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Abstract. In this paper, we introduced the notion of multiplier in lattice implication algebra, and considered the properties of multipliers in lattice implication algebras. We give a set of conditions which are equivalent to be an identity multiplier. Also, we characterized the fixed set $Fix_f(L)$ and $Kerf$ by multipliers. Moreover, we prove that if f is a multiplier of a lattice implication algebra, every filter F is a f-invariant.

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1. Introduction

The concept of lattice implication algebra was proposed by Xu [1], in order to establish an alternative logic knowledge representation. Also, in [2], Xu and Qin discussed the properties lattice H implication algebras, and gave some equivalent conditions about lattice H implication algebras. Xu and Qin [3] introduced the notion of filters in a lattice implication, and investigated their properties. The present author [4, 5] introduced the notion of derivation and f-derivation in lattice implications algebras and obtained some related results. In this paper, we introduced the notion of multiplier in lattice implication algebra, and considered the properties of multipliers in lattice implication algebras. We give a set of conditions which are equivalent to be an identity multiplier. Also, we characterized the fixed set $Fix_f(L)$ and $Kerf$ by multipliers. Moreover, we prove that if f is a multiplier of a lattice implication algebra, every filter F is a f-invariant.

2. Preliminaries

A lattice implication algebra is an algebra $(L;\wedge,\vee,',\rightarrow,0,1)$ of type $(2,2,1,2,0,0)$, where $(L;\wedge,\vee,0,1)$ is a bounded lattice, " $'$ " is an order-reversing involution and " $\rightarrow$ " is a binary operation, satisfying the following axioms: for all $x,y,z\in L$,

(L1) $x\rightarrow(y\rightarrow z)=y\rightarrow(x\rightarrow z)$,
(L2) $x\rightarrow x=1$,
(L3) $x\rightarrow y=y'\rightarrow x'$,
(L4) $x\rightarrow y=y\rightarrow x=1\Rightarrow x=y$,
(L5) $(x\rightarrow y)\rightarrow y=(y\rightarrow x)\rightarrow x$,
(L6) $(x\vee y)\rightarrow z=(x\rightarrow z)\wedge(y\rightarrow z)$,
(L7) $(x\wedge y)\rightarrow z=(x\rightarrow z)\vee(y\rightarrow z)$.

If L satisfies conditions (L1) – (L5), we say that L is a quasi lattice implication algebra. A lattice implication algebra L is called a *lattice H implication algebra*, if it satisfies $x\vee y\vee((x\wedge y)\rightarrow z)=1$ for all $x,y,z\in L$ (see [1]).

In the sequel the binary operation " $\rightarrow$ " will be denoted by juxtaposition. We can define a partial ordering " $\leq$ " on a lattice implication algebra L by $x\leq y$ if and only if $x\rightarrow y=1$ for all $x,y\in L$.

Theorem 2.1. *In a lattice implication algebra L, the following hold:*

(u1) $0\rightarrow x=1$, $1\rightarrow x=x$ *and* $x\rightarrow 1=1$,
(u2) $x\rightarrow y\leq(y\rightarrow z)\rightarrow(x\rightarrow z)$,
(u3) $x\leq y$ *implies* $y\rightarrow z\leq x\rightarrow z$ *and* $z\rightarrow x\leq z\rightarrow y$,
(u4) $x'=x\rightarrow 0$.
(u5) $x\vee y=(x\rightarrow y)\rightarrow y$,
(u6) $((y\rightarrow x)\rightarrow y')'=x\wedge y=((x\rightarrow y)\rightarrow x')'$,
(u7) $x\leq(x\rightarrow y)\rightarrow y$,

for all $x,y,z\in L$ (see [1]).

Definition 2.2. In a lattice H implication algebra L, the following hold, for all $x,y,z\in L$,

(u8) $x\rightarrow(x\rightarrow y)=x\rightarrow y$,
(u9) $x\rightarrow(y\rightarrow z)=(x\rightarrow y)\rightarrow(x\rightarrow z)$ (see [1]).

Definition 2.3. A subset F of a lattice implication algebra L is called a *filter* of L it satisfies,

(F1) $1\in F$,
(F2) $x\in F$ and $x\rightarrow y\in F$ imply $y\in F$, for all $x,y\in L$ (see [3]).

