Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 21, No. 2, (April 2021) pp. 179–193
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.21.2.179

The ring of soft $\hat{p}$-adic numbers

Fawzan Sidky, M. E. El-Shafei, M. K. Tahat

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 21, No. 2, April 2021

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 21, No. 2, (April 2021) pp. 179–193
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.21.2.179

@FMI

The ring of soft $\hat{p}$-adic numbers

Fawzan Sidky, M. E. El-Shafei, M. K. Tahat

Received 23 December 2020; Revised 14 January 2021; Accepted 31 January 2021

Abstract. We have studied and investigated the soft integers as a significant subset of soft real numbers, as we have developed many well-known theorems over integers to deal with the soft theory like the Chinese remainder theorem over the soft integers beside many other theories. Also, we have studied and investigated the p-adic numbers in the aspect of the soft theory to introduce the ring of soft $\hat{p}$-adic numbers and study some of their properties.

2020 AMS Classification: Primary 16W80, 11F85

Keywords: p-adic Numbers, Soft Sets, Soft integers, Soft topology, Soft $\hat{p}$-adic numbers, Soft$\hat{p}$-adic metric space.

Corresponding Author: M. K. Tahat (just.tahat@gmail.com)

1. Introduction

In 1999, Molodtsov [1] proposed the soft set theory as a new approach to managing uncertainties as he introduced the concept of the soft set to be a set associated with a set of parameters applied in several directions, that makes it a comprehensive extension for the theories of fuzzy sets [2], vague sets [3], and rough sets [4]. All these theories have their applications in many directions such as physics, biology, and computer science. Whereas the soft set theory is the most recent among these theories, nevertheless it has achieved fascinating results in all branches of mathematics and other sciences. Moreover, many authors studied the relationship between all of these theories such as [5, 6, 7, 8].

In [9], the authors defined new types of belong and non-belong relations and utilized them to define strong types of soft separation axioms. Then, Al-shami and El-Shafei [10, 11] presented two applications of these relations on the fields of soft separation axioms and decision-making problems. The interrelations between soft topological space and its parametric topological spaces were investigated in [12]. Recently, the concepts of soft topological ordered spaces and sum of soft topological spaces have been introduced in [13] and [14], respectively. Recently, many concepts have developed in soft theory. For instance, Acar et al [15] studied the soft algebraic structures of rings by introducing the idea of soft rings. Shabir and Naz [16] studied the soft topological structures by adding the notion of soft topology, which has been extensively studied and investigated by several authors like [17, 18, 19]. After that, some authors went to examine the connection between the soft topological structures and the soft algebraic structures such as the concepts of soft topological soft groups and rings [20], soft topological soft modules [21] and soft topological rings [22].

Furthermore, Das and Samanta [23] introduced the notions of soft real numbers and soft complex numbers and studied their fundamental properties; like the normal differentiation and partial differentiation of soft functions. In the same year, they have introduced the concept of soft metric space and discussed the sequences on soft real numbers in the papers [24, 25].

After that and based on the concept of soft real numbers, many papers have appeared, discussing the significant role of soft real numbers in extended many concepts and main theorems in the soft setting in all mathematics areas. For example, Das and Samanta [26] introduced the notion of integral of a soft function of complex numbers and established Cauchy's theorem, and Thakur and Samanta [27] discussed the differentiability of functions of soft real numbers, also he extended fundamental theories in soft setting, like Rolle's theorem and Lagrange's mean value.

On the other hand, over the last century, p-adic numbers [28] and p-adic analysis [29] have come to play a central role in modern number theory. This importance comes from the fact that they afford a natural and powerful language for talking about congruences between integers, and allow the use of methods borrowed from calculus and analysis for studying such problems. More recently, p-adic numbers have shown up in other areas of mathematics, and even in physics.

Our motive is to complete the gaping in the studies of the applications and extensions of the soft set theory to the number theory and abstract algebra.

In this paper, in Section 2, we presented well-known results of the essential preliminaries related to soft set and soft real numbers. In Section 3, the soft prime integers have been introduced; also, the fundamental theorem of soft arithmetic has been stated and proved such that for all nonzero soft integer there exist a unique soft factorization of soft prime integers. Moreover, some of the fundamental theorems like division theorem extended to soft integer numbers. In Section 4, we have defined $\hat{p}$-adic soft absolute value over rings like the rings of soft real numbers and soft rational numbers. In Section 5, we have investigated the p-adic numbers in the aspect of the soft theory to introduce the ring of soft $\hat{p}$-adic numbers and study some of their properties.

2. Preliminaries

In this section we give some background material that will be made use of in the paper, especially some terminology and notation. For more basic material on p-adic numbers (See [28, 29]) and on ring theory (See [30]). In this paper, E is assumed to be a nonempty set of parameters. Let X and Y denote initial universal sets. Mainly in this section, we remember some concepts and results related to soft sets.

Definition 2.1 ([1])**.** A *soft set* F_A over X is defined to be a mapping $F_A : A \longrightarrow P(X)$, where $A \subseteq E$.

Let $S(X)$ denotes the class of all soft sets over X. Let $F_A \in S(X)$. We may write $F_A = \{(a, F_A(a)) \mid a \in A\}$ and if $A = E$, we write F instead of F_A. If F_A is defined such that $F_A(a) = \phi, \forall a \in A$, then F_A is called a *null soft set* over X and denoted by $\tilde{\phi_A}$. And if F_A is defined such that $F_A(a) = X, \forall a \in A$, then F_A is called an *absolute soft set* over X, and denoted by $\tilde{X_A}$ (See [31]). If $F_A(a)$ is a singleton set for each $a \in A$, then F_A is called a *singleton soft set* over X. We denote the class of all singleton soft sets over X by $\hat{X}$. Note that if $X \subseteq Y$, then $\hat{X} \subseteq \hat{Y}$.

Let $F_A,\ G_B \in S(X)$. Then F_A is called a *soft subset* of G_B, if $A \subseteq B$ and $F_A(a) \subseteq G_B(a)\ \forall a \in A$. In this case, we write $F_A \subseteq G_B$.

If $F_A \in \hat{\mathbb{R}}$, then F_A is called a *soft real number* (See [23]). Similarly, one can define *soft integers* and *soft rational numbers*.

We will use the notations $\hat{n}, \hat{m}, \hat{k}, \cdots$ to denote soft real numbers. If $\hat{n}(a) = \{n\}, \forall a \in A$, then $\hat{n}$ is denoted by $\tilde{n}$.

Definition 2.2 ([32]). The *Cartesian product* of $F_A \in S(X)$ and $G_B \in S(Y)$ is defined to be the soft set $(F_A \times G_B)_{(A\times B)} \in S(X \times Y)$, such that $(F_A \times G_B)_{(A\times B)}(a, b) = F_A(a) \times G_B(b), \forall (a, b) \in A \times B$. We write shortly $F_A \times G_B$ instead of $(F_A \times G_B)_{(A\times B)}$. Moreover, $R \in S(X \times Y)$ is called a *soft relation from F_A to G_B*, if $R \subseteq (F_A \times G_B)$.

2.1. **Soft real numbers.** Mainly in this subsection, we remember some concepts and results related to soft real numbers. From now on, we will assume that all soft sets defined on the set of parameters E. For simplicity we take $E = \{1, 2, ..., s\}$. If $\hat{n} \in \hat{\mathbb{R}}$, we put $\hat{n}(i) = \{n_i\}$ and $(\hat{n})_i = n_i, \forall i \in E$, and $\hat{n} = (n_1, n_2, \cdots, n_s)$.

Definition 2.3 ([23]). Suppose that $\hat{n}, \hat{m} \in \hat{\mathbb{R}},\ k \in \mathbb{N}$ and $\{\hat{n}_j\}_{j\in J}$ is a class of soft real numbers.

(i) The *addition* of $\hat{n}$ and $\hat{m}$ is defined to be the soft real number $\hat{n} \ +\ \hat{m}$, such that
$$(\hat{n} \ +\ \hat{m})(i) = \{n_i + m_i\} \ \text{ for each } i \in E.$$

(ii) The *multiplication* of $\hat{n}$ and $\hat{m}$ is defined to be the soft real number $\hat{n} \ \cdot\ \hat{m}$ (or $\hat{n}\hat{m}$) such that
$$(\hat{n} \ \cdot\ \hat{m})(i) = \{n_i m_i\} \ \text{ for each } i \in E.$$

(iii) The *division* of $\hat{n}$ by $\hat{m}$, where $m_i \neq 0, \forall i \in E$, is defined to be the soft real number $\dfrac{\hat{n}}{\hat{m}}$ such that
$$(\frac{\hat{n}}{\hat{m}})(i) = \{\frac{n_i}{m_i}\} \ \text{ for each } i \in E.$$

(iv) A *soft power* of $\hat{n}$ is a soft real number $\hat{n}^{\hat{m}}$ such that
$$\hat{n}^{\hat{m}}(i) = \{(n_i)^{m_i}\}, \text{ for each } i \in E.$$

The *multiplication* of $\{\hat{n}_j\}_{j\in J}$ is denoted by $\prod_{j\in J} \hat{n}_j$. And the soft real number $\hat{n} \cdot \hat{n} \cdot \cdots \cdot \hat{n}$ (k times) is denoted by $\hat{n}^k$. Note that $\hat{n}^k = \hat{n}^{\tilde{k}}$.

