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A new approach to separation and regularity axioms via fuzzy soft sets

I. M. Taha

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Abstract. In the present work, a new form of separation and regularity axioms via fuzzy soft sets are introduced in a fuzzy soft topological space based on the sense of Aygünoğlu. Also, these axioms are introduced by using soft quasi-coincidence and some interesting properties of them are specified. Moreover, the relations of these axioms with each other are investigated with the help of examples.

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Keywords: Fuzzy soft set, Fuzzy soft topology, Soft quasi-coincidence, Fuzzy soft R_i $(i = 0, 1, 2, 3)$ spaces, Fuzzy soft T_i $(i = 1, 2, 2\frac{1}{2}, 3, 4)$ spaces.

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1. Introduction

Soft set theory, introduced by Molodtsov [15] in 1999 as a general mathematical tool for modeling uncertainties. By a soft set we mean a pair (F, E), where E is a set interpreted as the set of parameters and the mapping $F : E \longrightarrow P(X)$ is referred to as the soft structure on X. After the introduction of the notion of soft sets several researchers improved this concept. The soft set theory has been applied to many different fields with great success. Maji et al. [13] introduced the concept of fuzzy soft set which combines fuzzy sets [25] and soft sets [15]. Soft set and fuzzy soft set theories have a rich potential for applications in several directions. So far, lots of spectacular and creative researches about the theories of soft set and fuzzy soft set have been considered by some scholars (See [1, 2, 3, 4, 5, 6, 10, 11, 13, 16, 18, 19, 20]. Also, Aygünoğlu et al. [5] studied the topological structure of fuzzy soft sets based on the sense of Šostak [22]. Shabir et al. [21] and Georgiou et al. [8] defined and studied some soft separation axioms, soft θ-continuity and soft connectedness in soft topological spaces using (ordinary) points of a topological space X. Later, Hussain et al. [9] introduced and studied soft separation axioms using soft points defined by

Zorlutuna ; for more details the reader is referred to [8, 9, 17, 21, 24]. In this paper, fuzzy soft separation and regularity axioms are introduced by using a new approach in a fuzzy soft topological space based on the sense of Aygünoğlu et al. [5]. Further, the relations of these axioms with each other are investigated and some interesting properties of them are specified.

2. Preliminaries

Throughout this paper, X refers to an initial universe, E is the set of all parameters for X and $A \subseteq E$, I^X is the set of all fuzzy sets on X (where, $I = [0, 1]$, $I_0 = (0, 1]$) and for $\alpha \in I$, $\underline{\alpha}(x) = \alpha$, for all $x \in X$. A fuzzy point x_t for $t \in I_0$ is an element of I^X such that $x_t(y) = t$, if y=x and x_t(y)=0, if y$\neq$ x. The family of all fuzzy points in X is denoted by $P_t(X)$. For $\lambda \in I^X$ a fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$.

Definition 2.1 ([1, 5, 13])**.** A fuzzy soft set f_A over X is a mapping from E into I^X such that $f_A(e)$ is a fuzzy set on X, for each $e \in A$ and $f_A(e) = \underline{0}$, if $e \notin A$, where $\underline{0}$ is zero function on X. The fuzzy set $f_A(e)$, for each $e \in E$, is called an element of the fuzzy soft set f_A. $\widetilde{(X, E)}$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe [12].

Definition 2.2 ([14])**.** A fuzzy soft point e_{x_t} over X is a fuzzy soft set defined as follows: $e_{x_t}(e^*) = x_t$, if $e^* = e$ and $e_{x_t}(e^*) = \underline{0}$, if $e^* \in E - \{e\}$. where x_t is a fuzzy point. A fuzzy soft point e_{x_t} is said to belong to a fuzzy soft set f_A, denoted by $e_{x_t} \tilde{\in} f_A$ if $t < f_A(e)(x)$. Two fuzzy soft points e_{x_t} and $e^*_{y_s}$ are said to be distinct, denoted by $e_{x_t} \neq e^*_{y_s}$ if $x \neq y$ or $e \neq e^*$. The family of all fuzzy soft points in X is denoted by $\widetilde{P_t(X)}$.

Definition 2.3 ([3])**.** A fuzzy soft point $e_{x_t} \in \widetilde{P_t(X)}$ is said to be soft quasi-coincident with a fuzzy soft set $f_A \in \widetilde{(X, E)}$ denoted by $e_{x_t} \widetilde{q} f_A$, if $t + f_A(e)(x) > 1$. A fuzzy soft set $f_A \in \widetilde{(X, E)}$ is said to be soft quasi-coincident with a fuzzy soft set $g_B \in \widetilde{(X, E)}$ denoted by $f_A \widetilde{q} g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$. If f_A is not soft quasi-coincident with g_B, $f_A \widetilde{\not{q}} g_B$.

