Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 21, No. 3, (June 2021) pp. 279–293
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.21.3.279

@FMI

Applications of infinite matrices and λ-convergence for fuzzy sequence spaces

Kuldip Raj, Ayhan Esi, Sonali Sharma

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 21, No. 3, June 2021

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 21, No. 3, (June 2021) pp. 279–293
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.21.3.279

@FMI

Applications of infinite matrices and λ-convergence for fuzzy sequence spaces

Kuldip Raj, Ayhan Esi, Sonali Sharma

Received 20 December 2020; Revised 21 February 2021; Accepted 8 March 2021

Abstract. In the present paper we study some applications of infinite matrices and λ-convergence of order α to introduce some Ideal convergent sequence spaces of fuzzy numbers by means of Orlicz function. We make an effort to study some algebraic and topological properties of these spaces. We also study some interesting inclusions relation between these spaces. Finally, we have prove that these spaces are normal as well as monotone and convergence free. We shall prove these results with the help of examples.

2020 AMS Classification: 40A05, 40A30

Keywords: Ideal, $I-$convergent, Infinite matrix, Orlicz function, Fuzzy number, λ-convergence of order α, Difference sequence space.

Corresponding Author: Ayhan Esi (aesi23@hotmail.com)

1. Introduction

An ideal convergence is a generalization of statistical convergence. The ideal convergence plays a vital role not only in pure mathematics but also in other branches of science especially in computer science, information theory, biological science, dynamical systems, geographic information systems, and motion planning in robotics. Kostyrko et al. [1] introduced the notion of $I-$convergence based on the structure of admissible ideal I of subsets of natural number $\mathbb{N}$. For more details about ideal convergence see [1, 2, 3, 4, 5, 6].

Let X be a non empty set. Then a family of sets $I \subseteq 2^X$ (Power set of X) is said to be an *ideal*, if I is additive, that is, $A,\ B \in I \Rightarrow A \cup B \in I$ and $A \in I,\ B \subseteq A \Rightarrow B \in I$.

A non empty family of sets $F \subseteq 2^X$ is said to be *filter* on X, if and only if $\varnothing \notin F$ for $A,\ B \in F$, we have $A \cap B \in F$ and for each $A \in F$ and $A \subseteq B$ implies $B \in F$. An ideal $I \subseteq 2^X$ is called *non-trivial*, if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called *admissible*, if $\{\{x\} : x \in X\} \subseteq I$. A non-trivial ideal is *maximal*, if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset.

The theory of sequences of fuzzy numbers was first introduced by Matloka [7]. He introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties and showed that every convergent sequence of fuzzy numbers is bounded. In [8], Nanda studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Nuray and Savaş [9] defined the concepts of statistical convergence and statistically Cauchy for sequence of fuzzy numbers. They proved that a sequence of fuzzy numbers is statistically convergent if and only if it is statistically Cauchy whereas Savaş and Mursaleen [10] studied some equivalent alternative conditions for a sequence of fuzzy numbers to be statistically Cauchy. Recently, Qiu et al. [11, 12] studied algebraic and topological properties of the quotient space of fuzzy number based on Mareš equivalence relations. Its is shown that every fuzzy number has only one Mareš core and equivalence fuzzy number have the same Mareš cores. They have also introduced a new concept of convergence under which the quotient space is complete. As an application, it is shown that if we identify every fuzzy number with corresponding equivalence class, there would be more differentiable fuzzy number than what is found in the literature. In [13], Qiu et al. studied symmetric fuzzy numbers and additive equivalence of fuzzy numbers. The fuzzy complex analysis studied in [14]. For more details about fuzzy convergence see [15, 16, 17, 18].

Given any interval A, we shall denote its end points by $\underline{A}$, $\overline{A}$ and by D the set of all closed bounded intervals on $\mathbb{R}$, that is, $D = \{A \subset \mathbb{R} : A = [\underline{A}, \overline{A}]\}$. For $A,\ B \in D$, we define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$ and $d(A, B) = \max\{[\underline{A} - \underline{B}],\ [\overline{A} - \overline{B}]\}$. It is easy to see that d defines a Hausdorff metric on D and (D, d) is a complete metric space. Also, $\leq$ is a partial order on D.

A *fuzzy number* is a fuzzy set on the real axis, i.e., a mapping $X : \mathbb{R} \to [0, 1]$ which satisfies the following four conditions:
(i) X is normal, i.e., there exist an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$,
(ii) X is fuzzy convex, i.e., for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$,

$$X(\lambda x + (1 - \lambda)y) \geq \min[X(x), X(y)],$$

(iii) X is upper semi-continuous,
(iv) the closure of $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by $[X]^0$, is compact.