Proposition 2.4 ([23]). *Let $\hat{n}, \hat{m} \in \hat{\mathbb{R}}$ and $k, r \in \mathbb{N}$. Then*
(1) $\hat{n}^k \ \cdot\ \hat{n}^r = \hat{n}^{k+r}$,
(2) $(\hat{n}^k)^r = \hat{n}^{kr}$,
(3) $(\hat{n} \cdot \hat{m})^k = \hat{n}^k \ \cdot\ \hat{m}^k$.

Definition 2.5 ([33])**.** Suppose that $\hat{n}, \hat{m} \in \hat{\mathbb{R}}$. Then

(i) $\hat{n}$ is said to be *smaller than or equal* (resp. *smaller than*) $\hat{m}$ and denoted by $\hat{n} \leq \hat{m}$ (resp. $\hat{n} < \hat{m}$), if $n_i \leq m_i$ (resp. $n_i < m_i$) $\forall i \in E$,

(ii) $\hat{n}$ is called a *positive* (resp.*negative*) *soft real number*, if $n_i > 0$ (resp. $n_i < 0$) $\forall i \in E$. The set of all positive soft real numbers is denoted by $\hat{\mathbb{R}}^+$.

Definition 2.6 ([33])**.** Suppose that $\hat{n}, \hat{m} \in \hat{\mathbb{R}}$, $k \in \mathbb{N}$ and $\{\hat{n}_j\}_{j\in J}$ is a class of soft real numbers.

(i) The *additive inverse* of $\hat{n}$ is defined to be the soft real number $-\hat{n}$ such that $(-\hat{n})_i = -n_i$ for all $i \in E$.

(ii) The *multiplicative inverse* of $\hat{n}$ such that $n_i \neq 0$ $\forall i \in E$, is defined to be the soft real number $\hat{n}^{-1}$ such that $(\hat{n}^{-1})_i = (n_i)^{-1}$ for all $i \in E$.

(iii) The *meet* (or *infimum*) of $\hat{n}$ and $\hat{m}$ is defined to be the soft real number $\hat{n} \wedge \hat{m}$ (or $\inf\{\hat{n}, \hat{m}\}$) such that $(\hat{n} \wedge \hat{m})_i = n_i \wedge m_i$ $\forall i \in E$. And the infimum of $\{\hat{n}_j\}_{j\in J}$ is defined to be the soft real number $\bigwedge\limits_{j\in J} \hat{n}_j$ such that $(\bigwedge\limits_{j\in J} \hat{n}_j)_i = \bigwedge\limits_{j\in J} (n_j)_i$ for all $i \in E$.

(iv) The *join* (or *supremum*) of $\hat{n}$ and $\hat{m}$ is defined to be the soft real number $\hat{n} \vee \hat{m}$ (or $\sup\{\hat{n}, \hat{m}\}$) such that $(\hat{n} \vee \hat{m})_i = n_i \vee m_i$, $\forall i \in E$. And the supremum of $\{\hat{n}_j\}_{j\in J}$ is defined to be the soft real number $\bigvee\limits_{j\in J} \hat{n}_j$ such that $(\bigvee\limits_{j\in J} \hat{n}_j)_i = \bigvee\limits_{j\in J} (n_j)_i$ for all $i \in E$.

3. On soft integers

Definition 3.1. The *usual absolute value* on $\hat{\mathbb{R}}$ is defined to be the function $|\ | : \hat{\mathbb{R}} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ such that for all $\hat{n} \in \hat{\mathbb{R}}$, $(|\hat{n}|)_i = |n_i|$ for all $i \in E$, where $|n_i|$ is the usual absolute value of n_i.

Note that $\hat{\mathbb{Z}}$, $\hat{\mathbb{Q}}$ and $\hat{\mathbb{R}}$ are commutative rings with zero element $\tilde{0}$ and one element $\tilde{1}$ with respect to addition and multiplication of soft real numbers.

Proposition 3.2. *Let $\hat{n}$ and $\hat{m}$ be two soft positive integers. Then*
(1) $\hat{n} + \hat{m} \geq \hat{n} \vee \hat{m}$,
(2) $-(\hat{n} + \hat{m}) \leq -\hat{n} \wedge -\hat{m}$.

Proof. Let $\hat{n}$ and $\hat{m}$ be two soft positive integers.

(1) Let $i \in E$. Then $(\hat{n} + \hat{m})_i = n_i + m_i$, where $n_i, m_i \in \mathbb{Z}^+$ and $(\hat{n} \vee \hat{m})_i = n_i \vee m_i$. Since $n_i + m_i \geq n_i \vee m_i$ for each $i \in E$, it follows directly that $\hat{n} + \hat{m} \geq \hat{n} \vee \hat{m}$.

(2) It can be proven by the same manner as proof of (i). □

Definition 3.3. Let $\hat{a}, \hat{b}, \hat{p} \in \hat{\mathbb{Z}}$.

(i) We say that $\hat{a}$ *divides* $\hat{b}$ and write $\hat{a} \mid \hat{b}$, if there exists $\hat{c} \in \hat{\mathbb{Z}}$ such that $\hat{b} = \hat{a} \cdot \hat{c}$. Also, $\hat{a}$ and $\hat{c}$ are called *divisors* of $\hat{b}$ and $\hat{b}$ is called a *multiple* of $\hat{a}$. If $\hat{a}$ does not divide $\hat{b}$, we write $\hat{a} \nmid \hat{b}$.

(ii) If $\hat{p} \neq \tilde{1}$ and p_i is a prime or $p_i = 1$, $\forall i \in E$. Then $\hat{p}$ is called a *soft prime*. If a soft prime $\hat{p}$ has exactly one prime p_i, then it is called an i^{th} *soft prime*.

Remark 3.4. Let $\hat{a}, \hat{b}, \hat{p} \in \hat{\mathbb{Z}}$. Then
(1) $\hat{a} \mid \hat{b}$ if and only if $a_i \mid b_i$, $\forall i \in E$,
(2) $\hat{p}$ is an i^{th} soft prime if and only if $\tilde{1}$ and $\hat{p}$ are the only divisors of $\hat{p}$.

It is clear that the mapping

$$\hat{\mathbb{Z}} \to \mathbb{Z}^s$$
$$\hat{n} \mapsto (n_1, n_2, \cdots, n_s)$$

is an isomorphism of rings. Also, we have $\hat{\mathbb{Q}} \simeq \mathbb{Q}^s$ and $\hat{\mathbb{R}} \simeq \mathbb{R}^s$.

Proposition 3.5. *Let $\hat{a} \in \hat{\mathbb{Z}}$. For every order of the primes in the prime factorizations of $a_1, \cdots, a_s$ there exists a unique factorization of $\hat{a}$ into the product of soft powers of soft primes.*

Proof. Let $a_i = p_{1i}^{n_{1i}} p_{2i}^{n_{2i}} \cdots p_{r_i i}^{n_{r_i i}}$ be the prime factorization of a_i, $i = 1, 2, \cdots, s$ and let $r = \max\{r_1, r_2, \cdots, r_s\}$. Then
$\hat{a} = (\hat{p}_1)^{\hat{n}_1} (\hat{p}_2)^{\hat{n}_2} \cdots (\hat{p}_r)^{\hat{n}_r}$, where

$$\left(\hat{p}_j\right)_i = \begin{cases} p_{ji} & \text{if } j \le r_i \\ 1 & \text{if } j > r_i \end{cases} \quad \text{and} \quad \left(\hat{n}_j\right)_i = \begin{cases} n_{ji} & \text{if } j \le r_i \\ 1 & \text{if } j > r_i. \end{cases}$$

Note that each soft integer has a unique factorization into product of soft i^{th} primes, $i \in E$, because each soft prime is a product of i^{th} soft primes, $i \in E$. □

Example 3.6. Let $E = \{1, 2, 3\}$ and $\hat{a} = (30, 45, 18)$. Then
$a_1 = 30 = 2 \times 3 \times 5 = p_{11} \times p_{21} \times p_{31}$, $a_2 = 45 = 3^2 \times 5 = p_{12}^2 \times p_{22}$, $a_3 = 18 = 3^2 \times 2 = p_{13}^2 \times p_{23}$. Since $r_1 = 3, r_2 = r_3 = 2$, we have $r = 3$. Thus $\hat{a} = (\hat{p}_1)^{\hat{n}_1} (\hat{p}_2)^{\hat{n}_2} (\hat{p}_3)^{\hat{n}_3}$, where $\hat{p}_1 = (p_{11}, p_{12}, p_{13}) = (2, 3, 3)$, $\hat{p}_2 = (p_{21}, p_{22}, p_{23}) = (3, 5, 2)$,$\hat{p}_3 = (p_{31}, p_{32}, p_{33}) = (5, 1, 1)$, $\hat{n}_1 = (n_{11}, n_{12}, n_{13}) = (1, 2, 2)$, $\hat{n}_2 = (n_{21}, n_{22}, n_{23}) = (1, 1, 1)$, $\hat{n}_3 = (n_{31}, n_{32}, n_{33}) = (1, 1, 1)$.