Definition 2.4. Let L_1 and L_2 be lattice implication algebras.

(i) A mapping $f:L_1\rightarrow L_2$ is an *implication homomorphism*, if $f(x\rightarrow y)=f(x)\rightarrow f(y)$ for all $x,y\in L_1$.
(ii) A mapping $f:L_1\rightarrow L_2$ is an *lattice implication homomorphism*, if $f(x\vee y)=f(x)\vee f(y), f(x\wedge y)=f(x)\wedge f(y), f(x')=f(x)'$ for all $x,y\in L_1$ (see [2])

3. Multipliers of lattice implication algebras

In what follows, let L denote a lattice implication algebra unless otherwise specified.

Definition 3.1. Let L be a lattice implication algebra. A map $f : L \to L$ is called a *multiplier* of L, if

$$f(x \to y) = x \to f(y)$$

for all $x, y \in L$.

Example 3.2. Let L be a lattice implication algebra. Then

(1) $f(x) = 1$ is a multiplier of L,
(2) $f(x) = x$ is a multiplier of L.

Example 3.3. Let $L := \{0, a, b, c, 1\}$. Define the partial order relation on L as $0 < a < b < c < 1$, and define

$$x \wedge y := \min\{x, y\},\ x \vee y := \max\{x, y\}$$

for all $x, y \in L$ and "$'$" and "$\to$" as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Define a map $f : L \to L$ by

$$f(x) = \begin{cases} 1 & \text{if } x = c, 1 \\ b & \text{if } x = a \\ a & \text{if } x = 0 \\ c & \text{if } x = b. \end{cases}$$

Then it is easy to check that f is a multiplier of lattice implication algebra L.

Example 3.4. Let $L := \{0, a, b, 1\}$ be a set with the Cayley table.

x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$, we have $x' = x \to 0$. The operations $\wedge$ and $\vee$ on L are defined as follows:

$$x \vee y = (x \to y) \to y, \quad x \wedge y = ((x' \to y') \to y')'.$$

Then $(L, \vee, \wedge, ', \to)$ is a lattice implication algebra. Define a map $f : L \to L$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1, b \\ a & \text{if } x = 0 \\ b & \text{if } x = a. \end{cases}$$

Then it is easy to check that f is a multiplier of a lattice implication algebra L.

Proposition 3.5. *Let f be a multiplier of L. Then the following conditions hold:*

(1) $f(1) = 1$,
(2) $x \leq f(x)$ *for all* $x \in L$,
(3) $f(x) = f(x) \vee x$ *for all* $x \in L$.

Proof. (1) Let f be a multiplier of L. Then we have

$$f(1) = f(0 \to 1) = 0 \to f(1) = 1.$$

(2) Let f be a multiplier of L. Then by (1), for all $x \in L$,

$$x \to f(x) = f(x \to x) = f(1) = 1$$

which implies $x \leq f(x)$.

(3) For all $x \in L$, we have

$$f(x) \vee x = (f(x) \to x) \to x = (x \to f(x)) \to f(x) = 1 \to f(x) = f(x)$$

from (2). □

Let L be a lattice implication algebra and let f be a multiplier of L. If $x \leq y$ implies $f(x) \leq f(y)$, f is said to be *isotone.*

Example 3.6. In Example 3.4, f is an isotone multiplier of L.

Proposition 3.7. *Let f be a multiplier of L and let f be an implication homomorphism on L. Then f is isotone.*

Proof. Let $x \leq y$ for all $x, y \in L$. Then $x \to y = 1$. Now $f(x) \to f(y) = f(x \to y) = f(1) = 1$, which implies $f(x) \leq f(y)$. This completes the proof. □

Proposition 3.8. *Let f be a multiplier of L. Then we have $f(x) \to f(y) \leq f(x \to y)$ for all $x, y \in L$.*

Proof. Since $x \leq f(x)$ for all $x \in L$, it follows from (u3) that $f(x) \to f(y) \leq x \to f(y) = f(x \to y)$ for all $x, y \in L$.