The properties (i) to (iv) imply that for each $\alpha \in [0, 1]$, the $\alpha-$level set

$$X^\alpha = \{x \in \mathbb{R} : X(x) > \alpha\} = [\underline{X}^\alpha,\ \overline{X}^\alpha]$$

is a nonempty compact convex subset of $\mathbb{R}$. Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by $\overline{d}(x, y) = \sup_{\alpha \in [0,1]} d(X^\alpha, Y^\alpha)$. Puri and Ralescu [19] proved that $(L(\mathbb{R}),\ \overline{d})$ is a complete metric space. For $X, Y \in L(\mathbb{R})$, we define $X \leq Y$ if and only if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0, 1]$, we say that $X < Y$, if $X \leq Y$ and there exist $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$. The fuzzy numbers X and Y are said to be *incomparable*, if neither $X \leq Y$ nor $Y \leq X$. For any $X, Y, Z \in L(\mathbb{R})$, the linear structure of $L(\mathbb{R})$ induced addition $X + Y$ and scalar multiplication λX, $\lambda \in \mathbb{R}$, in terms of $\alpha-$level sets, by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda[X]^\alpha$$

for each $\alpha \in [0, 1]$.

The notion of difference sequence spaces was introduced by Kızmaz [20], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [25] by introducing the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Later the concept have been studied by Bektaş et al. [36] and Et et al. [21]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [24] who studied the spaces $\ell_\infty(\Delta_\nu)$, $c(\Delta_\nu)$ and $c_0(\Delta_\nu)$. Recently, Esi et al. [22] and Tripathy et al. [26] have introduced a new type of generalized difference operators and unified those as follows.

Let ν, m be non-negative integers. Then for Z a given sequence space, we have

$$Z(\Delta^m_\nu) = \{x = (x_k) \in w : (\Delta^m_\nu x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞,
where $\Delta^m_\nu x = (\Delta^m_\nu x_k) = (\Delta^{m-1}_\nu x_k - \Delta^{m-1}_\nu x_{k+1})$ and $\Delta^0_\nu x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta^m_\nu x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+\nu i}.$$

Taking $\nu = 1$, we get the spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Çolak [25]. Taking $m = \nu = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [20].

An *Orlicz function* M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [27] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an *Orlicz sequence space*. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [27] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a *Musielak-Orlicz function* (See [28, 29]). For more details about sequence spaces (See [30, 31, 32, 33]) and references therein.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that *A defines a matrix mapping from X into Y*, if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ is in Y, where

$$A_n(x) = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

converges for each $n \in \mathbb{N}$. By (X, Y) we denote the class of all matrices A such that $A : X \to Y$.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

The approach for a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors (See [34]).

Definition 1.1. A sequence $X = (X_k)$ of fuzzy numbers is said to be $I-$*convergent* to a fuzzy number X_0, if for every $\varepsilon > 0$ such that

$$A = \{k \in \mathbb{N} : \overline{d}(X_k, X_0) \geq \varepsilon\} \in I.$$

The fuzzy number X_0 is called $I-$*limit* of the sequence (X_k) of fuzzy numbers and we write $I - \lim X_k = X_0$.

Definition 1.2. A sequence $X = (X_k)$ of fuzzy numbers is said to be $I-$*bounded*, if there exists $M > 0$ such that

$$\{k \in \mathbb{N} : \overline{d}(X_k, \overline{0}) > M\} \in I.$$

Let E_F be denote the sequence space of fuzzy numbers.

Definition 1.3. A sequence space E_F is said to be *solid*(or *normal*), if $(Y_k) \in E_F$ whenever $(X_k) \in E_F$ and $\overline{d}(Y_k, \overline{0}) \leq \overline{d}(X_k, \overline{0})$ for all $k \in \mathbb{N}$.

Example 1.4. If we take $I = I_F = \{A \subseteq \mathbb{N} : A \text{ is a finite subset }\}$. Then I_F is a nontrival admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the usual convergence.

Example 1.5. If we take $I = I_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$. where $\delta(A)$ denote the asymptotic density of the set A. Then I_δ is a non-trival admissible ideal of $\mathbb{N}$ and the corresponding convergence coincide with the statistical convergence.

Lemma 1.6 ([35])**.** *If $\overline{d}$ is a translation invariant metric. Then*
(1) $\overline{d}(X + Y, \overline{0}) \leq \overline{d}(X, \overline{0}) + \overline{d}(Y, \overline{0})$,
(2) $\overline{d}(\lambda X, \overline{0}) \leq |\lambda| \overline{d}(X, \overline{0})$, $|\lambda| > 1$.

