl}).$$

Otherwise, we get

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\|\Delta x_{kl}-x_*\|\geq r+\varepsilon\}\notin\mathcal{I}_2,$$

for $\varepsilon:=\frac{\|x_*-c\|-r}{3}$. Because of c is an $\mathcal{I}_2$-cluster point of (Δx_{kl}), this contradicts with the fact that $x_*\in\mathcal{I}_2\text{-LIM}^r(\Delta x_{kl})$.

(2) From (3.10), we have

$$\mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right) \subseteq \bigcap_{c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)} \overline{B}_r\left(c\right). \tag{3.12}$$

Let

$$y \in \bigcap_{c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)} \overline{B}_r\left(c\right).$$

Then, we have $\|y-c\| \leq r$, for all $c \in \mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)$ which is same as

$$\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \overline{B}_r\left(y\right), \text{ i.e.,}$$

$$\bigcap_{c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)} \overline{B}_r\left(c\right) = \left\{x_* \in \mathbb{R}^n : \mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \overline{B}_r\left(x_*\right)\right\}. \tag{3.13}$$

Now, let $y \notin \mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right)$. Then there exists an $\varepsilon > 0$ such that

$$\left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\Delta x_{kl} - y\| \geq r+\varepsilon\right\} \notin \mathcal{I}_2$$

which gives the existence of an $\mathcal{I}_2$-cluster point c of the sequence (Δx_{kl}) with $\|y-c\| \geq r+\varepsilon$, that is,

$$\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \nsubseteq \overline{B}_r\left(y\right) \text{ and } y \notin \left\{x_* \in \mathbb{R}^n : \mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \overline{B}_r\left(x_*\right)\right\}.$$

Thus, $y \in \mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right)$ obtains from $y \in \left\{x_* \in \mathbb{R}^n : \mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \overline{B}_r\left(x_*\right)\right\}$, i.e.,

$$\left\{x_* \in \mathbb{R}^n : \mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \overline{B}_r\left(x_*\right)\right\} \subseteq \mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right) \tag{3.14}$$

So, the relations (3.12)–(3.14) ensure that (3.11) holds, i.e.,

$$\mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right) = \bigcap_{c\in\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)} \overline{B}_r\left(c\right) = \left\{x_* \in \mathbb{R}^n : \mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \overline{B}_r\left(x_*\right)\right\}.$$

□

Theorem 3.9. *Let (Δx_{kl}) be an $\mathcal{I}_2$-bounded sequences. If $r \geq diam\left(\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)\right)$, then we have $\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \mathcal{I}_2\text{-}LIM^r\left(\Delta x_{kl}\right)$.*

Proof. Let $c \notin \mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right)$. Then there exist an $\varepsilon > 0$ such that

$$\left\{(k,l) \in \mathbb{N}\times\mathbb{N} : \|\Delta x_{kl} - c\| \geq r+\varepsilon\right\} \notin \mathcal{I}_2. \tag{3.15}$$

Since (Δx_{kl}) is $\mathcal{I}_2$-bounded and from (3.15), there exists an $\mathcal{I}_2$-cluster point c_1 such that

$$\|c-c_1\| > r+\varepsilon_1,$$

where $\varepsilon_1 := \frac{\varepsilon}{2}$. Thus, we get

$$diam\left(\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)\right) > r+\varepsilon_1,$$

which proves the theorem.

Also, the converse of the theorem holds, i.e., if $\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right) \subseteq \mathcal{I}_2\text{-LIM}^r\left(\Delta x_{kl}\right)$, then we have $r \geq diam\left(\mathcal{I}_2\left(\Gamma_{(\Delta x_{kl})}\right)\right)$. □

4. Conclusion

In the present paper, we have given a more general type of convergence for double sequences, that is, rough $\Delta\mathcal{I}_2$-convergence in a more general setting. This definition and results provide new tools to deal with the convergence problems of double sequences occurring in many branches of science.

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Ömer Kİşİ (okisi@bartin.edu.tr)
Bartın University, Faculty of Science, Depertmant of Mathematics, 74100, Bartın, Turkey