□

Proposition 3.9. *Let f be a multiplier of L and let f be non-expansive. Then f is an implication homomorphism on L.*

Proof. Let f be a multiplier of L and non-expansive. Then we have $f(x) \leq x$. Hence $f(x \to y) = x \to f(y) \leq f(x) \to f(y)$ by (u3) and so $f(x) \to f(y) = f(x \to y)$, from Proposition 3.8 and (L4). This completes the proof.

□

Proposition 3.10. *Let L be a lattice implication algebra with $x \to y = y \to x$ for all $x, y \in L$. Then every idempotent multiplier of L is an implication homomorphism on L.*

Proof. Let f be an idempotent multiplier of L. Then $f^2(x) = f(x)$ for all $x \in L$. Thus for all $x, y \in L$,

$$\begin{aligned} f(x \to y) &= f^2(x \to y) = f(f(x \to y)) \\ &= f(x \to f(y)) = f(f(y) \to x) \\ &= f(y) \to f(x) = f(x) \to f(y), \end{aligned}$$

which implies that f is an implication homomorphism on L.

□

Proposition 3.11. *Let L be a lattice implication algebra and let f be a multiplier of L. Then $f : L \to L$ is an identity map if it satisfies $f(x) \to y = x \to f(y)$ for all $x, y \in L$.*

Proof. Let $x \to f(y) = f(x) \to y$ for all $x, y \in L$. Then $f(x) = f(1 \to x) = 1 \to f(x) = f(1) \to x = 1 \to x = x$. Thus f is an identity map of L.

□

Theorem 3.12. *If f is a multiplier of L and f is an implication homomorphism on L, then f is idempotent.*

Proof. Let f be a multiplier of L. Then for all $x \in L$, we have

$$f(x) \to f(f(x)) = f(f(x) \to f(x)) = f(1) = 1.$$

Thus $f(x) \le f(f(x))$. Also, since f is an implication homomorphism on L, we have

$$f(f(x)) \to f(x) = f(f(x) \to x) = f(x) \to f(x) = 1.$$

So $f(f(x)) \le f(x)$. Hence $f(f(x)) = f(x)$.

□

Theorem 3.13. *Let L be a lattice implication algebra and f a multiplier of L. Then f is one to one if and only if f is an identity multiplier of L.*

Proof. Sufficiency is obvious. Suppose that f is one to one. For every $x \in L$, we have

$$f(f(x) \to x) = f(x) \to f(x) = 1 = f(1)$$

and so $f(x) \to x = 1$, i.e., $f(x) \le x$. Since $x \le f(x)$ for all $x \in L$, from Proposition 3.5, it follows that $f(x) = x$, which implies that f is the identity multiplier.

□

In general, every multiplier of L need not be identity. However, in the following theorem, we give a set of conditions which are equivalent to be an identity multiplier of L.

Theorem 3.14. *Let L be a lattice implication algebra. A multiplier f of L is an identity map if and only if the following conditions are satisfied for all $x, y \in L$,*

(1) *f is idempotent, i.e., $f^2(x) = f(x)$,*
(2) $f(x \to y) = f(x) \to f(y)$,
(3) $f^2(x) \to y = f(x) \to f(y)$.

Proof. The condition for necessary is trivial. For sufficiency, assume that (1) and (2) hold. Then for $x, y \in L$, we get $f(x) \to y = f^2(x) \to y = f(x) \to f(y) = f(x \to y)$. Also, by the definition of the multiplier, we have $f(x \to y) = x \to f(y)$. Thus

$$f(x \to y) = x \to f(y) = f(x) \to y.$$

So by the Proposition 3.11, f is an identity multiplier of L. □

Let L be a lattice implication algebra and f_1, f_2 two self-maps. We define $f_1 \circ f_2 : L \to L$ by

$$(f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all $x \in L$.

Proposition 3.15. *Let L be a lattice implication algebra and f_1, f_2 two multipliers of L. Then $f_1 \circ f_2$ is also a multiplier of L.*

Proof. Let L be a lattice implication algebra and f_1, f_2 two multipliers of L. Then we have

$$\begin{aligned}(f_1 \circ f_2)(a \to b) &= f_1(f_2(a \to b)) \\ &= f_1(a \to f_2(b)) \\ &= a \to f_1(f_2(b)) \\ &= a \to (f_1 \circ f_2)(b)\end{aligned}$$

for any $a, b \in L$. This completes the proof. □

We define $x \sqcup y$ by

$$x \sqcup y = (x \to y) \to y$$

for all $x, y \in X$.