Lemma 1.7. *A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [37]).*

Lemma 1.8 ([1])**.** *If $I \subset 2^{\mathbb{N}}$ is a maximal ideal then for each $A \in \mathbb{N}$, we have either $A \in I$ or $\mathbb{N} \setminus A \in I$.*

Throughout the paper w^F denote the class of all fuzzy real-valued sequences. Also $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and set of real numbers respectively.

Let I be an admissible ideal of $\mathbb{N}$. Suppose $p = (p_k)$ is a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of strictly positive real numbers for all

$k \in \mathbb{N}$, $A = (a_{nk})$ an infinite matrix, $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Let $\alpha \in [0, 1)$ be any real number, let $\lambda = (\lambda_n)$ be an increasing sequence of positive real numbers such that $\lambda_{n+1} = \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$. We define the following new sequence spaces in this paper:

$$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m] = \left\{(x_k) \in w^F : \forall \varepsilon > 0, \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I, \text{ for some } \rho > 0 \text{ and } X_0 \in L(\mathbb{R})\right\},$$

$$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0 = \left\{(x_k) \in w^F : \forall \varepsilon > 0, \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I, \text{ for some } \rho > 0\right\},$$

$$w_{\lambda^\alpha}^{F}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = \left\{(x_k) \in w^F : \sup_n \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\}$$

and

$$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = \left\{(x_k) \in w^F : \exists\, K > 0 \text{ such that } \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} \geq K\right\} \in I, \text{ for some } \rho > 0\right\},$$

where $I_n = [n - \lambda_n + 1, n]$.

Example 1.9. Let $X_k(t) = \overline{1}$ for $k = 2^q, \ \ q = 1, 2, 3, \ldots$.

$$X_k(t) = \begin{cases} \frac{k}{3}(t-2)+1 & \text{for } t \in \left[\frac{2k-3}{2}, 2\right] \\ -\frac{k}{3}(t-2)+1 & \text{for } \ t \in \left[2, \frac{2k-3}{2}\right] \\ 0 & \text{otherwise.} \end{cases}$$

If $m = v = 1$, then $\Delta_v^m X_k = \Delta X_k$

$$[\Delta_v^m X_k]^\alpha = \begin{cases} -1 - \frac{3}{k}(1-\alpha) & k = 2^q \\ 1 - \frac{3}{k}(1-\alpha) & k \neq 2^q. \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x$, $u = (u_k) = 1$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$ and $\lambda_n^\alpha = (1, 2, 3, \ldots)$, we have

$$\sup_n \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} < \infty.$$

Then $(X_k) \in w^F_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ but (X_k) is not an Ideal convergent.

Let us consider a few special cases of the above sequence spaces:

(i) If $M_k(x) = x$ for all $k \in \mathbb{N}$, then we have

$$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m] = w^{I(F)}_{\lambda^\alpha}[A, p, u, \Delta_v^m],$$
$$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_0 = w^{I(F)}_{\lambda^\alpha}[A, p, u, \Delta_v^m]_0,$$
$$w^{F}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = w^{F}_{\lambda^\alpha}[A, p, u, \Delta_v^m]_\infty,$$
$$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = w^{I(F)}_{\lambda^\alpha}[A, p, u, \Delta_v^m]_\infty.$$

(ii) If $p = (p_k) = 1$, for all k then we have

$$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m] = w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, u, \Delta_v^m],$$
$$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_0 = w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, u, \Delta_v^m]_0,$$
$$w^{F}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = w^{F}_{\lambda^\alpha}[A, \mathcal{M}, u, \Delta_v^m]_\infty,$$
$$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty = w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, u, \Delta_v^m]_\infty.$$

(iii) If we take $A = (C, 1)$, i.e., the Cesàro matrix, then the above classes of sequences are denoted by $w^{I(F)}_{\lambda^\alpha}[w, \mathcal{M}, p, u, \Delta_v^m]$, $w^{I(F)}_{\lambda^\alpha}[w, \mathcal{M}, p, u, \Delta_v^m]_0$, $w^{F}_{\lambda^\alpha}[w, \mathcal{M}, p, u, \Delta_v^m]_\infty$ and $w^{I(F)}_{\lambda^\alpha}[w, \mathcal{M}, p, u, \Delta_v^m]_\infty$, respectively.