All definitions and properties of fuzzy soft sets, fuzzy soft points and fuzzy soft topology are found in [1, 3, 5, 7, 11, 13, 23].

3. Separation and regularity axioms via fuzzy soft sets

Definition 3.1. Let $r \in I_0$. A fuzzy soft topological space (X, τ_E) is said to be:

(i) r-FSR_0, if for any distinct fuzzy soft points e_{x_t}, $e_{y_s} \in \widetilde{P_t(X)}$ with $e_{x_t} \widetilde{\not{q}} C_\tau(e, e_{y_s}, r)$ implies $e_{y_s} \widetilde{\not{q}} C_\tau(e, e_{x_t}, r)$ for all $e \in E$,

(ii) r-FSR_1, if for any distinct fuzzy soft points e_{x_t}, $e_{y_s} \in \widetilde{P_t(X)}$ with $e_{x_t} \widetilde{\not{q}} C_\tau(e, e_{y_s}, r)$ implies that there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $e_{x_t} \tilde{\in} (g_B)_1$, $e_{y_s} \tilde{\in} (g_B)_2$ and $(g_B)_1 \widetilde{\not{q}} (g_B)_2$ for all $e \in E$,

(iii) r-FSR_2, if $e_{x_t} \widetilde{\not{q}} f_A$ with $\tau_e(f^c_A) \geq r$ implies that there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $e_{x_t} \tilde{\in} (g_B)_1$, $f_A \sqsubseteq (g_B)_2$ and $(g_B)_1 \widetilde{\not{q}} (g_B)_2$ for all $e \in E$,

(iv) r-FSR_3, if $(f_A)_1 \widetilde{\not q} (f_A)_2$ with $\tau_e((f_A)_i^c) \geq r$ for $i \in \{1,2\}$ implies that there exist $(g_B)_i \in \widetilde{(X,E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1,2\}$ such that $(f_A)_i \sqsubseteq (g_B)_i$ and $(g_B)_1 \widetilde{\not q} (g_B)_2$ for all $e \in E$.

Definition 3.2. Let $r \in I_0$. A fuzzy soft topological space (X, τ_E) is said to be:

(i) r-FST_1, if $\tau_e(e^c_{x_t}) \geq r$ for each $e_{x_t} \in \widetilde{P_t(X)}$, $e \in E$,

(ii) r-FST_2 , if $e_{x_t} \widetilde{\not q} e_{y_s}$ implies that there exist $(g_B)_i \in \widetilde{(X,E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1,2\}$ such that $e_{x_t} \tilde{\in} (g_B)_1$, $e_{y_s} \tilde{\in} (g_B)_2$ and $(g_B)_1 \widetilde{\not q} (g_B)_2$ for all $e \in E$,

(iii) r-$FST_{2\frac{1}{2}}$, if $e_{x_t} \widetilde{\not q} e_{y_s}$ implies that there exist $(g_B)_i \in \widetilde{(X,E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1,2\}$ such that $e_{x_t} \tilde{\in} (g_B)_1$, $e_{y_s} \tilde{\in} (g_B)_2$ and $C_\tau(e,(g_B)_1,r) \widetilde{\not q} C_\tau(e,(g_B)_2,r)$ for all $e \in E$,

(iv) r-FST_3, if it is r-FSR_2 and r-FST_1,

(v) r-FST_4, if it is r-FSR_3 and r-FST_1.

Lemma 3.3. *Let (X, τ_E) be a fuzzy soft topological space. Then for each $e \in E$, the following statements hold:*

(1) *For each $f_A \in \widetilde{(X,E)}$ with $\tau_e(f_A) \geq r$, $f_A \widetilde{q} g_B$ iff $f_A \widetilde{q} C_\tau(e, g_B, r)$,*

(2) *$e_{x_t} \widetilde{q} C_\tau(e, f_A, r)$ iff $f_A \widetilde{q} g_B$, for every $g_B \in \widetilde{(X,E)}$ with $\tau_e(g_B) \geq r$ and $e_{x_t} \tilde{\in} g_B$.*

Proof. (1) ($\Rightarrow$) It is clear.

($\Leftarrow$) Suppose there exist $f_A \in \widetilde{(X,E)}$ with $\tau_e(f_A) \geq r$ such that $f_A \widetilde{\not q} g_B$, $g_B \sqsubseteq f_A^c$. Since $\tau_e(f_A) \geq r$, we have $C_\tau(e, g_B, r) \sqsubseteq f_A^c$. It follows that $f_A \widetilde{\not q} C_\tau(e, g_B, r)$.