Definition 3.7. Let $\hat{a}, \hat{b} \in \hat{\mathbb{Z}}$. If $\hat{d} = \sup\{\hat{m} \in \hat{\mathbb{Z}} : \hat{m} \mid \hat{a} \text{ and } \hat{m} \mid \hat{b}\}$, then $\hat{d}$ is called the *greatest common divisor* of $\hat{a}$ and $\hat{b}$ and denoted by $\gcd(\hat{a}, \hat{b})$. If $\gcd(\hat{a}, \hat{b}) = \tilde{1}$, then we say that $\hat{a}$ and $\hat{b}$ are *relatively soft prime integers*.

Note that $\hat{d} = \gcd(\hat{a}, \hat{b})$ if and only if $d_i = \gcd(a_i, b_i), \forall i \in E$.

Example 3.8. Let $E = \{1, 2, 3\}$, $\hat{a} = (36, 42, 24)$ and $\hat{b} = (28, 16, 48)$.
Then $\gcd(\hat{a}, \hat{b}) = (\gcd(36, 28), \gcd(42, 16), \gcd(24, 48)) = (4, 2, 24)$.

Proposition 3.9. *Let $\hat{a}, \hat{b} \in \hat{\mathbb{Z}}$ such that $a_i \neq 0$ and $b_i \neq 0, \forall i \in E$. Let $\hat{d}$ be the greatest common divisor of $\hat{a}$ and $\hat{b}$. Then there exist soft integers $\hat{x}$ and $\hat{y}$ such that $\hat{a} \cdot \hat{x} + \hat{b} \cdot \hat{y} = \hat{d}$.*

Proof. Let $\hat{d} = \gcd(\hat{a}, \hat{b})$. Then $d_i = \gcd(a_i, b_i)$, $\forall i \in E$. Thus there exists $x_i, y_i \in \mathbb{Z}$ such that $a_i x_i + b_i y_i = d_i$, $\forall i \in E$. Let $\hat{x} = (x_1, x_2, \cdots, x_s)$ and $\hat{y} = (y_1, y_2, \cdots, y_s)$. Then $\hat{a} \cdot \hat{x} + \hat{b} \cdot \hat{y} = \hat{d}$. □

Theorem 3.10 (Division theorem for soft real numbers)**.** *If $\hat{a}, \hat{b} \in \hat{\mathbb{Z}}$ such that $b_i \neq 0 \, \forall i \in E$, then there exist unique $\hat{q}, \hat{r} \in \hat{\mathbb{Z}}$ such that $\hat{a} = \hat{q} \cdot \hat{b} + \hat{r}$ and $\tilde{0} \le \hat{r} < |\hat{b}|$, where $|\,|$ is the usual absolute value on $\hat{\mathbb{R}}$.*

Here $\hat{q}$ is called the *quotient of the division $\hat{a}$ by $\hat{b}$ and $\hat{r}$* is called the *remainder*.

Proof. Let $\hat{a}, \hat{b} \in \hat{\mathbb{Z}}$ such that $b_i \neq 0$, $\forall i \in E$. Then there exist unique $q_i, r_i \in \mathbb{Z}$ such that

$$a_i = q_i b_i + r_i \text{ and } 0 \le r_i < |\, b_i \,| \quad \forall i \in E.$$

Thus there exist unique $\hat{q} = (q_1, q_2, \cdots q_s)$ and $\hat{r} = (r_1, r_2, \cdots r_s)$ such that $\hat{a} = \hat{q} \cdot \hat{b} + \hat{r}$ and $\tilde{0} \le \hat{r} < |\, \hat{b} \,|$. □

Definition 3.11. Let $\hat{m} \in \hat{\mathbb{Z}}$ be a positive soft integer. The *congruence modulo* $\hat{m}$ is the relation $\cdots \equiv \cdots \pmod{\hat{m}}$ on the set $\hat{\mathbb{Z}}$ defined as follows:

$$\hat{x} \equiv \hat{y} \pmod{\hat{m}} \Leftrightarrow \hat{m} \mid (\hat{x} - \hat{y}),$$

where $\hat{x} - \hat{y} := \hat{x} + (-\hat{y})$.

Remark 3.12. Note that

(1) $\hat{x} = \hat{m} \cdot \hat{q} + \hat{r}$ if and only if $\hat{x} \equiv \hat{r} \pmod{\hat{m}}$,

(2) $\hat{x} \equiv \hat{y} \pmod{\hat{m}} \Leftrightarrow x_i \equiv y_i \pmod{m_i}$, $\forall i \in E$,

(3) the congruence modulo $\hat{m}$ is an equivalence relation. We denote the equivalence class (or residue class) of $\hat{a} \in \hat{\mathbb{Z}}$ by $[\hat{a}]_{\hat{m}}$, i.e.

$$\begin{aligned} [\hat{a}]_{\hat{m}} &= \{\hat{x} \mid \hat{x} \equiv \hat{a} \pmod{\hat{m}}\} \\ &= \{\hat{x} \mid x_i \equiv a_i \pmod{m_i}, \forall i \in E\} \\ &= \left\{\hat{x} \mid x_i \in [a_i]_{m_i}, \forall i \in E\right\}, \end{aligned}$$

where $[a_i]_{m_i}$ is the equivalence class of a_i modulo m_i.

Let $\hat{\mathbb{Z}}_{\hat{m}} = \left\{[\hat{a}]_{\hat{m}} \mid \hat{a} \in \hat{\mathbb{Z}}\right\}$ and ϕ be a mapping defined as follows:

$$\begin{aligned} \phi : \hat{\mathbb{Z}}_{\hat{m}} &\to \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s} \\ [\hat{a}]_{\hat{m}} &\mapsto \left([a_1]_{m_1}, [a_2]_{m_2}, \cdots, [a_s]_{m_s}\right), \end{aligned}$$

Since ϕ is a bijective mapping, $\mid \hat{\mathbb{Z}}_{\hat{m}} \mid = m_1 m_2 \cdots m_s$. Also, we can write

$$[\hat{a}]_{\hat{m}} = [a_1]_{m_1} \times [a_2]_{m_2} \times \cdots \times [a_s]_{m_s}.$$

Note that if $[\hat{a}]_{\hat{m}} \in \hat{\mathbb{Z}}_{\hat{m}}$, such that $\hat{a} \not< \hat{m}$, then there is $\hat{b} \in \hat{\mathbb{Z}}$ such that $\hat{b} < \hat{m}$ and $\hat{b} \equiv \hat{a} \pmod{\hat{m}}$. Thus we have $\hat{\mathbb{Z}}_{\hat{m}} = \left\{[\hat{a}]_{\hat{m}} \mid \tilde{0} \leq \hat{a} < \hat{m}\right\}$.

Definition 3.13. Let $\hat{m}$ be a positive soft integer. The operations of *addition modulo* $\hat{m}$ and the *multiplication modulo* $\hat{m}$ on $\hat{\mathbb{Z}}_{\hat{m}}$ are defined respectively as follows:for all $[\hat{a}]_{\hat{m}}, \left[\hat{b}\right]_{\hat{m}} \in \hat{\mathbb{Z}}_{\hat{m}}$,

$$[\hat{a}]_{\hat{m}} + \left[\hat{b}\right]_{\hat{m}} = \left[\hat{a} + \hat{b}\right]_{\hat{m}}, \ [\hat{a}]_{\hat{m}} \cdot \left[\hat{b}\right]_{\hat{m}} = \left[\hat{a} \cdot \hat{b}\right]_{\hat{m}}.$$

It is clear that $\hat{\mathbb{Z}}_{\hat{m}}$ is a commutative ring under addition and multiplication modulo $\hat{m}$ with zero element $\left[\tilde{0}\right]_{\hat{m}}$ and one element $\left[\tilde{1}\right]_{\hat{m}}$. Let $\hat{m}\hat{\mathbb{Z}} = \{\hat{m}\hat{a} \mid \hat{a} \in \hat{\mathbb{Z}}\}$. Then $\hat{m}\hat{\mathbb{Z}}$ is a subgroup of $\hat{\mathbb{Z}}$. It is clear that $\hat{m}\hat{\mathbb{Z}}$ is an ideal of $\hat{\mathbb{Z}}$. The rings $\hat{\mathbb{Z}}/\hat{m}\hat{\mathbb{Z}}$, $\hat{\mathbb{Z}}_{\hat{m}}$, and $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$ are isomorphic. Also, if $\hat{p} \in \hat{\mathbb{Z}}$ such that p_i is a prime number for all $i \in E$, then $\hat{\mathbb{Z}}_{\hat{p}} \setminus \{[\tilde{0}]_{\hat{p}}\}$ is a multiplicative group with respect to the multiplication modulo $\hat{p}$. Note that

$$\hat{\mathbb{Z}}_{\hat{p}} \setminus \{[\tilde{0}]_{\hat{p}}\} \simeq \mathbb{Z}_{p_1} \setminus \{[0]_{p_1}\} \times \mathbb{Z}_{p_2} \setminus \{[0]_{p_2}\} \times \cdots \times \mathbb{Z}_{p_s} \setminus \{[0]_{p_s}\}.$$

Note that the congruence relation modulo $\hat{m}$ in the form

$$\hat{a} \cdot \hat{x} \equiv \hat{b} \pmod{\hat{m}}$$

is called a *linear congruence modulo* $\hat{m}$.