Let L be a lattice implication algebra and f_1, f_2 two self maps. We define $f_1 \sqcup f_2 : L \to L$ by

$$(f_1 \sqcup f_2)(x) = f_1(x) \sqcup f_2(x)$$

for all $x \in L$.

Proposition 3.16. *Let L be a lattice H implication algebra and let f_1, f_2 be two multipliers of L. Then $f_1 \sqcup f_2$ is also a multiplier of L.*

Proof. Let L be a lattice H implication algebra and let f_1, f_2 be two multipliers of L. Then we have, for all $a, b \in L$,

$$\begin{aligned}(f_1 \sqcup f_2)(a \to b) = f_1(a \to b) \sqcup f_2(a \to b) &= a \to f_1(b) \sqcup a \to f_2(b) \\ &= ((a \to f_1(b)) \to (a \to f_1(b))) \to (a \to f_2(b)) \\ &= (a \to (f_1(b) \to f_2(b))) \to (a \to f_2(b)) \\ &= a \to ((f_1(b) \to f_2(b)) \to f_2(b)) \\ &= a \to (f_1(b) \sqcup f_2(b)) \\ &= a \to (f_1 \sqcup f_2)(b).\end{aligned}$$

This completes the proof. □

Let L be a lattice implication algebra and f_1, f_2 two self-maps. We define $f_1 \vee f_2 : L \to L$ by

$$(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$$

for all $x \in L$.

Proposition 3.17. *Let L be a lattice H implication algebra and f_1, f_2 two multipliers of L. Then $f_1 \vee f_2$ is also a multiplier of L.*

Proof. Let L be a lattice H implication algebra and f_1, f_2 two multipliers of L. Then we have

$$\begin{aligned}(f_1 \vee f_2)(a \to b) &= f_1(a \to b) \vee f_2(a \to b) = (a \to f_1(b)) \vee (a \to f_2(b)) \\ &= ((a \to f_1(b)) \to (a \to f_2(b))) \to (a \to f_2(b)) \\ &= (a \to (f_1(b) \to f_2(b))) \to (a \to f_2(b)) \\ &= a \to ((f_1(b) \to f_2(b)) \to f_2(b)) \\ &= a \to (f_1(b) \vee f_2(b)) \\ &= a \to (f_1 \vee f_2)(b)\end{aligned}$$

for any $a, b \in L$. This completes the proof. □

Let L_1 and L_2 be two lattice implication algebras. Then $L_1 \times L_2$ is also a lattice implication algebra with respect to the point-wise operation given by

$$(a, b) \to (c, d) = (a \to c, b \to d)$$

for all $a, c \in L_1$ and $b, d \in L_2$.

Theorem 3.18. *Let L_1 and L_2 be two lattice implication algebras. Define a map $f : L_1 \times L_2 \to L_1 \times L_2$ by $f(x, y) = (x, 1)$ for all $(x, y) \in L_1 \times L_2$. Then f is a multiplier of $L_1 \times L_2$ with respect to the point-wise operation.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in L_1 \times L_2$. The we have

$$\begin{aligned}f((x_1, y_1) \to (x_2, y_2)) &= f(x_1 \to x_2, y_1 \to y_2) \\ &= (x_1 \to x_2, 1) \\ &= (x_1 \to x_2, y_1 \to 1) \\ &= (x_1, y_1) \to (x_2, 1) \\ &= (x_1, y_1) \to f(x_2, y_2).\end{aligned}$$

Therefore f is a multiplier of the direct product $L_1 \times L_2$. □

Let f be a multiplier of L. Define a set $Fix_f(L)$ by

$$Fix_f(L) := \{x \in L \mid f(x) = x\}$$

for all $x \in L$.