(2) ($\Rightarrow$) Suppose $e_{x_t} \widetilde{q} C_\tau(e, f_A, r)$, $g_B \widetilde{q} C_\tau(e, f_A, r)$ with $e_{x_t} \tilde{\in} g_B$. By (1), we have $g_B \widetilde{q} f_A$ for each $g_B \in \widetilde{(X,E)}$ with $\tau_e(g_B) \geq r$ and $e_{x_t} \tilde{\in} g_B$.

($\Leftarrow$) Suppose $e_{x_t} \widetilde{\not q} C_\tau(e, f_A, r)$. Then $e_{x_t} \tilde{\in} (C_\tau(e, f_A, r))^c$. Since $f_A \sqsubseteq C_\tau(e, f_A, r)$,

$$f_A \widetilde{\not q} (C_\tau(e, f_A, r))^c,$$

where $\tau_e((C_\tau(e, f_A, r))^c) \geq r$. □

Theorem 3.4. *Let (X, τ_E) be a fuzzy soft topological space and $r \in I_0$. Then for each $e \in E$, the following statements are equivalent:*

(1) *(X, τ_E) is r-FSR_0,*

(2) *if $e_{x_t} \widetilde{\not q} f_A = C_\tau(e, f_A, r)$, then there exists $g_B \in \widetilde{(X,E)}$ with $\tau_e(g_B) \geq r$ such that $e_{x_t} \widetilde{\not q} g_B$,*

(3) *if $e_{x_t} \widetilde{\not q} f_A = C_\tau(e, f_A, r)$, then $C_\tau(e, e_{x_t}, r) \widetilde{\not q} C_\tau(e, f_A, r)$,*

(4) *if $e_{x_t} \widetilde{\not q} C_\tau(e, e_{y_s}, r)$, then $C_\tau(e, e_{x_t}, r) \widetilde{\not q} C_\tau(e, e_{y_s}, r)$,*

(5) *if $e_{x_t} \widetilde{\not q} C_\tau(e, g_B, r)$, then $C_\tau(e, e_{x_t}, r) \widetilde{\not q} g_B$.*

Proof. (1) $\Rightarrow$ (2) Let $e_{x_t} \widetilde{\not q} f_A = C_\tau(e, f_A, r)$. Since $C_\tau(e, e_{y_s}, r) \sqsubseteq C_\tau(e, f_A, r)$ for each $e_{y_s} \tilde{\in} f_A$, $e_{x_t} \widetilde{\not q} C_\tau(e, e_{y_s}, r)$ and $e_{y_s} \widetilde{\not q} C_\tau(e, e_{x_t}, r)$, because (X, τ_E) is r-FSR_0. For each $e_{y_s} \widetilde{\not q} C_\tau(e, e_{x_t}, r)$, by Lemma 3.3 (2), there exists $(h_C)_{e_{y_s}} \in \widetilde{(X,E)}$ with $\tau_e((h_C)_{e_{y_s}}) \geq r$ such that $e_{x_t} \widetilde{\not q} (h_C)_{e_{y_s}}$, $e_{y_s} \tilde{\in} (h_C)_{e_{y_s}}$. Let $g_B = \bigsqcup_{e_{y_s} \tilde{\in} f_A} \{(h_C)_{e_{y_s}} : e_{x_t} \widetilde{\not q} (h_C)_{e_{y_s}}\}$. Then we have $e_{x_t} \widetilde{\not q} g_B$ and $\tau_e(g_B) \geq r$.

(2) $\Rightarrow$ (3) Let $e_{x_t} \widetilde{\not q} f_A = C_\tau(e, f_A, r)$. Then by (2), there exists $g_B \in \widetilde{(X,E)}$ with $\tau_e(g_B) \geq r$ such that $e_{x_t} \widetilde{\not q} g_B$ and $f_A \sqsubseteq g_B$. Since $e_{x_t} \widetilde{\not q} g_B$, $e_{x_t} \tilde{\in} (g_B)^c$. Thus $C_\tau(e, e_{x_t}, r) \sqsubseteq (g_B)^c \sqsubseteq (f_A)^c$. So $C_\tau(e, e_{x_t}, r) \widetilde{\not q} C_\tau(e, f_A, r)$.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are easily proved.