Theorem 3.14. *If* $\gcd(\hat{a}, \hat{m}) = \tilde{1}$, *then the linear congruence*

$$\hat{a} \cdot \hat{x} \equiv \hat{b} \pmod{\hat{m}}$$

has a unique solution $\hat{x} = \hat{b} \cdot \hat{r} \pmod{\hat{m}}$, *where* $\hat{r} \cdot \hat{a} \equiv \tilde{1} \pmod{\hat{m}}$.

Proof. Since $\gcd(\hat{a}, \hat{m}) = \tilde{1}$, we have $\gcd(a_i, m_i) = 1, \forall i \in E$. And since $\hat{a} \cdot \hat{x} \equiv \hat{b} \pmod{\hat{m}}$, we have $a_i x_i \equiv b_i \pmod{m_i}$, $\forall i \in E$. Then there exists a unique solution $x_i \equiv b_i r_i \pmod{m_i}$, where $r_i a_i \equiv 1 \pmod{m_i}$ for the linear congruence $a_i x_i \equiv b_i \pmod{m_i}$. Thus there exists a unique solution $\hat{x} = \hat{b} \cdot \hat{r} \pmod{\hat{m}}$ for the soft congruence $\hat{a} \cdot \hat{x} \equiv \hat{b} \pmod{\hat{m}}$, where $\hat{r} \cdot \hat{a} \equiv \tilde{1} \pmod{\hat{m}}$. □

Theorem 3.15 (Chinese soft remainder theorem for soft numbers)**.** *Let $\hat{m}_1, \hat{m}_2, \cdots, \hat{m}_r$ be soft integers such that $(\hat{m}_j)_i \neq 0$, for all $i \in E$ and $j = 1, 2, \cdots, r$. Assume that* $\gcd(\hat{m}_l, \hat{m}_k) = \tilde{1}$, *for all $1 \leq l \neq k \leq r$. Then for any soft integers $\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_r$, the congruences*

$$\hat{x} \equiv \hat{a}_j \pmod{\hat{m}_j}, \quad j = 1, 2, \cdots, r$$

have a common solution, which is unique modulo $\hat{m}$, where $\hat{m} = \hat{m}_1 \hat{m}_2 \cdot \cdot \cdot \hat{m}_r$.

Proof. Since $\gcd(\hat{m}_l, \hat{m}_k) = \tilde{1}$ for all $l, k \in \{1, 2, 3, \cdots, r\}$ with $l \neq k$, we have $\gcd((m_l)_i, (m_k)_i) = 1$, $\forall i \in E$. Since $\hat{x} \equiv \hat{a}_j \pmod{\hat{m}_j}$, $j = 1, 2, \cdots, r$, we have $x_i \equiv (a_j)_i \pmod{(m_j)_i}$ $j = 1, 2, \cdots, r$, $\forall i \in E$. Then it follows, from the Chinese remainder theorem, that there exists a common solution which is unique modulo $(m_1)_i (m_2)_i \cdots (m_r)_i$. Thus the congruences $\hat{x} \equiv \hat{a}_j \pmod{\hat{m}_j}$ $j = 1, 2, \cdots, r$ have a common solution which is unique modulo $\hat{m}_1 \, \hat{m}_2 \cdots \hat{m}_r$. □

Let $\hat{m}_1, \hat{m}_2, \cdots, \hat{m}_r$ be positive soft integers such that $\gcd(\hat{m}_i, \hat{m}_j) = \tilde{1}$ for all different i, j in $\{1, 2, \cdots, r\}$. Let $\hat{m} = \hat{m}_1 \hat{m}_2 \cdots \hat{m}_r$. Then it follows, from Theorem 3.15, that

$$\begin{aligned} \varphi : \hat{\mathbb{Z}}_{\hat{m}} &\to \mathbb{Z}_{\hat{m}_1} \times \mathbb{Z}_{\hat{m}_2} \times \cdots \times \mathbb{Z}_{\hat{m}_r} \\ [\hat{a}]_{\hat{m}} &\mapsto \Big([\hat{a}]_{\hat{m}_1}, [\hat{a}]_{\hat{m}_2}, \cdots, [\hat{a}]_{\hat{m}_r}\Big), \end{aligned}$$

is an isomorphism of rings.

4. The $\hat{p}$-adic soft absolute value

Definition 4.1. Let $\hat{p} \in \hat{\mathbb{Z}}$ be a soft prime. The *$\hat{p}$-adic valuation on* $\hat{\mathbb{Z}}$ is the function $\nu_{\hat{p}} : \hat{\mathbb{Z}} \to \widehat{\mathbb{N}_0 \cup \{\infty\}}$ such that for all $\hat{n} \in \hat{\mathbb{Z}}$,

$$\Big(\nu_{\hat{p}}(\hat{n})\Big)_i = \begin{cases} \nu_{p_i}(n_i) & \text{if } p_i \text{ is a prime} \\ 0 & \text{if } p_i = 1, \end{cases}$$

where ν_{p_i} is the p_i-adic valuation on $\mathbb{Z}$.

Note that if $n_i \neq 0$, then $n_i = p_i^{\nu_{p_i}(n_i)} m_i$, where $p_i \nmid m_i$. Thus if $n_i \neq 0$ for all $i \in E$, then $\hat{n} = \hat{p}^{\nu_{\hat{p}}(\hat{n})} \hat{m}$, where $\hat{p} \nmid \hat{m}$.

Example 4.2. Let $s = 3$, $\hat{n} = (20, 0, 18)$ and $\hat{p} = (1, 3, 2)$. Then $\nu_{\hat{p}}(\hat{n}) = (0, \nu_3(0), \nu_2(18)) = (0, \infty, 1)$

Definition 4.3. Let $\hat{p} \in \hat{\mathbb{Z}}$ be a soft prime. The *$\hat{p}$-adic valuation on* $\hat{\mathbb{Q}}$ is the function $\nu_{\hat{p}} : \hat{\mathbb{Q}} \to \widehat{\mathbb{Z} \cup \{\infty\}}$ such that for all $\hat{n} \in \hat{\mathbb{Q}}$,

$$\Big(\nu_{\hat{p}}(\hat{n})\Big)_i = \begin{cases} \nu_{p_i}(n_i) & \text{if } p_i \text{ is a prime,} \\ 0 & \text{if } p_i = 1, \end{cases}$$

where ν_{p_i} is the p_i-adic valuation on $\mathbb{Q}$.

Note that since $\nu_p\left(\frac{a}{b}\right) = \nu_p(a) - \nu_p(b)$, where $a, b \in \mathbb{Z}$, $b \neq 0$ and p is a prime, we have

$$\nu_{\hat{p}}\left(\frac{\hat{a}}{\hat{b}}\right) = \nu_{\hat{p}}(\hat{a}) - \nu_{\hat{p}}(\hat{b}),$$

where $\hat{a}, \hat{b} \in \hat{\mathbb{Z}}$, $b_i \neq 0$ for all $i \in E$.

Also, since for each $i \in E$, $\frac{a_i}{b_i} = p_i^{\nu_{p_i}\left(\frac{a_i}{b_i}\right)} \dfrac{c_i}{d_i}$, where $a_i, b_i, c_i, d_i \neq 0$ and $p_i \nmid c_i d_i$, we have

$$\frac{\hat{a}}{\hat{b}} = \hat{p}^{\nu_{\hat{p}}\left(\frac{\hat{a}}{\hat{b}}\right)} \frac{\hat{c}}{\hat{d}},$$

where $a_i, b_i, c_i, d_i \neq 0$ for all $i \in E$ and $\hat{p} \nmid \hat{c}\hat{d}$.