(iv) If $I = I_F$, then we obtain

$w^{F}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m] =$

$$\left\{(x_k) \in w^F : \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0 \right.$$
$$\left. \text{and } X_0 \in L(\mathbb{R})\right\},$$

$w^{F}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_0 =$

$$\left\{(x_k) \in w^F : \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} = 0, \text{ for some } \rho > 0\right\},$$

$w^{F}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty =$

$$\left\{(x_k) \in w^F : \lim_{n,r\to\infty} \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0\right\}.$$

(v) If $I = I_\delta$ is an admissible ideal of $\mathbb{N}$, then we have

$w^{I(F)}_{\lambda^\alpha}[A, \mathcal{M}, p, u, \Delta_v^m] =$

$$\left\{(x_k) \in w^F : \forall\varepsilon > 0, \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I_\delta,\right.$$

$$\text{for some } \rho > 0 \text{ and } X_0 \in L(\mathbb{R})\Bigg\},$$

$$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0 =$$

$$\Bigg\{(x_k) \in w^F : \forall \varepsilon > 0, \Bigg\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\Bigg\} \in I_\delta,$$

$$\text{for some } \rho > 0\Bigg\},$$

and

$$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty =$$

$$\Bigg\{(x_k) \in w^F : \exists\, K > 0 \text{ such that } \Bigg\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k}$$

$$\geq K\Bigg\} \in I_\delta, \quad \text{for some } \rho > 0\Bigg\}.$$

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$ and let $D = \max\left\{1, 2^{H-1}\right\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1.3}$$

Also, $|a_k|^{p_k} \leq \max\left\{1, |a|^H\right\}$ for all $a \in \mathbb{C}$.

The main aim of this paper is to introduced and study some $\lambda-$convergent sequence spaces of fuzzy numbers by using the relation of an infinite matrix and a sequence of Orlicz functions. We also make an effort to study some properties like linearity, paranorm, solidity and some interesting inclusion relations between the above defined spaces.

2. Main Results

In this section, we study some topological properties and some inclusion relations between the sequence spaces which we have defined above.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the spaces $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$, $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ and $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ are linear spaces over the complex field $\mathbb{C}$.*

Proof. We shall prove the result for the space $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_n^m]_0$ only and others can be proved in the similar way. Let $X = (X_k)$ and $Y = (Y_k)$ be two elements in

$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_n^m]_0$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$A_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I.$$

Let β and γ be two scalars. Then by using the inequality (1.3) and continuity of the function $\mathcal{M} = (M_k)$, we have

$$\begin{aligned}
&\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(\beta u_k \Delta_v^m X_k + \gamma u_k \Delta_v^m Y_k, \overline{0})}{|\beta|\rho_1 + |\gamma|\rho_2} \right) \right]^{p_k} \\
&\leq D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[\frac{|\beta|}{|\beta|\rho_1 + |\gamma|\rho_2} M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \\
&+ \ D \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[\frac{|\gamma|}{|\beta|\rho_1 + |\gamma|\rho_2} M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k} \\
&\leq DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \\
&+ \ DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k},
\end{aligned}$$

where $K = \max \left\{ 1, \left(\frac{|\beta|}{|\beta|\rho_1 + |\gamma|\rho_2} \right)^H, \left(\frac{|\gamma|}{|\beta|\rho_1 + |\gamma|\rho_2} \right)^H \right\}$.

From the above relation we obtain the following:

$$\begin{aligned}
&\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(\beta u_k \Delta_v^m X_k + \gamma u_k \Delta_v^m Y_k, \overline{0})}{|\beta|\rho_1 + |\gamma|\rho_2} \right) \right]^{p_k} \geq \varepsilon \right\} \subseteq \\
&\left\{ n \in \mathbb{N} : DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, \overline{0})}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \\
&\cup \left\{ n \in \mathbb{N} : DK \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m Y_k, \overline{0})}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \right\} \in I.
\end{aligned}$$

This completes the proof. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then the spaces $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$, $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$*

and $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty$ *are paranormed spaces with the paranorm* g_Δ *defined by*

$$g_\Delta(X) = \inf\left\{(\rho)^{\frac{p_n}{H}} : \left(\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1, \right.$$

$$\left. \text{for some } \rho > 0,\ n = 1, 2, \ldots.\ r \in \mathbb{N}\right\},$$

where $H = \max\{1, \sup_k p_k\}$.