(1) $\Rightarrow$ (5) Suppose $C_\tau(e, e_{x_t}, r)\widetilde{q}g_B$. Then there exists $e_{y_s}\tilde{\in}g_B$ such that

$$e_{y_s}\widetilde{q}C_\tau(e, e_{x_t}, r).$$

Since (X, τ_E) is r-FSR_0, $e_{x_t}\widetilde{q}C_\tau(e, e_{y_s}, r)$. Since $C_\tau(e, e_{y_s}, r) \sqsubseteq C_\tau(e, g_B, r)$,

$$e_{x_t}\widetilde{q}C_\tau(e, g_B, r).$$

(5) $\Rightarrow$ (1) It easily proved. □

Theorem 3.5. *Let (X, τ_E) be a fuzzy soft topological space. Then for each $e \in E$,*

$$(r\text{-}FSR_3 \text{ and } r\text{-}FSR_0) \Rightarrow^{(1)} r\text{-}FSR_2 \Rightarrow^{(2)} r\text{-}FSR_1 \Rightarrow^{(3)} r\text{-}FSR_0.$$

Proof. (1) Let (X, τ_E) be r-FSR_3 and r-FSR_0. Then for $e_{x_t}\,\widetilde{\not q}f_A$ with $\tau_e((f_A)^c) \geq r$, $e_{x_t}\,\widetilde{\not q}f_A = C_\tau(e, f_A, r)$. Thus By r-FSR_0 of (X, τ_E) and by Theorem 3.4 (3), we have $C_\tau(e, e_{x_t}, r)\,\widetilde{\not q}f_A = C_\tau(e, f_A, r)$. So by r-FSR_3 of (X, τ_E), there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $(g_B)_1\,\widetilde{\not q}(g_B)_2$, $e_{x_t}\tilde{\in}C_\tau(e, e_{x_t}, r) \sqsubseteq (g_B)_1$ and $f_A \sqsubseteq (g_B)_2$ for all $e \in E$. Hence (X, τ_E) is r-FSR_2.

(2) Let (X, τ_E) be r-FSR_2 and $e_{x_t}\,\widetilde{\not q}C_\tau(e, e_{y_s}, r)$. Then by r-FSR_2 of (X, τ_E), there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $(g_B)_1\,\widetilde{\not q}(g_B)_2$, $e_{x_t}\tilde{\in}(g_B)_1$ and $e_{y_s}\tilde{\in}C_\tau(e, e_{y_s}, r) \sqsubseteq (g_B)_2$ for all $e \in E$. Thus (X, τ_E) is r-FSR_1.

(3) Let (X, τ_E) be r-FSR_1 and $e_{x_t}\,\widetilde{\not q}C_\tau(e, e_{y_s}, r)$ for any distinct fuzzy soft points $e_{x_t}, e_{y_s} \in \widetilde{P_t(X)}$ and $e \in E$. Then by r-FSR_1 of (X, τ_E), there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $(g_B)_1\,\widetilde{\not q}(g_B)_2$, $e_{x_t}\tilde{\in}(g_B)_1$ and $e_{y_s}\tilde{\in}(g_B)_2$. Thus $e_{x_t}\tilde{\in}(g_B)_1 \sqsubseteq ((g_B)_2)^c$. By the definition of C_τ, $C_\tau(e, e_{x_t}, r) \sqsubseteq ((g_B)_2)^c \sqsubseteq (e_{y_s})^c$ and $e_{y_s}\,\widetilde{\not q}C_\tau(e, e_{x_t}, r)$. So (X, τ_E) is r-FSR_0. □

Theorem 3.6. *Let (X, τ_E) be a fuzzy soft topological space. Then for each $e \in E$, the following statements hold:*

(a) r-$FST_4 \Rightarrow^{(1)} r$-$FST_3 \Rightarrow^{(2)} r$-$FST_{2\frac{1}{2}} \Rightarrow^{(3)} r$-$FST_2 \Rightarrow^{(4)} r$-$FST_1$,

(b) r-$FST_2 \Rightarrow r$-FSR_1 *space.*

Proof. (a) (1) and (2) are easily proved.

(3) Let (X, τ_E) be r-$FST_{2\frac{1}{2}}$ and $e_{x_t}\,\widetilde{\not q}e_{y_s}$. Then there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $C_\tau(e, (g_B)_1, r)\,\widetilde{\not q}C_\tau(e, (g_B)_2, r)$, $e_{x_t}\tilde{\in}(g_B)_1$ and $e_{y_s}\tilde{\in}(g_B)_2$ for all $e \in E$. Thus $(g_B)_1\,\widetilde{\not q}(g_B)_2$ and (X, τ_E) is r-FST_2.