Proposition 4.4. *Let $\hat{p}$ be a soft prime and $\hat{x}, \hat{y} \in \hat{\mathbb{Q}}$. Then*
(1) $\nu_{\hat{p}}(\hat{x}\hat{y}) = \nu_{\hat{p}}(\hat{x}) + \nu_{\hat{p}}(\hat{y})$,
(2) $\nu_{\hat{p}}(\hat{x} + \hat{y}) = \nu_{\hat{p}}(\hat{x}) \wedge \nu_{\hat{p}}(\hat{y})$.

Proof. (1) For all $i \in E$, we have

$$\left(\nu_{\hat{p}}(\hat{x}\hat{y})\right)_i = \begin{cases} \nu_{p_i}(x_i y_i) & \text{if } p_i \text{ is a prime} \\ 0 & \text{if } p_i = 1 \end{cases}$$
$$= \begin{cases} \nu_{p_i}(x_i) + \nu_{p_i}(y_i) & \text{if } p_i \text{ is a prime} \\ 0 & \text{if } p_i = 1. \end{cases}$$

Then $\nu_{\hat{p}}(\hat{x}\hat{y}) = \nu_{\hat{p}}(\hat{x}) + \nu_{\hat{p}}(\hat{y})$.

(2) For all $i \in E$, we have

$$\left(\nu_{\hat{p}}(\hat{x} + \hat{y})\right)_i = \begin{cases} \nu_{p_i}(x_i + y_i) & \text{if } p_i \text{ is a prime} \\ 0 & \text{if } p_i = 1 \end{cases}$$
$$= \begin{cases} \nu_{p_i}(x_i) \wedge \nu_{p_i}(y_i) & \text{if } p_i \text{ is a prime} \\ 0 & \text{if } p_i = 1. \end{cases}$$

Then $\nu_{\hat{p}}(\hat{x} + \hat{y}) = \nu_{\hat{p}}(\hat{x}) \wedge \nu_{\hat{p}}(\hat{y})$.

□

Definition 4.5 ([29]). Let R be a ring with identity.

(1) An *absolute value on* R is a function $|\,|\colon R \to \mathbb{R}^+ \cup \{0\}$ such that for all $x, y \in R$,
  (i) $|x| = 0 \iff x = 0$,
  (ii) $|xy| = |x||y|$,
  (iii) $|x + y| \leq |x| + |y|$.

(2) An absolute value on R is called *non-archimedian*, if it satisfies the condition $|x + y| \leq |x| \vee |y|\ \forall x, y \in R$.

Definition 4.6. Let R be a ring with identity.

(1) A function $|\,|\colon \hat{R} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ is called a *soft absolute value on* the ring $\hat{R}$ if it satisfies the following conditions: for all $\hat{x}, \hat{y} \in \hat{R}$,
  (i) $(|\ \hat{x}\ |)_i = 0 \iff x_i = 0$,
  (ii) $|\ \hat{x}\hat{y}\ | = |\ \hat{x}\ |\,|\ \hat{y}\ |$,
  (iii) $|\ \hat{x} + \hat{y}\ | \leq |\ \hat{x}\ | + |\ \hat{y}\ |$.

(2) A soft absolute value on $\hat{R}$ is called *non-archimedian*, if it satisfies the condition $|\hat{x}+\hat{y}| \leq |\hat{x}| \vee |\hat{y}|, \forall \hat{x}, \hat{y} \in \hat{R}$.

Remark 4.7. Let $|\ |$ be an absolute value on $\mathbb{R}$. Then the function $|\ |: \hat{\mathbb{R}} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ defined by $(|\ \hat{n}\ |)_i = |\ n_i\ |$ for all $i \in E$ and $\hat{n} \in \hat{\mathbb{R}}$, is a soft absolute value on $\hat{\mathbb{R}}$

Definition 4.8. Let $\hat{p}$ be a soft prime. The *$\hat{p}$-adic soft absolute value on* $\hat{\mathbb{Q}}$ is the function $|\ |_{\hat{p}}: \hat{\mathbb{Q}} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ such that for all $\hat{n} \in \hat{\mathbb{Q}}$ and $i \in E$,

$$\left(|\ \hat{n}\ |_{\hat{p}}\right)_i = \begin{cases} |\ n_i\ |_{p_i} & \text{if } p_i \text{ is a prime} \\ |\ n_i\ | & \text{if } p_i = 1. \end{cases}$$

Where $|\ |_{p_i}$ is the p_i-adic absolute value on $\mathbb{Q}$ and $|\ |$ is the trivial absolute value on $\mathbb{Q}$.

Remark 4.9. Let $\hat{p}$ be a soft prime and $\hat{n} \in \hat{\mathbb{Q}}$. Then $|\ \hat{n}\ |_{\hat{p}} = \hat{p}^{-\nu_{\hat{p}}(\hat{n})}\hat{n}_{\hat{p}}$ such that $\hat{n}_{\hat{p}} \in \hat{\mathbb{N}}_0$ is defined by:

$$(\hat{n}_{\hat{p}})_i = \begin{cases} 1 & \text{if } p_i \text{ is a prime} \\ |\ n_i\ | & \text{if } p_i = 1, \end{cases}$$

where $|\ |$ is the trivial absolute value on $\mathbb{Q}$.

Proposition 4.10. *The $\hat{p}$-adic soft absolute value on $\hat{\mathbb{Q}}$ is a non-archimedean soft absolute value.*

Proof. Let $|\ |_{\hat{p}}$ be the $\hat{p}$-adic soft absolute value on $\hat{\mathbb{Q}}$. Let $\hat{m}, \hat{n} \in \hat{\mathbb{Q}}$.

(i) It is clear that $\left(|\ \hat{m}\ |_{\hat{p}}\right)_i = 0 \iff m_i = 0$.

(ii)

$$\begin{aligned} \left(|\ \hat{m}\hat{n}\ |_{\hat{p}}\right)_i &= \begin{cases} |\ m_i n_i\ |_{p_i} & \text{if } p_i \text{ is a prime} \\ |\ m_i n_i\ | & \text{if } p_i = 1 \end{cases} \\ &= \begin{cases} |\ m_i\ |_{p_i}|\ n_i\ |_{p_i} & \text{if } p_i \text{ is a prime} \\ |\ m_i\ ||\ n_i\ | & \text{if } p_i = 1 \end{cases} \\ &= \left(|\ \hat{m}\ |_{\hat{p}}|\ \hat{n}\ |_{\hat{p}}\right)_i \quad \forall i \in E. \end{aligned}$$

(iii)

$$\begin{aligned} \left(|\ \hat{m}+\hat{n}\ |_{\hat{p}}\right)_i &= \begin{cases} |\ m_i + n_i\ |_{p_i} & \text{if } p_i \text{ is a prime} \\ |\ m_i + n_i\ | & \text{if } p_i = 1 \end{cases} \\ &\leq \begin{cases} |\ m_i\ |_{p_i} + |\ n_i\ |_{p_i} & \text{if } p_i \text{ is a prime} \\ |\ m_i\ | + |\ n_i\ | & \text{if } p_i = 1 \end{cases} \\ &= \left(|\ \hat{m}\ |_{\hat{p}} + |\ \hat{n}\ |_{\hat{p}}\right)_i \quad \forall i \in E. \end{aligned}$$

(iv)

$$\Big(\mid \hat{m}+\hat{n} \mid_{\hat{p}}\Big)_i = \begin{cases} \mid m_i + n_i \mid_{p_i} & \text{if } p_i \text{ is a prime} \\ \mid m_i + n_i \mid & \text{if } p_i = 1 \end{cases}$$
$$\leq \begin{cases} \mid m_i \mid_{p_i} \vee \mid n_i \mid_{p_i} & \text{if } p_i \text{ is a prime} \\ \mid m_i \mid \vee \mid n_i \mid & \text{if } p_i = 1 \end{cases}$$
$$= \Big(\mid \hat{m} \mid_{\hat{p}} \vee \mid \hat{n} \mid_{\hat{p}}\Big)_i \quad \forall i \in E.$$

Then $\mid \hat{m}+\hat{n} \mid_{\hat{p}} \leq \mid \hat{m} \mid_{\hat{p}} \vee \mid \hat{n} \mid_{\hat{p}}$. □

Definition 4.11. Let R be a ring with identity.

(1) [29] A function $d : R \times R \to \mathbb{R}^+ \cup \{0\}$ is called a *metric on* R, if the following conditions are satisfied: for all $x, y, z \in R$,
   (i) $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$,
   (ii) $d(x, y) = d(y, x)$,
   (iii) $d(x, y) \leq d(x, z) + d(z, y)$.

(2) A function $d : \hat{R} \times \hat{R} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ is called a *soft metric on* $\hat{R}$, if the following conditions are satisfied: for all $\hat{x}, \hat{y}, \hat{z} \in \hat{R}$,
   (i) $d(\hat{x}, \hat{y}) \geq \tilde{0}$ and $d(\hat{x}, \hat{y}) = \tilde{0} \iff \hat{x} = \hat{y}$,
   (ii) $d(\hat{x}, \hat{y}) = d(\hat{y}, \hat{x})$,
   (iii) $d(\hat{x}, \hat{y}) \leq d(\hat{x}, \hat{z}) + d(\hat{z}, \hat{y})$.