Proposition 3.19. *Let f be a multiplier of L. If $x \in Fix_f(L)$. Then we have*

$$\overbrace{(f \circ f \circ f \cdots \circ f)}^{n}(x) = x.$$

Proof. By definition of $Fix_f(L)$, the proof is straightforward. □

Proposition 3.20. *Let L be a lattice implication algebra and let f be a multiplier of L. Then we have the following properties:*

(1) *if $x \in L$ and $y \in Fix_f(L)$, we have $x \to y \in Fix_f(L)$,*
(2) *if $y \in Fix_f(L)$, $x \vee y \in Fix_f(L)$ for all $x \in L$.*

Proof. (1) Let $x \in L$ and $y \in Fix_f(L)$. Then we have $f(y) = y$. Thus we get

$$\begin{aligned} f(x \to y) &= x \to f(y) \\ &= x \to y. \end{aligned}$$

(2) Let $x \in L$ and $y \in Fix_f(L)$. Then we get

$$\begin{aligned} f(x \vee y) &= f((x \to y) \to y) \\ &= (x \to y) \to f(y) \\ &= (x \to y) \to y) = x \vee y. \end{aligned}$$

□

Proposition 3.21. *Let L be a lattice implication algebra and let f be a multiplier of L. If $x \le y$ and $x \in Fix_f(L)$, then we have $y \in Fix_f(L)$.*

Proof. Let $x \le y$ and $x \in Fix_f(L)$. Then we have $x \to y = 1$ and $f(x) = x$. Thus we get

$$f(y) = f((1 \to y) = f((x \to y) \to y) = f(x \vee y) = x \vee y = y$$

from Proposition 3.20 (2). □

Let us recall from Proposition 3.16 that the composition of two multipliers f and g of an almost distributive lattice L is a multiplier of L where $(f \circ g)(x) = f(g(x))$ for all $x \in L$.

Theorem 3.22. *Let f and g be two idempotent multipliers of L such that $f \circ g = g \circ f$. Then the following conditions are equivalent:*

(1) $f = g$,
(2) $f(L) = g(L)$,
(3) $Fix_f(L) = Fix_g(L)$.

Proof. (1) ⇒ (2): It is obvious.

(2) ⇒ (3): Assume that $f(L) = g(L)$. Let $x \in Fix_f(L)$. Then $x = f(x) \in f(L) = g(L)$. Thus $x = g(y)$ for some $y \in L$. Now $g(x) = g(g(y)) = g^2(y) = g(y) = x$. So $x \in Fix_g(L)$. Hence $Fix_f(L) \subseteq Fix_g(L)$. Similarly, we can obtain $Fix_g(L) \subseteq Fix_f(L)$. Therefore $Fix_f(L) = Fix_g(L)$.

(3) ⇒ (1): Assume that $Fix_f(L) = Fix_g(L)$. Let $x \in L$. Since $f(x) \in Fix_f(L) = Fix_g(L)$, we have $g(f(x)) = f(x)$. Also, we obtain $g(x) \in Fix_g(L) = Fix_f(L)$. Then we get $f(g(x)) = g(x)$. Thus we have

$$f(x) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = g(x).$$

So f and g are equal in the sense of mappings.

□

Let L be a lattice implication algebra. Then, for each $a \in L$, we define a map $f_a : L \to L$ by

$$f_a(x) = a \to x$$

for all $x \in L$.

Proposition 3.23. *For each $a \in L$, the map f_a is a multiplier of L.*

We call the multiplier f_a of Proposition 3.23 as simple multiplier. Let us denote $S(L)$ by the set of all simple multipliers on L.

Proof. Suppose that f_a is a map defined by $f_a(x) = a \to x$ for each $x \in L$. Then for any $x, y \in L$, we have

$$f_a(x \to y) = a \to (x \to y) = x \to (a \to y) = x \to f_a(y).$$

Thus f_a is a multiplier of L.

□

Proposition 3.24. *Let L be a lattice H-implication algebra. Then the map f_a is an isotone multiplier of L.*

Proof. Let $x, y \in L$ be such that $x \leq y$. Then we get $x \to y = 1$. Thus $f_a(x) \to f_a(y) = (a \to x) \to (a \to y) = a \to (x \to y) = a \to 1 = 1$, which implies $f_a(x) \leq f_a(y)$. This completes the proof.

□

Proposition 3.25. *Let L be a lattice H-implication algebra. Then the map f_a is an implication homomorphism on L.*

Proof. Let $x, y \in L$. Then we get $f_a(x \to y) = a \to (x \to y) = (a \to x) \to (a \to y) = f_a(x) \to f_a(y)$ for every $x, y \in L$. This completes the proof.