(4) Let (X, τ_E) be r-FST_2 and $e_{x_t}\,\widetilde{\not q}e_{y_s}$. Then there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $e_{x_t}\tilde{\in}(g_B)_1$, $e_{y_s}\tilde{\in}(g_B)_2$ and $(g_B)_1\,\widetilde{\not q}(g_B)_2$. Thus $e_{y_s}\tilde{\in}(g_B)_2 \sqsubseteq ((g_B)_1)^c \sqsubseteq (e_{x_t})^c$. So

$$(e_{x_t})^c = \bigsqcup_{e_{y_s}\tilde{\in}(e_{x_t})^c} \{(g_B)_{e_{y_s}} : e_{y_s}\tilde{\in}(g_B)_{e_{y_s}},\ \tau_e((g_B)_{e_{y_s}}) \geq r\}.$$

Hence $\tau_e((e_{x_t})^c) \geq r$ and (X, τ_E) is r-FST_1.

(b) For any $e_{x_t}, e_{y_s} \in \widetilde{P_t(X)}$ such that $e_{x_t} \neq e_{y_s}$ and $e_{x_t}\,\widetilde{\not q}C_\tau(e, e_{y_s}, r)$, $e_{x_t}\,\widetilde{\not q}e_{y_s}$. Since (X, τ_E) is r-FST_2, there exist $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $e_{x_t}\tilde{\in}(g_B)_1$, $e_{y_s}\tilde{\in}(g_B)_2$ and $(g_B)_1\,\widetilde{\not q}(g_B)_2$. Then (X, τ_E) is r-FSR_1. □

Example 3.7. Let $X=\{x,y\}$, $E=\{e^1,e^2\}$ be the parameter set of X, $e\in E$ and $\alpha\in(0,1)$. Define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e^1}(h_E)=\begin{cases} 1 & \text{if} \quad h_E\in\{\Phi,\widetilde{E}\},\\ \frac{1}{2} & \text{if} \quad h_E\in\{e_{x_\alpha},e_{y_\alpha}\},\\ \frac{2}{3} & \text{if} \quad h_E\in\{e_{x_\alpha}\sqcup e_{y_\alpha}, e_{x_\alpha}\sqcup e_{y_1}, e_{x_1}\sqcup e_{y_\alpha}\},\\ 0 & \text{otherwise,}\end{cases}$$

$$\tau_{e^2}(h_E)=\begin{cases} 1 & \text{if} \quad h_E\in\{\Phi,\widetilde{E}\},\\ \frac{1}{3} & \text{if} \quad h_E\in\{e_{x_\alpha},e_{y_\alpha}\},\\ \frac{1}{2} & \text{if} \quad h_E\in\{e_{x_\alpha}\sqcup e_{y_\alpha}, e_{x_\alpha}\sqcup e_{y_1}, e_{x_1}\sqcup e_{y_\alpha}\},\\ 0 & \text{otherwise.}\end{cases}$$

(1) Then the converse of Theorem 3.5 (3) is not true. For $0<r\le\frac{1}{3}$, (X,τ_E) is r-FSR_0. But (X,τ_E) is not r-FSR_1.

(2) Then the converse of Theorem 3.5 (*a*) (4) is not true. For $0<r\le\frac{1}{3}$, (X,τ_E) is r-FST_1. But (X,τ_E) is not r-FST_2.

Theorem 3.8. *Let (X,τ_E) be a fuzzy soft topological space and $r\in I_0$. Then (X,τ_E) is r-FSR_1, if $e_{x_t}\widetilde{\not q}C_\tau(e,e_{y_s},r)$ implies that there exist $f_A,g_B\in\widetilde{(X,E)}$ with $\tau_e(f_A)\ge r$, $\tau_e(g_B)\ge r$ such that $f_A\widetilde{\not q}g_B$ and $C_\tau(e,e_{x_t},r)\sqsubseteq f_A$, $C_\tau(e,e_{y_s},r)\sqsubseteq g_B$ for each $e\in E$.*

Proof. Let $e_{x_t},\ e_{y_s}\in\widetilde{P_t(X)}$ such that $e_{x_t}\ne e_{y_s}$ and $e_{x_t}\widetilde{\not q}C_\tau(e,e_{y_s},r)$. Since there exist $f_A,g_B\in\widetilde{(X,E)}$ with $\tau_e(f_A)\ge r$, $\tau_e(g_B)\ge r$ such that $f_A\widetilde{\not q}g_B$ and $C_\tau(e,e_{x_t},r)\sqsubseteq f_A$, $C_\tau(e,e_{y_s},r)\sqsubseteq g_B$. But $e_{x_t}\tilde{\in}C_\tau(e,e_{x_t},r)$ and $e_{y_s}\tilde{\in}C_\tau(e,e_{y_s},r)$. Then $e_{x_t}\tilde{\in}f_A$ and $e_{y_s}\tilde{\in}g_B$. Thus (X,τ_E) is r-FSR_1. □