Proof. Clearly, $g_\Delta(-X) = g_\Delta(X)$ and $g_\Delta(\theta) = 0$. Let $X = (X_k)$ and $Y = (Y_k)$ be two elements in $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$. Then for every $\rho > 0$ we write

$$A_1 = \left\{\rho > 0 : \left(\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}$$

and

$$A_2 = \left\{\rho > 0 : \left(\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m Y_k, \overline{0})}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}.$$

Let $\rho_1 \in A_1$ and $\rho_2 \in A_2$. If $\rho = \rho_1 + \rho_2$, then we get the following

$$\begin{aligned}
&\left(\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m (X_k + Y_k), \overline{0}))}{\rho}\right)\right]\right) \\
&\qquad \leq \frac{\rho_1}{\rho_1+\rho_2}\left(\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho_1}\right)\right]\right) \\
&\qquad + \frac{\rho_2}{\rho_1+\rho_2}\left(\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m Y_k, \overline{0})}{\rho_2}\right)\right]\right).
\end{aligned}$$

Thus, we have

$$\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m (X_k + Y_k), \overline{0})}{\rho}\right)\right]^{p_k} \leq 1$$

and

$$\begin{aligned}
g_\Delta(X+Y) &= \inf\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \rho_1 \in A_1, \rho_2 \in A_2\} \\
&\leq \inf\{(\rho_1)^{\frac{p_n}{H}} : \rho_1 \in A_1\} + \inf\{(\rho_2)^{\frac{p_n}{H}} : \rho_2 \in A_2\} \\
&= g_\Delta(X) + g_\Delta(Y).
\end{aligned}$$

Let $t_k^m \to t$, where $t_k^m,\ t \in \mathbb{C}$, and let $g_\Delta(X_k^m - X_k) \to 0$ as $m \to \infty$. To prove that $g_\Delta(t_k^m X_k^m - tX_k) \to 0$ as $m \to \infty$. Let $t_k \to t$, where $t_k,\ t \in \mathbb{C}$, and $g_\Delta(X_k^m - X_k) \to 0$ as $m \to \infty$. We have

$$A_3 = \left\{\rho_k > 0 : \frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, \overline{0})}{\rho_k}\right)\right]^{p_k} \leq 1\right\}$$

and

$$A_4 = \left\{\rho'_k > 0 : \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m Y_k, \overline{0})}{\rho'_k}\right)\right]^{p_k} \leq 1\right\}.$$

If $\rho_k \in A_3$ and $\rho'_k \in A_4$ then by inequality (1.3) and continuity of the function $\mathcal{M} = (M_k)$, we have that

$$\begin{aligned} &M_k\left(\frac{\overline{d}(u_k\Delta_v^m(t^m X_k^m - tX, \overline{0}))}{|t^m - t|\rho_k + |t|\rho'_k}\right) \\ &\leq M_k\left(\frac{\overline{d}(u_k\Delta_v^m(t^m X_k^m - tX_k), \overline{0})}{|t^m - t|\rho_k + |t|\rho'_k}\right) + M_k\left(\frac{\overline{d}(u_k\Delta_v^m(tX_k - tX, \overline{0}))}{|t^m - t|\rho_k + |t|\rho'_k}\right) \\ &\leq \frac{|t^m - t|\rho_k}{|t^m - t|\rho_k + |t|\rho'_k} M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k^m, \overline{0})}{\rho_k}\right) \\ &+ \frac{|t|\rho'_k}{|t^m - t|\rho_k + |t|\rho'_k} M_k\left(\frac{\overline{d}(u_k\Delta_v^m(X_k^m - X_k), \overline{0})}{\rho'_k}\right). \end{aligned}$$

From the above inequality it follows that

$$\frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m(t^m X_k^m - tX), \overline{0})}{|t^m - t|\rho_k + |t|\rho'_k}\right)\right]^{p_k} \leq 1$$

and consequently,

$$\begin{aligned} g_\Delta(t_k^m X_k + tX) &= \inf\{(|t_k^m - t|\rho_k + |t|\rho'_k\})^{\frac{p_n}{H}} : \rho_k \in A_3, \rho'_k \in A_4\} \\ &\leq |t_k^m - t|\rho_k^{\frac{p_n}{H}} \inf\{(\rho_k)^{\frac{p_n}{H}} : \rho_k \in A_3\} + |t|\rho'_k \inf\{(\rho'_k)^{\frac{p_n}{H}} : \rho'_k \in A_4\} \\ &\leq \max\{|t|, |t|^{\frac{p_n}{H}}\} g_\Delta(X_k^m - X_k). \end{aligned}$$

Note that $g_\Delta(X_k^m) \leq g_\Delta(X^m) + g_\Delta(X_k^m - X^m)$, for all $k \in \mathbb{N}$. Hence, by our assumption the right hand tends to 0 as $m \to \infty$. This completes the proof. □

Theorem 2.3. (1) *Let* $0 < \inf p_k \leq p_k \leq 1$. *Then*

$$\begin{aligned} w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m] &\subseteq w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, u, \Delta_v^m], \\ w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0 &\subseteq w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]_0. \end{aligned}$$

(2) *Let* $1 \leq p_k \leq \sup p_k < \infty$. *Then*

$$\begin{aligned} w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, u, \Delta_v^m] &\subseteq w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m], \\ w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]_0 &\subseteq w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0. \end{aligned}$$

Proof. (1) Let $X = (X_k)$ be an element in $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Since $0 < \inf p_k \leq p_k \leq 1$, we have

$$\frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\right)\right] \leq \frac{1}{\lambda_n^\alpha} \sum_{k\in I_n} a_{nk} \left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k, X_0)}{\rho}\right)\right]^{p_k}.$$

Then we get

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right] \geq \varepsilon \right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

The other part can be proved in a similar way.