Remark 4.12. Let R be a ring with identity and $d : \hat{R} \times \hat{R} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ be defined by $d(\hat{x}, \hat{y}) =\mid \hat{x} - \hat{y} \mid$, for all $\hat{x}, \hat{y} \in \hat{R}$, where $\mid\ \mid$ is a soft absolute value on $\hat{R}$. Then d is soft metric on $\hat{R}$.

Remark 4.13. Let $\hat{p}$ be a soft prime. Let $d : \hat{\mathbb{Q}} \times \hat{\mathbb{Q}} \to \widehat{\mathbb{R}^+ \cup \{0\}}$ be a function defined by $d(\hat{x}, \hat{y}) =\mid \hat{x} - \hat{y} \mid_{\hat{p}}$, for all $\hat{x}, \hat{y} \in \hat{\mathbb{Q}}$, where $\mid\ \mid_{\hat{p}}$ is the $\hat{p}$-adic soft absolute value on $\hat{\mathbb{Q}}$. Then d is soft metric on $\hat{\mathbb{Q}}$. This follows directly from Proposition 4.10 and Remark 4.12.

Up now on we consider that any soft metric d on $\hat{R}$, where R is a a ring with identity, is defined by $d(\hat{x}, \hat{y}) =\mid \hat{x} - \hat{y} \mid$ for all $\hat{x}, \hat{y} \in \hat{R}$, where $\mid\ \mid$ is a soft absolute value on $\hat{R}$.

5. The ring of soft $\hat{p}$-adic numbers

Definition 5.1. Let $\mid\ \mid$ be a soft absolute value on $\hat{\mathbb{R}}$ defined as in Remark 4.7. A sequence of soft real numbers $(\hat{x}_n)$ is called a *Cauchy sequence*, if for every $\hat{\epsilon} \in \hat{\mathbb{R}}^+$, there exists $M \in \mathbb{N}$ such that $\mid \hat{x}_m - \hat{x}_n \mid < \hat{\epsilon}$, $\forall m, n \geq M$.

Put $(\hat{x}_n)_i = x_{ni}$, $\forall i \in E$. Note that each sequence $(\hat{x}_n)$ of soft real numbers corresponds to the sequences $(x_{ni}), i = 1, 2, \cdots, s$.

Note also that the sequence $(\hat{x}_n)$ of soft real numbers is a Cauchy sequence if and only if (x_{ni}) is a Cauchy sequence, $i = 1, 2, \cdots, s$.

Proposition 5.2. *Let* $\mid\ \mid$ *be a non-archimedean soft absolute value on* $\hat{\mathbb{Q}}$. *Let* $(\hat{x}_n)$ *be a sequence with* $\hat{x}_n \in \hat{\mathbb{Q}}$. *Then* $(\hat{x}_n)$ *is a Cauchy sequence if and only if* $\lim\limits_{n\to\infty} \mid \hat{x}_{n+1} - \hat{x}_n \mid = \tilde{0}$.

Proof. Let $(\hat{x}_n)$ be a Cauchy sequence. Then for any $\hat{\epsilon} > \tilde{0}$, there exists $k \in \mathbb{N}$ such that

$$\mid \hat{x}_m - \hat{x}_n \mid < \hat{\epsilon}, \quad \forall m, n > k.$$

Let $m > n$ and $m = n + r$. Since $|\ |$ is a non-archimedean absolute value, we have

$$| \hat{x}_m - \hat{x}_n | = | \hat{x}_{n+r} - \hat{x}_n | = | \hat{x}_{n+r} - \hat{x}_{n+r-1} + \hat{x}_{n+r-1} - \hat{x}_{n+r-2} , \cdots , \hat{x}_{n+1} - \hat{x}_n |$$
$$\leq \sup\{| \hat{x}_{n+r} - \hat{x}_{n+r-1} |, | \hat{x}_{n+r-1} - \hat{x}_{n+r-2} |, \cdots , | \hat{x}_{n+1} - \hat{x}_n |\}.$$

Then we have

$$| x_{mi} - x_{ni} | \leq \sup\{| x_{(n+r)i} - x_{(n+r-1)i} |, | x_{(n+r-1)i} - x_{(n+r-2)i} |, \cdots , | x_{(n+1)i} - x_{ni} |\} \forall i \in E.$$

Thus $\lim_{n\to\infty} | x_{(n+1)i} - x_{ni} | = \lim_{n\to\infty} (| \hat{x}_{n+1} - \hat{x}_n |)_i = 0\ \forall i \in E$. So $\lim_{n\to\infty} | \hat{x}_{n+1} - \hat{x}_n | = \tilde{0}$. The inverse direction is clear. □

Definition 5.3. Let $\hat{p}$ be a soft prime. Let

$$R_{\hat{p}} = \{(\hat{x}_n) \mid \hat{x}_n \in \hat{\mathbb{Q}} \text{ and } (\hat{x}_n) \text{ is a Cauchy sequence with respect to } |\ |_{\hat{p}}\}$$

We define the *addition* and *multiplication* on $R_{\hat{p}}$ as fallows: for all $(\hat{x}_n), (\hat{y}_n) \in R_{\hat{p}}$,

$$(\hat{x}_n) + (\hat{y}_n = (\hat{x}_n + \hat{y}_n),\ (\hat{x}_n)(\hat{y}_n) = (\hat{x}_n\hat{y}_n).$$

Remark 5.4. Let $\hat{p}$ be a soft prime. Let

$$R_{p_i} = \{(x_n) \mid x_n \in \mathbb{Q} \text{ and } (x_n) \text{ is a Cauchy sequence with respect to } |\ |_{p_i}\}.$$

If $p_i = 1$, then $|\ |_1$ denotes the trivial absolute value on $\mathbb{Q}$. Thus R_{p_i} is a ring for every $i = 1, 2, \cdots , s$.

Let $\psi : R_{\hat{p}} \to R_{p_1} \times R_{p_2} \times \cdots \times R_{p_s}$ such that $\psi((\hat{x}_n)) = ((x_{n1}), (x_{n2}), \cdots , (x_{ns}))$, $\forall (\hat{x}_n) \in R_{\hat{p}}$. Then it is clear that ψ is an isomorphism of rings.

Proposition 5.5. *$R_{\hat{p}}$ is a ring with identity element* $(\tilde{1})$.

Proof. Let $(\hat{x}_n), (\hat{y}_n) \in R_{\hat{p}}$. The sequence $(\hat{x}_n + \hat{y}_n)$ corresponds to the sequences $(x_{n1} + y_{n1}), (x_{n2} + y_{n2}), \cdots , (x_{ns} + y_{ns})$. Since $(x_{ni} + y_{ni}) = (x_{ni}) + (y_{ni})$ is a Cauchy sequence with respect to $|\ |_{p_i}$ (if $p_i = 1$, then $|\ |_1$ denotes the trivial absolute value on $\mathbb{Q}$), we have $(\hat{x}_n + \hat{y}_n) \in R_{\hat{p}}$. Similarly, $(\hat{x}_n\hat{y}_n) \in R_{\hat{p}}$. Then it is easy to see that $R_{\hat{p}}$ is a ring with zero element $(\tilde{0})$ and one element $(\tilde{1})$. □

Proposition 5.6. *Let $\hat{p}$ be a soft prime. Then*

$$M_{\hat{p}} = \left\{(\hat{x}_n) \in R_{\hat{p}} \mid \lim_{n\to\infty} |\hat{x}_n|_{\hat{p}} = \tilde{0}\right\}$$

is an ideal of $R_{\hat{p}}$.