Theorem 3.9. *Let (X,τ_E) be r-FSR_0. If $\tau_e(C_\tau(e,e_{x_t},r))\ge r$ for each $e_{x_t}\in\widetilde{Pt(X)}$, $e\in E$ and $r\in I_0$, then (X,τ_E) is r-FSR_1.*

Proof. Let $e_{x_t},\ e_{y_s}\in\widetilde{P_t(X)}$ such that $e_{x_t}\ne e_{y_s}$ and $e_{x_t}\widetilde{\not q}C_\tau(e,e_{y_s},r)$. Then by Theorem 3.4 (4), $C_\tau(e,e_{x_t},r)\widetilde{\not q}C_\tau(e,e_{y_s},r)$. Since $\tau_e(C_\tau(e,e_{x_t},r))\ge r$ for each $e_{x_t}\in\widetilde{Pt(X)}$ and $e\in E$, (X,τ_E) is r-FSR_1. □

Theorem 3.10. *Let (X,τ_E) be a fuzzy soft topological space and $r\in I_0$. Then for each $e\in E$, the following statements are equivalent:*

(1) *(X,τ_E) is r-FSR_2,*

(2) *if $e_{x_t}\tilde{\in}f_A$ for each $f_A\in\widetilde{(X,E)}$ with $\tau_e(f_A)\ge r$, there exists $g_B\in\widetilde{(X,E)}$ with $\tau_e(g_B)\ge r$ such that $e_{x_t}\tilde{\in}g_B\sqsubseteq C_\tau(e,g_B,r)\sqsubseteq f_A$,*

(3) *if $e_{x_t}\widetilde{\not q}f_A$ for each $f_A\in\widetilde{(X,E)}$ with $\tau_e((f_A)^c)\ge r$, there exist $(g_B)_i\in\widetilde{(X,E)}$ for $i\in\{1,2\}$ with $\tau_e((g_B)_i)\ge r$ such that $e_{x_t}\tilde{\in}(g_B)_1$, $f_A\sqsubseteq(g_B)_2$ and $C_\tau(e,(g_B)_1,r)\widetilde{\not q}C_\tau(e,(g_B)_2,r)$.*

Proof. (1)⇒ (2) Let $e_{x_t}\tilde{\in}f_A$ for each $f_A\in\widetilde{(X,E)}$ with $\tau_e(f_A)\ge r$. Then $e_{x_t}\widetilde{\not q}$ $(f_A)^c$. Since (X,τ_E) is r-FSR_2, there exist $(g_B)_i\in\widetilde{(X,E)}$ with $\tau_e((g_B)_i)\ge r$ for $i\in\{1,2\}$ such that$e_{x_t}\tilde{\in}(g_B)_1$, $(f_A)^c\sqsubseteq(g_B)_2$ and $(g_B)_1\widetilde{\not q}(g_B)_2$ for all $e\in E$. Thus $e_{x_t}\tilde{\in}(g_B)_1\sqsubseteq((g_B)_2)^c\sqsubseteq f_A$. Since $C_\tau(e,(g_B)_1,r)\sqsubseteq((g_B)_2)^c$, $e_{x_t}\tilde{\in}(g_B)_1\sqsubseteq C_\tau(e,(g_B)_1,r)\sqsubseteq f_A$.

(2) $\Rightarrow$ (3) Let $e_{x_t}\tilde{\not q}f_A$ for each $f_A \in \widetilde{(X,E)}$ with $\tau_e((f_A)^c) \geq r$. Then $e_{x_t}\tilde{\in}(f_A)^c$. Thus by (2), there exists $g_B \in \widetilde{(X,E)}$ with $\tau_e(g_B) \geq r$ such that $e_{x_t}\tilde{\in}g_B \sqsubseteq C_\tau(e, g_B, r) \sqsubseteq (f_A)^c$. Since $\tau_e(g_B) \geq r$ and $e_{x_t}\tilde{\in}g_B$, there exists $(g_B)_1 \in \widetilde{(X,E)}$ with $\tau_e((g_B)_1) \geq r$ such that $e_{x_t}\tilde{\in}(g_B)_1 \sqsubseteq C_\tau(e,(g_B)_1, r) \sqsubseteq g_B \sqsubseteq C_\tau(e, g_B, r) \sqsubseteq (f_A)^c$. It implies $f_A \sqsubseteq I_\tau(e,(g_B)^c, r) \sqsubseteq (g_B)^c$. Put $(g_B)_2 = I_\tau(e,(g_B)^c, r)$. So $\tau_e((g_B)_2) \geq r$. Hence $C_\tau(e,(g_B)_2, r) \sqsubseteq (g_B)^c \sqsubseteq (C_\tau(e,(g_B)_1, r))^c$ and $C_\tau(e,(g_B)_1, r)$ $\tilde{\not q}C_\tau(e,(g_B)_2, r)$.