(2) Let $X = (X_k)$ be an element in $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, u, \Delta_v^m]$. Since $1 \leq p_k \leq \sup p_k < \infty$, for each $0 < \varepsilon < 1$, there exists a positive integer n_0 such that

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right] \leq \varepsilon < 1 \text{ for all } n \geq n_0.$$

This implies that

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \leq \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right].$$

Then we have

$$\left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right] \geq \varepsilon \right\} \in I.$$

The other part can be proved in the similar way. This completes the proof. □

Theorem 2.4. *Let $X = (X_k)$ be a sequence of Fuzzy numbers, $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then*

$$w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0 \subset w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m] \subset w_{\lambda^\alpha}^{F}[A, \mathcal{M}, p, u, \Delta_v^m]_\infty.$$

Proof. The inclusion $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0 \subset w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is obvious. Let $X = (X_k) \in w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Then there is some fuzzy number X_0, such that

$$\frac{1}{\lambda_n^\alpha} \sum_{k \in I_n} a_{nk} \left[M_k \left(\frac{\overline{d}(u_k \Delta_v^m X_k, X_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon.$$

Now by inequality (1.3), we have

$$\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,\overline{0})}{\rho}\right)\right]^{p_k} \le D\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k} + D\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(X_0,\overline{0})}{\rho}\right)\right]^{p_k}.$$

This implies that $X=(X_k)\in w^F_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]_\infty$. This completes the proof. □

Theorem 2.5. *Let $\mathcal{M}=(M_k)$ and $\mathcal{S}=(S_k)$ be a sequence of Orlicz functions. Then*

$$w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]\cap w^{I(F)}_{\lambda^\alpha}[A,\mathcal{S},p,u,\Delta_v^m]\subset w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M}+\mathcal{S},p,u,\Delta_v^m].$$

Proof. Let $X=(X_k)\in w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]\cap w^{I(F)}_{\lambda^\alpha}[A,\mathcal{S},p,u,\Delta_v^m]$ using the inequality (1.3), we have

$$\begin{aligned}
&\tfrac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[(M_k+S_k)\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}\\
&\quad=\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)+\ S_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}\\
&\quad\le D\left\{\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}\right.\\
&\quad\left.+\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n}^{\infty} a_{nk}\left[S_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,X_0)}{\rho}\right)\right]^{p_k}\right\}.
\end{aligned}$$

Then $X=(X_k)\in w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M}+\mathcal{S},p,u,\Delta_v^m]$. This completes the proof. □

Theorem 2.6. *The sequence spaces $w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]_0$ and $w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]_\infty$ are normal as well as monotone.*

Proof. We give the proof of the theorem for $w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]_0$ only. Let $X=(X_k)\in w^{I(F)}_{\lambda^\alpha}[A,\mathcal{M},p,u,\Delta_v^m]_0$ and $Y=(Y_k)$ be such that $\overline{d}(Y_k,\overline{0})\le\overline{d}(X_k,\overline{0})$ for all $k\in\mathbb{N}$. Then for given $\varepsilon>0$, we have

$$B=\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m X_k,\overline{0})}{\rho}\right)\right]^{p_k}\ge\varepsilon\right\}\in I,$$

again the set

$$B_1=\left\{n\in\mathbb{N}:\frac{1}{\lambda_n^\alpha}\sum_{k\in I_n} a_{nk}\left[M_k\left(\frac{\overline{d}(u_k\Delta_v^m Y_k,\overline{0})}{\rho}\right)\right]^{p_k}\ge\varepsilon\right\}\subseteq B.$$

Then $B_1 \in I$ and thus $Y = (Y_k) \in w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$. So the space $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is normal. Also from the Lemma 1.8, it follows that $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is monotone. This completes the proof. □

Theorem 2.7. *If I is not maximal ideal then the space $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is neither normal nor monotone.*

Example 2.8. Let us consider a sequence of fuzzy numbers

$$X_k(t) = \begin{cases} \frac{1+t}{2} & -1 \le t \le 1 \\ \frac{3-t}{2} & 1 \le t \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

If $m = 0$, then $\Delta_v^m X_k = 1$. Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x$, $u = (u_k) = 1$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$ and $\lambda_n^\alpha = (1, 2, 3, ...)$. Then we have $(X_k) \in w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Since I is not maximal by Lemma 1.9, their exist a subset K of $\mathbb{N}$ such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let us define sequence $Y = (Y_k)$ by

$$Y_k = \begin{cases} X_k & k \in K \\ 0 & \text{otherwise.} \end{cases}$$

Then (Y_k) belongs to the canonical pre image of the k-step spaces of $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. But $Y_k \notin w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Thus $w_{\lambda^\alpha}^{I(F)}[A,\mathcal{M},p,u,\Delta_v^m]$ is not monotone. So by Lemma 1.8, $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not normal.

Theorem 2.9. *If I is neither maximal nor $I = I_F$ then the spaces $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ and $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ are not symmetric.*

Example 2.10. Let us consider a sequence of fuzzy numbers

$$X_k(t) = \begin{cases} 1+t & -1 \le t \le 0 \\ 1-t & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $m = 0$, then $\Delta_v^m X_k = 1$. Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x$, $u = (u_k) = 1$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$ and $\lambda_n^\alpha = (1, 2, 3, ...)$. Then for $k \in A \in I$ (an infinite set), $(X_k) \in w^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let us consider a sequence $Y = (Y_k)$, a rearrangement of the sequence space (X_k) is defined as

$$Y_k = \begin{cases} X_k & k \in K \\ 0 & \text{otherwise.} \end{cases}$$

Then $(Y_k) \in w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Thus $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not symmetric. Similarly, $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not symmetric.

Theorem 2.11. *The spaces $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ and $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ are not convergent free in general.*

Example 2.12. Let us consider a sequence of fuzzy numbers

$$X_k(t) = \begin{cases} \frac{1+t}{2} & -1 \le t \le 1 \\ \frac{3-t}{2} & 1 \le t \le 3 \\ 0 & \text{otherwise.} \end{cases}$$

If $m = 0$, then $\Delta_v^m X_k = 1$. Let $A = (C, 1)$, the Cesàro matrix, $\mathcal{M}(x) = x$, $u = (u_k) = 1$, $p = (p_k) = 1$, for all $k \in \mathbb{N}$ and $\lambda_n^\alpha = (1, 2, 3, ...)$. Then we have $(X_k) \in w^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. Let $Y_k(t) = \frac{1}{k}$ for all $k \in \mathbb{N}$. Then $(Y_k) \in w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$. But $X_k = 0$ does not imply $Y_k = 0$. Thus $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]$ is not convergent free. Similarly, $w_{\lambda^\alpha}^{I(F)}[A, \mathcal{M}, p, u, \Delta_v^m]_0$ is not convergent free.

References

[1] P. Kostyrko, T. Salat and W. Wilczynski, I-convergence, Real Analysis Exchange 26 (2000) 669–686.

[2] E. Savaş, Some $I-$convergent sequence spaces of fuzzy numbers defined by infinite matrix, Math. Comput. Appl. 18 (2013) 84–93.

[3] E. Savaş, Δ^m- strongly summable sequences spaces in 2-normed spaces defined by ideal convergence and an Orlicz function, Appl. Math. and Comput. 217 (2010) 271–276.

[4] S. A. Mohiuddine and B. Hazarika, Some Classes of ideal convergent sequences and generalized difference matrix operator, Filomat 31 (6) (2017) 1827–1834.

[5] A. Esi and B Hazarika, λ-ideal convergence in intuitionistic fuzzy 2-normed linear space, J. Intell. Fuzzy Syst. 24 (4) (2013) 725–732.

[6] A. Esi and B Hazarika, Lacunary summable sequence spaces of fuzzy numbers defined by ideal convergence and an Orlicz function, Afrika Matematika 25 (2) (2014) 331–343.

[7] M. Matloka, Sequences of fuzzy numbers, BUSEFAL 28 (1986) 28–37.

[8] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 123–126.

[9] F. Nuray and E. Savaş, Statistical convergence of fuzzy numbers, Math. Slovaca 45 (1995) 269–273.

[10] E. Savaş and Mursaleen, On statistically convergent double sequence of fuzzy numbers, Inform. Sci. 162 (2004) 183–192.

[11] D. Qiu, W. Zhang and C. Lu, On fuzzy differential equations in the quotient space of fuzzy number, Fuzzy Sets and Systems 295 (2016) 72–98.

[12] D. Qiu, C. Lu, W. Zhang and Y. Lan, Algebraic properties and topological properties of the quotient space of fuzzy numbers based on Mareš equivalence relation, Fuzzy Sets and Systems 245 (2014) 63–82.

[13] D. Qiu and W. Zhang, Symmetric fuzzy numbers and additive equivalence of fuzzy numbers, Soft Computing 17 (2013) 1471–1477.

[14] D. Qiu, L. Shu and Z. W. Mo, Notes on fuzzy complex analysis, Fuzzy Sets and Systems 160 (2009) 1578–1589.

[15] S. A. Mohiuddine, A. Asiri and B. Hazarika, Weighted statistical convergence through difference operator of sequences of fuzzy numbers with application to fuzzy approximation theorems, Int. J. Gen. Syst. 48 (5) (2019) 492–506.

[16] T. Yaying, B. Hazarika, S. A. Mohiuddine, M. Mursaleen and K. J. Ansari, Sequence spaces derived by the triple band generalized Fibonacci difference operator, Advances in Difference Equations (2020) 2020:639.

[17]] B. Hazarika, A. Alotaibi and S. A. Mohiuddine, Statistical convergence in measure for double sequences of fuzzy-valued functions, Soft Computing 24 (2020) 6613–6622.

[18] S. A. Mohiuddine, B. Hazarika and A. Alotaibi, On statistical convergence of double sequences of fuzzy valued functions, J. Intell. Fuzzy Syst. 32 (6) (2017) 4331–342.

[19] M. L. Puri and D. A. Ralescu, Differential of fuzzy numbers, J. Math. Anal. Appl. 91 (1983) 552–558.

[20] H. Kızmaz, On certain sequence spaces, Canad. Math. Bull. 24 (1981) 169–176.

[21] M. Et and A. Esi, On Kö the-Toeplitz duals of generalized difference sequence spaces, Bull. Malays. Math. Sci. Soc. 23 (2000) 25–32.

[22] A. Esi, B. C. Tripathy and B. Sarma, On some new type generalized difference sequence spaces, Math. Slovaca 57 (2007) 475–482.

[23] B. Hazarika and E. Savaş, Some I-convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz, Math. Comput. Modelling 54 (2011) 2986–2998.
[24] B. C. Tripathy and A. Esi, A new type of difference sequence spaces, Int. J. Sci. Tech. 1 (2006) 11–14.
[25] M. Et and R. Çolak, On generalized difference sequence spaces, Soochow. J. Math. 21 (1995) 377–386.
[26] B. C. Tripathy, A. Esi and B. Tripathy, On a new type of generalized difference Cesàro sequence spaces, Soochow J. Math. 31 (2005) 333–340.
[27] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 10 (1971) 379–390.
[28] L. Maligranda, Orlicz spaces and interpolation, Seminars in Mathematics 5, Polish Academy of Science 1989.
[29] J. Musielak, Orlicz spaces and modular spaces, Lecture notes in Mathematics 1034 1983.
[30] S. A. Mohiuddine, K. Raj and A. Alotaibi, Generalized spaces of double sequences for Orlicz functions and bounded regular matrices over n-normed spaces, J. Inequal. Appl. 2014 2014:332.
[31] K. Raj and A. Kilicman, On certain generalized paranormed spaces, J. Inequal. Appl. (2015) 2015: 37.
[32] K. Raj and C. Sharma, Applications of strongly convergent sequences to Fourier series by means of modulus functions, Acta Math. Hungar. 150 (2016) 396–411.
[33] K. Raj, S. Jamwal and S. K. Sharma, New classes of generalized sequence spaces defined by an Orlicz function, J. Comput. Anal. Appl. 15 (2013) 730–737.
[34] E. Savaş and M. Mursaleen, Matrix transformations in some sequence spaces, Istanb. Univ. Fen Fak. Mat. Derg. 52 (1993) 1–5.
[35] M. Mursaleen and M. Başarir, On some new sequence spaces of fuzzy numbers, Indian J. Pure Appl. Math. 34 (2003) 1351–1357.
[36] Ç. A. Bektaş, M. Et and R. Çolak, Generalized difference sequence spaces and their dual spaces, J. Math. Anal. Appl. 292 (2004) 423–432.
[37] P.K. Kamthan and M. Gupta, Sequence spaces and series, Marcel Dekker 1980.

Kuldip Raj (kuldipraj68@gmail.com)
Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India

Ayhan Esi (ayhan.esi@ozal.edu.tr)
Department of Basic Engineering Sciences, Engineering Faculty, Malatya Turgut Ozal University, Malatya, 44100, Turkey

Sonali Sharma (sonalisharma8082@gmail.com)
Department of Mathematics, Shri Mata Vaishno Devi University, Katra-182320, J&K, India