□

Proposition 3.26. *Let L be a lattice implication algebra. Then, for each $p \in L$, we have $\alpha_p(x \vee p) = 1$.*

Proof. For $p \in L$, we have

$$\begin{aligned}\alpha_p(x \vee p) &= \alpha_p((p \to x) \to x) = p \to ((p \to x) \to x)\\ &= (p \to x) \to (p \to x) = 1,\end{aligned}$$

for any $x \in L$. This completes the proof.

□

Theorem 3.27. *Let L be a lattice implication algebra. Then the following properties hold for all $p, q \in L$.*

(1) *The simple multiplier f_1 is an identity function on L.*
(2) *If $p \leq q$, then $f_q \leq f_p$.*
(3) *If $p \neq q$, then $f_q \neq f_p$.*
(4) *$f_p(x \sqcup p) = 1$, for all $x \in L$.*

Proof. (1) For every $x \in L$, we have $f_1(x) = 1 \to x = x$.

(2) Let $p \leq q$. Then we get $q \to x \leq p \to x$, Thus $f_q(x) \leq f_p(x)$ for all $x \in L$. That is, $f_q \leq f_p$.

(3) Let $f_q = f_p$. so $f_q(x) = f_p(x)$ for all $x \in L$. This implies $p \to x = q \to x$ for all $x \in L$. Now if $x = p$, then $p \to p = q \to p$, which implies $q \to p = 1$, that is $q \leq p$. if $x = q$, then $p \to q = q \to q$, which implies $p \to q = 1$, that is $p \leq q$. Thus $p = q$, which is a contradiction. So if $p \neq q$, then $f_q \neq f_p$.

(4) For every $p \in L$, we have

$$\begin{aligned} f_p(x \sqcup p) &= f_p(p \to x) \to x) = p \to ((p \to x) \to x) \\ &= (p \to x) \to (p \to x) = 1. \end{aligned}$$

□

For any $f_a, f_b \in S(L)$, define two binary operations

$$(f_p \wedge f_q)(x) = f_p(x) \wedge f_q(x), \quad (f_p \vee f_q)(x) = f_p(x) \vee f_q(x)$$

for any $p, q, x \in L$.

Lemma 3.28. *Let $f_p, f_q \in S(L)$. Then we have for all $p, q \in L$,*

$$f_p \wedge f_q \in S(L) \text{ and } f_p \vee f_q \in S(L).$$

Proof. Let $f_p, f_q \in S(L)$. Then we have for all $x \in L$,

$$\begin{aligned} (f_p \wedge f_q)(x) &= f_p(x) \wedge f_q(x) = (p \to x) \wedge (q \to x) \\ &= (p \vee q) \to x = f_{(p \vee q)}(x). \end{aligned}$$

Since $p \vee q \in L$, we have $f_p \wedge f_q \in S(L)$.

Also, for every $x \in L$, we get

$$\begin{aligned} (f_p \vee f_q)(x) &= f_p(x) \vee f_q(x) = (p \to x) \vee (q \to x) \\ &= (p \wedge q) \to x = f_{(p \wedge q)}(x). \end{aligned}$$

Since $p \wedge q \in L$, we have $f_p \vee f_q \in S(L)$.

□

By using the Lemma 3.28, we have the the following theorem.

Theorem 3.29. *Let L be a lattice implication algebra. Then $S(L)$ is a bounded $\wedge$-semilattice with top element f_0 and bottom element f_1.*

Proposition 3.30. *For any $p \in L$, the mapping $\beta_p(a) = p \to (p \to a)$ is a multiplier of L.*

Proof. Let $p \in L$. Then we have

$$\begin{aligned}\beta_p(a \to b) &= p \to (p \to (a \to b)) \\ &= p \to (a \to (p \to b)) \\ &= a \to (p \to (p \to b)) \\ &= a \to \beta_p(b)\end{aligned}$$

for all $a, b \in L$. This completes the proof. □

Proposition 3.31. *Let L be a lattice H implication algebra. For any $p \in L$, the multiplier $\beta_p(a) = p \to (p \to a)$ is a homomorphism of L.*