(3) $\Rightarrow$ (1). It is trivial. □

Example 3.11. Let $X = \{x, y, z\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Let $f_E,\ g_E \in \widetilde{(X,E)}$ be as follows: $f_{e_1}(x) = 1.0,\ f_{e_1}(y) = 0.0,\ f_{e_1}(z) = 1.0,\ f_{e_2}(x) = 0.0,\ f_{e_2}(y) = 1.0,\ f_{e_2}(z) = 0.0,\ g_{e_1}(x) = 0.0,\ g_{e_1}(y) = 1.0,\ g_{e_1}(z) = 0.0,\ g_{e_2}(x) = 1.0,\ g_{e_2}(y) = 0.0,\ g_{e_2}(z) = 1.0$. We define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(h_E) = \begin{cases} 1 & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2} & \text{if} \quad h_E = f_E, \\ \frac{2}{3} & \text{if} \quad h_E = g_E, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_{e_2}(h_E) = \begin{cases} 1 & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{4} & \text{if} \quad h_E = f_E, \\ \frac{1}{2} & \text{if} \quad h_E = g_E, \\ 0 & \text{otherwise}. \end{cases}$$

Then for $0 < r \leq \frac{1}{4}$, (X, τ_E) is r-FSR_2.

Theorem 3.12. *Let (X, τ_E) be a fuzzy soft topological space and $r \in I_0$. Then, for each $e \in E$ the following statements are equivalent:*

(1) *(X, τ_E) is r-FSR_3,*

(2) *if $h_C \sqsubseteq f_A$ for each $f_A, h_C \in \widetilde{(X,E)}$ with $\tau_e(f_A) \geq r$ and $\tau_e((h_C)^c) \geq r$, there exists $g_B \in \widetilde{(X,E)}$, $\tau_e(g_B) \geq r$ such that $h_C \sqsubseteq g_B \sqsubseteq C_\tau(e, g_B, r) \sqsubseteq f_A$.*

(3) *if $(f_A)_1 \tilde{\not q}(f_A)_2$ for each $(f_A)_i \in \widetilde{(X,E)}$ with $\tau_e((f_A)_i^c) \geq r$, there exist $(g_B)_i \in \widetilde{(X,E)}$ for $i \in \{1,2\}$, $\tau_e((g_B)_i) \geq r$ such that $(f_A)_i \sqsubseteq (g_B)_i$, $C_\tau(e,(g_B)_1, r)$ $\tilde{\not q}C_\tau(e,(g_B)_2, r)$.*

Proof. It is similarly proved as in Theorem 3.10. □

Example 3.13. Let $X = \{x, y, z\}$ be a classical set and $E = \{e_1, e_2\}$ be the parameter set of X. Let $f_E,\ g_E \in \widetilde{(X,E)}$ be as follows: $f_{e_1}(x) = 0.9,\ f_{e_1}(y) = 0.1,\ f_{e_1}(z) = 0.1,\ f_{e_2}(x) = 0.1,\ f_{e_2}(y) = 0.9,\ f_{e_2}(z) = 0.9,\ g_{e_1}(x) = 0.1,\ g_{e_1}(y) = 0.9,\ g_{e_1}(z) = 0.9,\ g_{e_2}(x) = 0.9,\ g_{e_2}(y) = 0.1,\ g_{e_2}(z) = 0.1$. We define fuzzy soft topology $\tau_E : E \longrightarrow [0,1]^{\widetilde{(X,E)}}$ as follows:

$$\tau_{e_1}(h_E) = \begin{cases} 1 & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ \frac{2}{3} & \text{if} \quad h_E = f_E, \\ \frac{1}{3} & \text{if} \quad h_E = g_E, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tau_{e_2}(h_E) = \begin{cases} 1 & \text{if} \quad h_E \in \{\Phi, \widetilde{E}\}, \\ \frac{1}{2} & \text{if} \quad h_E = f_E, \\ \frac{1}{4} & \text{if} \quad h_E = g_E, \\ 0 & \text{otherwise}, \end{cases}$$

Then for $0 < r \leq \frac{1}{4}$, (X, τ_E) is r-FSR_3.