Proof. Let $(\hat{x}_n) \in M_{\hat{p}}$. Since $\lim_{n\to\infty} |\hat{x}_n|_{\hat{p}} = \tilde{0}$ if and only if $\lim_{n\to\infty} |x_{n_i}|_{p_i} = 0, \forall i \in E$,

$$(x_{ni}) \in M_{p_i} = \left\{(x_n) \in R_{p_i} \mid \lim_{n\to\infty} |x_n|_{p_i} = 0\right\}.$$

Then $\psi(M_{\hat{p}}) = M_{p_1} \times M_{p_2} \times \cdots \times M_{p_s}$. Since M_{p_i} ($i = 1, 2, \cdots , s$) is an ideal of R_{p_i}, we have $M_{\hat{p}}$ is an ideal of $R_{\hat{p}}$ and $M_{\hat{p}} \simeq M_{p_1} \times M_{p_2} \times \cdots \times M_{p_s}$. □

The ring $\hat{\mathbb{Q}}_{\hat{p}} = R_{\hat{p}}/M_{\hat{p}}$ is called the *ring* of the soft $\hat{p}$-adic numbers. Note that

$$\begin{aligned}\hat{\mathbb{Q}}_{\hat{p}} = R_{\hat{p}}/M_{\hat{p}} &\simeq R_{p_1} \times R_{p_2} \times \cdots \times R_{p_s}/M_{p_1} \times M_{p_2} \times \cdots \times M_{p_s} \\ &\simeq R_{p_1}/M_{p_1} \times R_{p_2}/M_{p_2} \times \cdots \times R_{p_s}/M_{p_s} \\ &= \mathbb{Q}_{p_1} \times \mathbb{Q}_{p_2} \times \cdots \times \mathbb{Q}_{p_s}.\end{aligned}$$

Note that $\mathbb{Q}$ and $\hat{\mathbb{Q}}$ can be regarded as subrings of $R_{\hat{p}}$ and $\hat{\mathbb{Q}}_{\hat{p}}$, since

$$\begin{array}{c} \mathbb{Q} \to R_{\hat{p}} \\ x \mapsto (\tilde{x}) \end{array} \quad , \quad \begin{array}{c} \mathbb{Q} \to \hat{\mathbb{Q}}_{\hat{p}} \\ x \mapsto (\tilde{x}) + M_{\hat{p}} \end{array},$$

$$\begin{array}{c} \hat{\mathbb{Q}} \to R_{\hat{p}} \\ \hat{x} \mapsto (\hat{x}) \end{array} \quad \text{and} \quad \begin{array}{c} \hat{\mathbb{Q}} \to \hat{\mathbb{Q}}_{\hat{p}} \\ \hat{x} \mapsto (\hat{x}) + M_{\hat{p}} \end{array}$$

are monomorphisms of rings.

5.1. **The ring of $\hat{p}$-adic soft integers.** Note that $|\ \ |_p : \mathbb{Q}_p \to \mathbb{R}^+ \cup \{0\}$ is defined by $|(x_n) + M_p|_p = \lim_{n\to\infty} |x_n|_p \ \forall (x_n) \in R_p$ (see [29]).

Let $|\ \ |_{\hat{p}} : \hat{\mathbb{Q}}_{\hat{p}} \to \widehat{\mathbb{R}^+ \cup \{0\}}$, where $\hat{p}$ is a soft prime with p_i is a prime for all $i \in E$ such that

$$|(\hat{x}_n) + M_{\hat{p}}|_{\hat{p}} = \lim_{n\to\infty} |\hat{x}_n|_{\hat{p}} \quad , \forall (\hat{x}_n) \in R_{\hat{p}}.$$

Note that if $(\hat{x}_n) + M_{\hat{p}} = (\hat{y}_n) + M_{\hat{p}}$, then $\lim_{n\to\infty} |\hat{x}_n|_{\hat{p}} = \lim_{n\to\infty} |\hat{y}_n|_{\hat{p}}$ and so $|(\hat{x}_n) + M_{\hat{p}}|_{\hat{p}} = |(\hat{y}_n) + M_{\hat{p}}|_{\hat{p}}$. Thus $|\ \ |_{\hat{p}}$ is well defined. Moreover, note that $\left(\lim_{n\to\infty} |\hat{x}_n|_{\hat{p}}\right)_i = \lim_{n\to\infty} |x_{ni}|_{p_i} = |(x_{ni}) + M_{p_i}|_{p_i}$ $i = 1, 2. \cdots, s$. So $\left(|(\hat{x}_n) + M_{\hat{p}}|_{\hat{p}}\right)_i = |(x_{ni}) + M_{p_i}|_{p_i}$ $\forall i \in E$.

Proposition 5.7. *$|\ \ |_{\hat{p}}$ is a non-archimedean absolute value on $\hat{\mathbb{Q}}_{\hat{p}}$.*

Proof. Follows directly from the fact that $|\ \ |_p$ is a non-archimedean absolute value on $\mathbb{Q}_p$, where p is a prime (see [29]). And $\left(|(\hat{x}_n) + M_{\hat{p}}|_{\hat{p}}\right)_i = |(x_{ni}) + M_{p_i}|_{p_i}$ $\forall i \in E$. □

Definition 5.8. Let R be a ring with identity.

(i) A *soft open ball of center $\hat{a} \in \hat{R}$ and radius $\hat{r} \in \hat{\mathbb{R}}^+$* is the set

$$B(\hat{a}, \hat{r}) = \{\hat{x} \in \hat{R} \mid d(\hat{x}, \hat{a}) < \hat{r}\}.$$

(ii) A soft subset $\hat{S} \subset \hat{R}$ is called *dense in* $\hat{R}$, if $B(\hat{a}, \hat{r}) \cap \hat{S} \neq \hat{\phi}$ $\forall \hat{a} \in \hat{R}$ and $\hat{r} \in \hat{\mathbb{R}}^+$.

Proposition 5.9. *Let f be a mapping defined from $\hat{\mathbb{Q}}$ to $\hat{\mathbb{Q}}_{\hat{p}}$ such that $f(\hat{x}) = (\hat{x}) + M_{\hat{p}}$, $\forall \hat{x} \in \hat{\mathbb{Q}}$. Then $f(\hat{\mathbb{Q}})$ is dense in $\hat{\mathbb{Q}}_{\hat{p}}$.*

Proof. Let the metric on $\mathbb{Q}_p$ and $\hat{\mathbb{Q}}_{\hat{p}}$ be defined by $|\ |_p$ and $|\ |_{\hat{p}}$. Let $(\hat{x}_n) + M_{\hat{p}} \in \hat{\mathbb{Q}}_{\hat{p}}$ and $\hat{r} \in \hat{\mathbb{R}}^+$. Since the image under the mapping $f_i : \mathbb{Q} \to \mathbb{Q}_{p_i}$, $x \mapsto (x) + M_{p_i}$, is dense in $\mathbb{Q}_{p_i}$ for all $i \in E$, there exists $(y_i) + M_{p_i} \in B((x_{ni}) + M_{p_i}, r_i)$ for all $i \in E$. Then we get

$$\begin{aligned} d((x_{ni}) + M_{p_i}, (y_i) + M_{p_i}) &= |(x_{ni} - y_i) + M_{p_i}|_{p_i} \\ &= \lim_{n\to\infty} |(x_{ni} - y_i)|_{p_i} < r_i \quad \forall i \in E. \end{aligned}$$

Thus $|(\hat{x}_n - \hat{y}) + M_{\hat{p}}|_{\hat{p}} = \lim_{n\to\infty} |(\hat{x}_n - \hat{y})|_{\hat{p}} < \hat{r}$. Let $\hat{y} \in \mathbb{Q}$ such that $(\hat{y})_i = y_i$ $\forall i \in E$. Then $(\hat{y}) + M_{\hat{p}} \in B((\hat{x}_n) + M_{\hat{p}}, \hat{r})$. Thus $f(\hat{\mathbb{Q}})$ is dense in $\hat{\mathbb{Q}}_{\hat{p}}$. □

Note that for any prime $p, \mathbb{Z}_p = \left\{(x_n) + M_p \in \mathbb{Q}_p \mid |(x_n) + M_p|_p \leq 1\right\}$ is a commutative subring of $\mathbb{Q}_p$ (see [29]).

Assume that $\hat{\mathbb{Z}}_{\hat{p}} = \left\{(\hat{x}_n) + M_{\hat{p}} \in \hat{\mathbb{Q}}_{\hat{p}} \mid |(\hat{x}_n) + M_{\hat{p}}|_{\hat{p}} \leq \tilde{1}\right\}$. Then it is clear that $\hat{\mathbb{Z}}_{\hat{p}}$ is a commutative subring of $\hat{\mathbb{Q}}_{\hat{p}}$ with identity element $(\tilde{1}) + M_{\hat{p}}$.

Definition 5.10. $\hat{\mathbb{Z}}_{\hat{p}}$ is called the *ring of $\hat{p}$-adic soft integers*.

Proposition 5.11. *The followings hold:*

(1) $\hat{\mathbb{Z}}$ *is a subring of* $\hat{\mathbb{Z}}_{\hat{p}}$,

(2) $\hat{\mathbb{Q}} \cap \hat{\mathbb{Z}}_{\hat{p}} = \left\{ \left(\frac{\hat{a}}{\hat{b}} \right) + M_{\hat{p}} \mid \hat{p} \nmid \hat{b} \right\}$,

(3) $\hat{\mathbb{Z}}_{\hat{p}} \simeq \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_s}$,

(4) $I_{\hat{p}} = \left\{ (\hat{x}_n) + M_{\hat{p}} \in \hat{\mathbb{Q}}_{\hat{p}} \mid |(\hat{x}_n) + M_{\hat{p}}|_{\hat{p}} < \tilde{1} \right\}$ *is a principle ideal of* $\hat{\mathbb{Z}}_{\hat{p}}$ *generated by* $(\hat{p}) + M_{\hat{p}}$.