Proof. Let $p \in L$. Then we have

$$\begin{aligned}\beta_p(a \to b) &= p \to (p \to (a \to b)) \\ &= p \to ((p \to a) \to (p \to b)) \\ &= (p \to (p \to a)) \to (p \to (p \to b)) \\ &= \beta_p(a) \to \beta_p(b)\end{aligned}$$

for all $a, b \in L$. This completes the proof. □

Proposition 3.32. *Let L be a lattice implication algebra. If $a \le b$ for any $a, b \in L$, we have $\beta_p(a \to b) = 1$.*

Proof. Let a and b be such that $a \le b$. Then $a \to b = 1$. Thus we have $\beta_p(a \to b) = \beta_p(1) = p \to (p \to 1) = p \to 1 = 1$. This completes the proof. □

Let L be a lattice implication algebra and let f be a multiplier of L. Define a $Kerf$ by

$$Kerf = \{x \in L \mid f(x) = 1\}.$$

Proposition 3.33. *Let f be a multiplier of a lattice implication algebra L. If f is an implication homomorphism on L, $Kerf$ is a filter of L.*

Proof. Clearly, $1 \in Kerf$. Let $x, x \to y \in Kerf$. Then $f(x) = 1$ and $f(x \to y) = 1$. Thus we have

$$1 = f(x \to y) = f(x) \to f(y) = 1 \to f(y) = f(y),$$

which implies $y \in Kerf$. □

Proposition 3.34. *Let L be a lattice implication algebra and let f be a multiplier of L. If $y \in Kerf$, then we have $x \vee y \in Kerf$ for all $x \in L$.*

Proof. Let f be a multiplier of L and $y \in Kerf$. Then we get $f(y) = 1$, and thus

$$f(x \vee y) = f((x \to y) \to y) = (x \to y) \to f(y) = (x \to y) \to 1 = 1.$$

So we have $x \vee y \in Kerf$. □

Proposition 3.35. *Let L be a lattice implication algebra and f be a multiplier of L. If $x \leq y$ and $x \in Kerf$, then $y \in Kerf$.*

Proof. Let $x \leq y$ and $x \in Kerf$. Then we get $x \to y = 1$ and $f(x) = 1$, and thus

$$\begin{aligned} f(y) &= f(1 \to y) = f((x \to y) \to y) \\ &= f((y \to x) \to x) = (y \to x) \to f(x) \\ &= (y \to x) \to 1 = 1. \end{aligned}$$

So we have $y \in Kerf$.

□

Proposition 3.36. *Let L be a lattice implication algebra and let f be a multiplier of L. If $y \in Kerf$, then we have $x \to y \in Kerf$ for all $x \in L$.*

Proof. Let $y \in Kerf$. Then $f(y) = 1$. Thus we have

$$f(x \to y) = x \to f(y) = x \to 1 = 1.$$

So we get $x \to y \in Kerf$.

□

Definition 3.37. Let L be a lattice implication algebra. A nonempty subset F of L is said to be a f-invariant, if $f(F) \subseteq F$ where $f(F) = \{f(x) \mid x \in F\}$.

Theorem 3.38. *Let L be a lattice implication algebra and let f be a multiplier of L. Then every filter F is a f-invariant.*

Proof. Let F be a filter of L. Let $y \in f(F)$. Then $y = f(x)$ for some $x \in F$. It follows that $x \to y = x \to f(x) = 1 \in F$, which implies $y \in F$. Thus $f(F) \subseteq F$. So F is a f-invariant.

□

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References

[1] Y. Xu, Lattice implication algebras, J. Southwest Jiaotong Univ. 1 (1993) 20–27.

[2] Y. Xu and K. Y. Qin, Lattice H implication algebras and lattice implication algebra classes, J. Hebei Mining and Civil Engineering Institute 3 (1992) 139–143.

[3] Y. Xu and K. Y. Qin, On filters of lattice implication algebras, J. Fuzzy Math. 1 (2) (1993) 251–260.

[4] S. D. Lee and K. H. Kim, On derivations of lattice implication algebras, Ars Combinatoria, 108 (2013) 279–288.

[5] Y. H. Yon and K. H. Kim, On f-derivations of lattice implication algebras, Ars Combinatoria, 110 (2013) 205–215.

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