Definition 3.14. Let (X, τ) and (Y, τ^*) be fuzzy soft topological space's and $e \in E$. Then a fuzzy soft mapping φ_ψ from $\widetilde{(X, E)}$ into $\widetilde{(Y, F)}$ is called:

(i) fuzzy soft open, if $\tau_e(f_A) \leq \tau^*_{\psi(e)}(\varphi_\psi(f_A))$ for every $f_A \in \widetilde{(X, E)}$,

(ii) fuzzy soft closed, if $\tau_e((f_A)^c) \leq \tau^*_{\psi(e)}((\varphi_\psi(f_A))^c)$ for every $f_A \in \widetilde{(X, E)}$.

Theorem 3.15. *Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X, E)}$ into $\widetilde{(Y, F)}$ is fuzzy soft continuous, fuzzy soft open, bijective and (X, τ_E) is r-FSR_2 (resp. r-FSR_3), then (Y, τ^*_F) is r-FSR_2 (resp. r-FSR_3).*

Proof. Let $e_{y_s} \widetilde{\not{q}} g_B$ with $\tau^*_{\psi(e)}((g_B)^c) \geq r$. Then $\tau_e((\varphi_\psi^{-1}(g_B))^c) \geq r$, because φ_ψ is a fuzzy soft continuous mapping. Put $e_{y_s} = \varphi_\psi(e_{x_s})$. Then $e_{x_s} \widetilde{\not{q}} \varphi_\psi^{-1}(g_B)$.Then by r-FSR_2 of (X, τ_E), there exists $(g_B)_i \in \widetilde{(X, E)}$ with $\tau_e((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $e_{x_s} \tilde{\in} (g_B)_1$, $\varphi_\psi^{-1}(g_B) \sqsubseteq (g_B)_2$ and $(g_B)_1 \widetilde{\not{q}} (g_B)_2$. Since φ_ψ is a fuzzy soft open and bijective mapping, $e_{y_s} \tilde{\in} \varphi_\psi((g_B)_1)$, $g_B \sqsubseteq \varphi_\psi(\varphi_\psi^{-1}(g_B)) \sqsubseteq \varphi_\psi((g_B)_2)$, $\varphi_\psi((g_B)_1) \widetilde{\not{q}}$ $\varphi_\psi((g_B)_2)$. Thus (Y, τ^*_F) is r-FSR_2. Other case is similarly proved. □

Theorem 3.16. *Let (X, τ_E) and (Y, τ^*_F) be fuzzy soft topological space's. If a fuzzy soft mapping φ_ψ from $\widetilde{(X, E)}$ into $\widetilde{(Y, F)}$ is fuzzy soft continuous, fuzzy soft closed, bijective and (Y, τ^*_F) is r-FSR_2 (resp. r-FSR_3), then (X, τ_E) is r-FSR_2 (resp. r-FSR_3).*

Proof. Let $e_{x_t} \widetilde{\not{q}} f_A$ with $\tau_e((f_A)^c) \geq r$. Since φ_ψ is a fuzzy soft closed and bijective mapping, $\varphi_\psi(e_{x_t}) \widetilde{\not{q}} \varphi_\psi(f_A)$ with $\tau^*_{\psi(e)}((\varphi_\psi(f_A))^c) \geq r$. Then by r-FSR_2 of (Y, τ^*_F), there exist $(g_B)_i \in \widetilde{(Y, F)}$ with $\tau^*_{\psi(e)}((g_B)_i) \geq r$ for $i \in \{1, 2\}$ such that $\varphi_\psi(e_{x_t}) \tilde{\in} (g_B)_1$, $\varphi_\psi(f_A) \sqsubseteq (g_B)_2$ and $(g_B)_1 \ \widetilde{\not{q}} (g_B)_2$. Thus $e_{x_t} \tilde{\in} \varphi_\psi^{-1}((g_B)_1)$, $f_A \sqsubseteq \varphi_\psi^{-1}((g_B)_2)$ with $\tau_e(\varphi_\psi^{-1}((g_B)_i)) \geq r$ for $i \in \{1, 2\}$ and $\varphi_\psi^{-1}((g_B)_1) \widetilde{\not{q}} \varphi_\psi^{-1}((g_B)_2)$ (since φ_ψ is a fuzzy soft continuous mapping). So (X, τ_E) is r-FSR_2. Other case is similarly proved. □

4. Conclusions

In our theoretical work, a new form of separation and regularity axioms called fuzzy soft R_i, $(i = 0, 1, 2, 3)$ and fuzzy soft T_i, $(i = 1, 2, 2\frac{1}{2}, 3, 4)$ spaces are introduced in a fuzzy soft topological space based on the sense of Aygünoğlu. Furthermore, the relations of these axioms with each other are investigated and some interesting properties of them are specified.

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Compliance with ethical standards

Conflict of interest: The author declare that they have no conflict of interest.

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