Proof. (1) Since the mapping $\hat{\mathbb{Z}} \to \hat{\mathbb{Z}}_{\hat{p}}$, $\hat{x} \mapsto (\hat{x}) + M_{\hat{p}}$, is a monomorphism of rings, we have the required.

(2) Note first that $\hat{\mathbb{Q}}$ is a subring of $\hat{\mathbb{Q}}_{\hat{p}}$ and $\mathbb{Q}$ is a subfield of $\mathbb{Q}_p$ for any prime p. Now, we have

$$\begin{aligned} (\hat{x}_n) + M_{\hat{p}} \in \hat{\mathbb{Q}} \cap \hat{\mathbb{Z}}_{\hat{p}} &\iff |(\hat{x}) + M_{\hat{p}}|_{\hat{p}} \leq \tilde{1} \\ &\iff |(x_i) + M_{p_i}|_{p_i} \leq 1, \quad i = 1, 2, \cdots, s \\ &\iff (x_i) + M_{p_i} \in \mathbb{Q} \cap \mathbb{Z}_{p_i} \ \ (= \{\frac{a_i}{b_i} + M_{p_i} \in \mathbb{Q}_{p_i} \mid p_i \nmid b_i\}), \quad i = 1, 2, \cdots, s \\ &\iff x_i = \frac{a_i}{b_i}, \ p_i \nmid b_i, \quad i = 1, 2, \cdots, s. \end{aligned}$$

Then $\hat{\mathbb{Q}} \cap \hat{\mathbb{Z}}_{\hat{p}} = \left\{ \left(\frac{\hat{a}}{\hat{b}} \right) + M_{\hat{p}} \mid \hat{p} \nmid \hat{b} \right\}$.

(3) Since

$$\begin{aligned} \hat{\mathbb{Z}}_{\hat{p}} &\to \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_s} \\ (\hat{x}_n) &\mapsto ((x_{n1}) + M_{p_1}, (x_{n2}) + M_{p_2}, \cdots, (x_{ns}) + M_{p_s}) \end{aligned}$$

is an isomorphism of rings, we have the required.

(4) $I_{p_i} = \left\{ (x_{ni}) + M_{p_i} \in \mathbb{Q}_{p_i} \mid |(x_{ni}) + M_{p_i}|_{p_i} < 1 \right\}$ is a principle ideal of $\mathbb{Z}_{p_i}$ generated by $(p_i) + M_{p_i}$. Then it follows from (3) that $I_{\hat{p}} \simeq I_{p_1} \times I_{p_2} \times \cdots \times I_{p_s}$. Thus $I_{\hat{p}}$ is a principle ideal of $\hat{\mathbb{Z}}_{\hat{p}}$ generated by $(\hat{p}) + M_{\hat{p}}$. □

Proposition 5.12. *The image of* $\hat{\mathbb{Z}}$ *under the mapping*

$$\begin{aligned} \hat{\mathbb{Z}} &\to \hat{\mathbb{Z}}_{\hat{p}} \\ \hat{x} &\mapsto (\hat{x}) + M_{\hat{p}} \end{aligned}$$

is dense in $\hat{\mathbb{Z}}_{\hat{p}}$.

Proof. Similar to proof of Proposition 5.9. □

6. Conclusions

We have produced the concept of the soft prime integers; also, the fundamental theorem of soft arithmetic has been stated and proved such that for all nonzero soft integer there exist a unique soft factorization of soft prime integers. Moreover, some of the fundamental theorems like division theorem extended to soft integer numbers. We have defined $\hat{p}$-adic soft absolute value over rings like the rings of soft real numbers and soft rational numbers. We have investigated the p-adic numbers in the aspect of the soft theory to introduce the ring of soft $\hat{p}$-adic numbers and study some of their properties.

Compliance with Ethical Standards

Conflict of Interest: The authors declare that they have no conflict of interest.

This article does not contain any studies with human participants performed by any of the authors.

References

[1] D. Molodtsov, Soft set theory–first results, Comput. Math. Appl. 37 (4) (1999) 19–31.
[2] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[3] W. L. Gau and D. J. Buehrer, Vague sets, IEEE Transactions on Systems, Man and Cybernetics 23 (2) (1993) 610–614.
[4] Z. Pawlak, Rough sets, Internat. J. Comput. Inform. Sci. 11 (5) (1982) 341–356.
[5] D. Meng, X. Zhang and K. Qin, Soft rough fuzzy sets and soft fuzzy rough sets, Comput. Math. Appl. 62 (12) (2011) 4635 – 4645.
[6] E. Türkmen and A. Pancar, On some new operations in soft module theory, Neural. Comput. Appl. 22 (6) (2013) 1233–1237.
[7] H. Aktaş and N. Çağman, Soft decision making methods based on fuzzy sets and soft sets, J. Intell. Fuzzy Syst. 30 (5) (2016) 2797–2803.
[8] F. Smarandache, Extension of soft set to hypersoft set, and then to plithogenic hypersoft set, Neutrosophic Sets Syst. 22 (2018) 168–170.
[9] M. E. El-Shafei and T. M. Al-shami, Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem, Comput. Appl. Math. (2020) 39:138.
[10] T. M. Al-shami and M. E. El-Shafei, Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone, Soft Comput. 24 (7) (2020) 5377–5387.
[11] M. E. El-Shafei, M. Abo-Elhamayel and T. M. Al-shami, Partial soft separation axioms and soft compact spaces, Filomat 32 (13) (2018) 4755–4771.
[12] T. M. Al-shami and L. D. Kočinac, The equivalence between the enriched and extended soft topologies, Appl. Comput. Math. 18 (2) (2019) 149–162.
[13] T. M. Al-shami, M. E. El-Shafei and M. Abo-Elhamayel, On soft topological ordered spaces, J. King Saud Univ. Sci. 31 (4) (2019) 556–566
[14] T. M. Al-shami, L. D. Kočinac and B. A. Asaad, Sum of soft topological spaces, Mathematics 2020,8,990.
[15] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Comput. Math. Appl. 59 (11) (2010) 3458–3463.
[16] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (7) (2011) 1786–1799.
[17] A. Aygünoğlu and H. Aygün, Some notes on soft topological spaces, Neural. Comput. Appl. 21 (1) (2012) 113–119.
[18] S. Nazmul and S. K. Samanta, Neighbourhood properties of soft topological spaces, Ann. Fuzzy Math. Inform. 6 (1) (2013) 1–15.
[19] A. Kandil and O. A. Tantawy, S. A. El-Sheikh and S. A. Hazza, Some types of pairwise soft sets and the associated soft topologies, J. Intell. Fuzzy Syst. 32 (1) (2017) 1007–1018.

[20] M. K. Tahat, F. Sidky and M. Abo-Elhamayel, Soft topological soft groups and soft rings, Soft Comput. 22 (21) (2018) 7143–7156.

[21] F. Sidky, M. E. El-Shafei and M. K. Tahat, Soft topological soft modules, Ann. Fuzzy Math. Inform. 20 (3) (2020) 257–272.

[22] M. K. Tahat, F. Sidky and M. Abo-Elhamayel, Soft topological rings. J. King Saud Univ. Sci. 31 (4) (2019) 1127–1136.

[23] S. Das and S. K. Samanta, Soft real sets, soft real numbers and their properties, J. Fuzzy Math. 20 (3) (2012) 551–576.

[24] S. Das and S. K. Samanta, On soft metric spaces, J. Fuzzy Math. 21 (3) (2013) 707–734.

[25] S. Das and S. K. Samanta, Soft metric, Ann. Fuzzy Math. Inform. 6 (1) (2013) 77–94.

[26] S. Das and S. K. Samanta, On soft complex sets and soft complex numbers, J. Fuzzy Math. 21 (1) (2013) 195–216.

[27] R. Thakur and S. K. Samanta, A study on some fundamental properties of continuity and differentiability of functions of soft real numbers, Adv. Fuzzy Syst. 2018 (2018) 8 pages.

[28] K. Hensel, Über eine neue begründung der theorie der algebraischen zahlen. Jahresbericht der Deutschen Mathematiker-Vereinigung 6 (1897) 83–88.

[29] F. Q. Gouvêa, p-adic numbers. In: p-adic Numbers, (1997) 43–85, Springer.

[30] I. M. Isaacs, Algebra: a graduate course, vol 100, American Mathematical Soc. 1994

[31] P. K. Maji, R. Biswas and A.R. Roy, Soft set theory, Comput. . Appl. 45 (4) (2003) 555–562.

[32] K. V. Babitha and J. J. Sunil, Soft set relations and functions, Comput. Math. Appl. 60 (7) (2010) 1840–1849.

[33] R. Hassanand and O.Tantawy, Soft real analysis, Journal of Progressive Research in Mathematics 8 (1) (2016) 1207–1219.

Fawzan Sidky (fawzan1@gmail.com)
Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

M. E. El-Shafei (meshafei@hotmail.com)
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt

M. K. Tahat (just.tahat@gmail.com)
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt