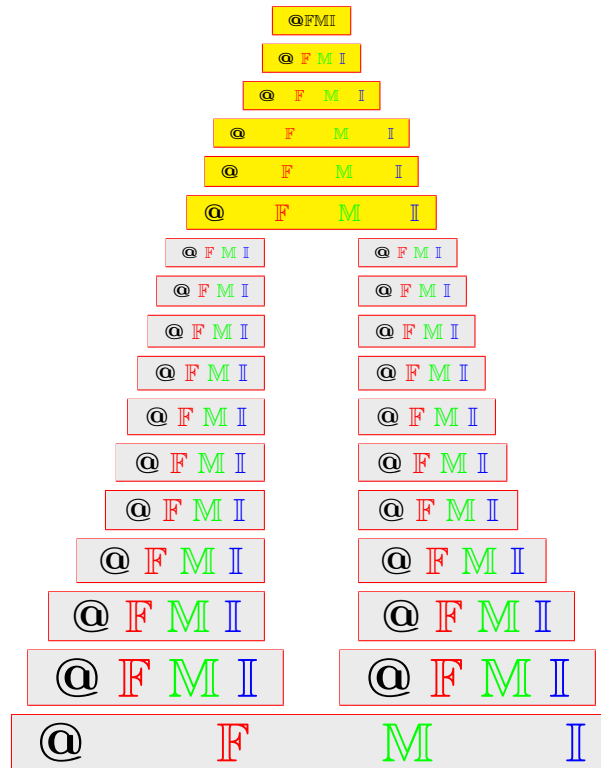


Fuzzy completeness and various operations in co-residuated lattices

JU-MOK OH, YONG CHAN KIM



Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 21, No. 3, June 2021

Fuzzy completeness and various operations in co-residuated lattices

JU-MOK OH, YONG CHAN KIM

Received 3 February 2021; Revised 28 February 2021; Accepted 11 March 2021

ABSTRACT. In this paper, we introduce the notions of distance functions, Alexandrov topologies and fuzzy complete lattices an extension of Zhang’s fuzzy completeness based on complete co-residuated lattices. Moreover, we investigate the properties of join (meet) preserving maps and various operations as extensions of Zadeh powerset operations. We give their examples.

2020 AMS Classification: 03E72, 54A40, 54B10

Keywords: Complete co-residuated lattices, Distance functions, Fuzzy completes, Alexandrov topologies.

Corresponding Author: Yong Chan Kim (yck@gwnu.ac.kr)

1. INTRODUCTION

As an algebraic structure for many valued logic, a complete residuated lattice is an important mathematical tool (See [1, 2, 3, 4, 5, 6, 7]). For an extension of classical rough sets introduced by Pawlak [8], many researchers [1, 2, 9, 10] developed L -lower and L -upper approximation operators in complete residuated lattices. By using the concepts of lower and upper approximation operators, fuzzy concepts, information systems and decision rules are investigated in complete residuated lattices (See [1, 2, 4, 9]).

Zhang and Fan [11] introduced the notion of fuzzy complete lattices using fuzzy partially order on a frame as generalizations of usual complete lattices. Based on residuated lattices as an extension of frame, Zhang [12], and Zhang and Xie [13] introduced the notions of partially order, join, meet and fuzzy completeness. Bělohlávek [1, 2] introduced concept lattices for information systems and fuzzy closure operators using Galois connections in a complete residuated lattice. Georgescu and Popescuc [14, 15] introduced fuzzy attribute-oriented formal concept lattices for information

systems and fuzzy closure operators using adjunctions in a complete residuated lattice. Rodabaugh [5] interpreted Zadeh’s powersets operators from fuzzy sets to fuzzy sets as adjoint functions. Join (meet) preserving maps, fuzzy equations, fuzzy rough sets, fuzzy concepts (See [1, 2, 9, 10, 16, 17, 18]), Dedekind-MacNeille completion (See [11, 13]) and powerset operators (See [12]) are easily handle by using adjoint and Galois connection.

On the other hand, Zheng and Wang [19] introduced a complete co-residuated lattice as the generalization of t-conorm. Junsheng and Qing [4] investigated $(\odot, \&)$ -generalized fuzzy rough set on $(L, \vee, \wedge, \odot, \&, 0, 1)$, where $(L, \vee, \wedge, \&, 0, 1)$ is a complete residuated lattice and $(L, \vee, \wedge, \odot, 0, 1)$ is complete co-residuated lattice in a sense of Zheng and Wang [19]. Kim and Ko et al. [20] studied preserving maps and approximation operators in complete co-residuated lattices.

In this paper, we study the distance functions instead of fuzzy partially ordered sets. We define Alexandrov topologies, join preserving maps as the sense of a distance function.

This paper is organized as follows. In Section 2, we recall the definitions of complete co-residuated lattices and distance spaces. Moreover, we give their examples and properties. In Section 3, using distance functions, Alexandrov topologies and fuzzy complete lattices as an extension of Zhang’s complete residuated lattices are introduced. In Theorem 3.6, we show that the Alexandrov topology induced by a distance function is a fuzzy complete lattice. In Theorem 3.8, we investigate the (dual) embedding maps and join(meet) preserving maps from sets to Alexandrov topologies. In Theorem 3.10, as extensions of Zadeh’s powersets operators from fuzzy sets to fuzzy sets, four types of operations are investigated. We give their examples.

2. PRELIMINARIES

Definition 2.1 ([4, 19, 20]). An algebra $(L, \wedge, \vee, \oplus, \perp, \top)$ is called a *complete co-residuated lattice*, if it satisfies the following conditions:

- (C1) $L = (L, \vee, \wedge, \perp, \top)$ is a complete lattice, where \perp is the bottom element and \top is the top element,
- (C2) $a = a \oplus \perp$, $a \oplus b = b \oplus a$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ for all $a, b, c \in L$,
- (C3) $(\bigwedge_{i \in \Gamma} a_i) \oplus b = \bigwedge_{i \in \Gamma} (a_i \oplus b)$.

Let (L, \leq, \oplus) be a complete co-residuated lattice. For each $x, y \in L$, we define

$$x \ominus y = \bigwedge \{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ iff $x \geq (z \ominus y)$.

For $\alpha \in L$, $A \in L^X$, we denote $(\alpha \ominus A), (\alpha \oplus A), \alpha_X \in L^X$ as $(\alpha \ominus A)(x) = \alpha \ominus A(x)$, $(\alpha \oplus A)(x) = \alpha \oplus A(x)$, $\alpha_X(x) = \alpha$.

Put $n(x) = \top \ominus x$. The condition $n(n(x)) = x$ for each $x \in L$ is called a *double negative law*.

Remark 2.2 ([20]). (1) An infinitely distributive lattice $(L, \leq, \vee, \wedge, \oplus = \vee, \perp, \top)$ is a complete co-residuated lattice. In particular, the unit interval $([0, 1], \leq, \vee, \wedge, \oplus =$

$\vee, 0, 1$) is a complete co-residuated lattice, where

$$x \ominus y = \bigwedge \{z \in L \mid y \vee z \geq x\} \\ = \begin{cases} 0, & \text{if } y \geq x, \\ x, & \text{if } y \not\geq x. \end{cases}$$

(2) The unit interval with a right-continuous t-conorm \oplus [6], $([0, 1], \leq, \oplus)$, is a complete co-residuated lattice.

(3) $([1, \infty], \leq, \vee, \oplus = \cdot, \wedge, 1, \infty)$ is a complete co-residuated lattice, where

$$x \ominus y = \bigwedge \{z \in [1, \infty] \mid yz \geq x\} \\ = \begin{cases} 1, & \text{if } y \geq x, \\ \frac{x}{y}, & \text{if } y \not\geq x. \end{cases}$$

$$\infty \cdot a = a \cdot \infty = \infty, \forall a \in [1, \infty], \infty \ominus \infty = 1.$$

(4) $([0, \infty], \leq, \vee, \oplus = +, \wedge, 0, \infty)$ is a complete co-residuated lattice where

$$y \ominus x = \bigwedge \{z \in [0, \infty] \mid x + z \geq y\} \\ = \bigwedge \{z \in [0, \infty] \mid z \geq -x + y\} = (y - x) \vee 0, \\ \infty + a = a + \infty = \infty, \forall a \in [0, \infty], \infty \ominus \infty = 0.$$

(5) $([0, 1], \leq, \vee, \oplus, \wedge, 0, 1)$ is a complete co-residuated lattice, where

$$x \oplus y = (x^p + y^p)^{\frac{1}{p}} \wedge 1, \quad 1 \leq p < \infty, \\ x \ominus y = \bigwedge \{z \in [0, 1] \mid (z^p + y^p)^{\frac{1}{p}} \geq x\} \\ = \bigwedge \{z \in [0, 1] \mid z \geq (x^p - y^p)^{\frac{1}{p}}\} = (x^p - y^p)^{\frac{1}{p}} \vee 0,$$

(6) Let $P(X)$ be the collection of all subsets of X . Then $(P(X), \subset, \cup, \cap, \oplus = \cup, \emptyset, X)$ is a complete co-residuated lattice.

Lemma 2.3 ([20]). *Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

- (1) *If $y \leq z$, $x \oplus y \leq x \oplus z$, $y \ominus x \leq z \ominus x$ and $x \ominus z \leq x \ominus y$.*
- (2) *$(\bigvee_{i \in \Gamma} x_i) \ominus y = \bigvee_{i \in \Gamma} (x_i \ominus y)$ and $x \ominus (\bigwedge_{i \in \Gamma} y_i) = \bigvee_{i \in \Gamma} (x \ominus y_i)$.*
- (3) *$(\bigwedge_{i \in \Gamma} x_i) \ominus y \leq \bigwedge_{i \in \Gamma} (x_i \ominus y)$.*
- (4) *$x \ominus (\bigvee_{i \in \Gamma} y_i) \leq \bigwedge_{i \in \Gamma} (x \ominus y_i)$.*
- (5) *$x \ominus x = \perp$, $x \ominus \perp = x$ and $\perp \ominus x = \perp$. Moreover, $x \ominus y = \perp$ iff $x \leq y$.*
- (6) *$y \oplus (x \ominus y) \geq x$, $y \geq x \ominus (x \ominus y)$ and $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.*
- (7) *$x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.*
- (8) *$x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$, $x \ominus y \geq (x \ominus z) \ominus (y \ominus z)$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$ and $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.*
- (9) *$x \oplus y = \perp$ iff $x = \perp$ and $y = \perp$.*
- (10) *$(x \oplus y) \ominus z \leq x \oplus (y \ominus z)$ and $(x \ominus y) \oplus z \geq x \ominus (y \oplus z)$.*
- (11) *If L satisfies a double negative law and $n(x) = \top \ominus x$, then $n(x \oplus y) = n(x) \ominus y = n(y) \ominus x$ and $x \ominus y = n(y) \ominus n(x)$.*

Definition 2.4 ([20]). Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. Let X be a set. A function $d_X : X \times X \rightarrow L$ is called a *distance function*, if it satisfies the following conditions:

- (M1) $d_X(x, x) = \perp$ for all $x \in X$,
- (M2) $d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z)$ for all $x, y, z \in X$,

(M3) if $d_X(x, y) = d_X(y, x) = \perp$, then $x = y$.
 The pair (X, d_X) is called a *distance space*.

Remark 2.5 ([20]). (1) We define a distance function $d_X : X \times X \rightarrow [0, \infty]$. Then (X, d_X) is called a *pseudo-quasi-metric space*.

(2) Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. Define a function $d_L : L \times L \rightarrow L$ as $d_L(x, y) = x \ominus y$. By Lemma 2.3 (5) and (6), (L, d_L) is a distance space. For $\tau \subset L^X$, we define a function $d_\tau : \tau \times \tau \rightarrow L$ as

$$d_\tau(A, B) = \bigvee_{x \in X} (A(x) \ominus B(x)).$$

Then (τ, d_τ) is a distance space.

3. FUZZY COMPLETENESS AND VARIOUS OPERATIONS IN CO-RESIDUATED LATTICES

In this section, we assume $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ is a complete co-residuated lattice.

Definition 3.1 ([20]). Let (X, d_X) be a distance space and $A \in L^X$.

(i) A point x_0 is called a *fuzzy join* of A , denoted by $x_0 = \sqcup_X A$, if it satisfies

- (J1) $A(x) \geq d_X(x, x_0)$,
- (J2) $\bigvee_{x \in X} (d_X(x, y) \ominus A(x)) \geq d_X(x_0, y)$.

The pair (X, d_X) is called *fuzzy join complete*, if $\sqcup_X A$ exists for each $A \in L^X$.

(ii) A point x_1 is called a *fuzzy meet* of A , denoted by $x_1 = \sqcap_X A$, if it satisfies

- (M1) $A(x) \geq d_X(x_1, x)$,
- (M2) $\bigvee_{x \in X} (d_X(y, x) \ominus A(x)) \geq d_X(y, x_1)$.

The pair (X, d_X) is called *fuzzy meet complete*, if $\sqcap_X A$ exists for each $A \in L^X$.

The pair (X, d_X) is called *fuzzy complete*, if $\sqcap_X A$ and $\sqcup_X A$ exists for each $A \in L^X$.

Theorem 3.2 ([20]). Let (X, d_X) be a distance space and $\Phi \in L^X$.

- (1) A point x_0 is a fuzzy join of Φ iff $\bigvee_{x \in X} (d_X(x, y) \ominus \Phi(x)) = d_X(x_0, y)$.
- (2) A point x_1 is a fuzzy meet of Φ iff $\bigvee_{x \in X} (d_X(y, x) \ominus \Phi(x)) = d_X(y, x_1)$.
- (3) If $\sqcup_X \Phi$ is a fuzzy join of $\Phi \in L^X$, then it is unique. Moreover, if $\sqcap_X \Phi$ is a fuzzy meet of $\Phi \in L^X$, then it is unique.

Example 3.3. Let (X, d_X) be a distance spaces and $A \in L^X$.

(1) Since $\bigvee_{x \in X} (d_X(x, y) \ominus d_X(x, z)) = d(z, y)$, by Theorem 3.2,

$$z = \sqcup_X (d_X)^z,$$

where $(d_X)^z(x) = d_X(x, z)$.

(2) Since $\bigvee_{x \in X} (d_X(y, x) \ominus d_X(z, x)) = d_X(y, z)$, by Theorem 3.3,

$$z = \sqcap_X (d_X)_z,$$

where $(d_X)_z(x) = d_X(z, x)$.

Remark 3.4. Let (L^X, d_{L^X}) be a function space and $\Phi \in L^{L^X}$.

(1) Since $\sqcup_{L^X} \Phi$ is a join of Φ , we see that for all $\Phi \in L^{L^X}$,

$$\begin{aligned} d_{L^X}(\sqcup_{L^X} \Phi, B) &= \bigvee_{A \in L^X} (d_{L^X}(A, B) \ominus \Phi(A)) \\ &= \bigvee_{A \in L^X} d_{L^X}(A \ominus \Phi(A), B) \end{aligned}$$

$$= d_{L^X}(\bigvee_{A \in L^X} (A \ominus \Phi(A)), B).$$

Then by Theorem 3.3, we have

$$\sqcup_{L^X} \Phi = \bigvee_{A \in L^X} (A \ominus \Phi(A)).$$

(2) Since $\sqcup_{L^X} \Phi$ is a meet of Φ , it follows that for all $\Phi \in L^{L^X}$,

$$\begin{aligned} d_{L^X}(B, \sqcup_{L^X} \Phi) &= \bigvee_{A \in L^X} (d_{L^X}(B, A) \ominus \Phi(A)) \\ &= \bigvee_{A \in L^X} d_{L^X}(B, \Phi(A) \oplus A) \\ &= d_{L^X}(B, \bigwedge_{A \in L^X} (\Phi(A) \oplus A)). \end{aligned}$$

Then by Theorem 3.3, $\sqcup_{L^X} \Phi = \bigwedge_{A \in L^X} (\Phi(A) \oplus A)$.

Definition 3.5. A subset $\tau \subset L^X$ is called an *Alexandrov topology* on X , if it satisfies the following conditions:

(A1) if $A_i \in \tau$ for all $i \in I$, then $\bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau$,

(A2) if $A \in \tau$ and $\alpha \in L$, then $\alpha_X, A \ominus \alpha, A \oplus \alpha \in \tau$.

The pair (X, τ) is called an *Alexandrov topological space* on X .

Theorem 3.6. Let (X, d_X) be a distance space. We define

$$\begin{aligned} \tau_{d_X} &= \{A \in L^X \mid A(x) \oplus d_X(x, y) \geq A(y)\}, \\ \tau_{d_X^{-1}} &= \{A \in L^X \mid A(x) \oplus d_X(y, x) \geq A(y)\}. \end{aligned}$$

(1) τ_{d_X} and $\tau_{d_X^{-1}}$ are Alexandrov topologies.

(2) $(\tau_{d_X}, d_{\tau_{d_X}})$ is a complete lattice.

(3) $(\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ is a complete lattice.

(4) $\tau_{d_X} = \{\bigvee_{x \in X} A(x) \oplus d_X(x, -) \mid A \in L^X\}$.

(5) $\tau_{d_X^{-1}} = \{\bigvee_{x \in X} A(x) \oplus d_X(-, x) \mid A \in L^X\}$.

Proof. (1) For $A \in \tau, \alpha_X, \alpha \oplus A, A \ominus \alpha, (d_X)_z \in \tau$ from:

$$\begin{aligned} (\alpha \oplus A(x)) \oplus d_X(x, y) &\geq (\alpha \oplus A(y)) \\ (A(x) \ominus \alpha) \oplus d_X(x, y) &\geq (A(y) \ominus \alpha) \\ (d_X)_z(x) \oplus d_X(x, y) &\leq (d_X)_z(y). \end{aligned}$$

Then τ is an Alexandrov topology with $(d_X)_z \in \tau$.

(2) For $\Phi \in L^{\tau_{d_X}}$, let $A_0 = \bigwedge_{A \in \tau_{d_X}} (\Phi(A) \oplus A)$ and $A_1 = \bigvee_{A \in \tau_{d_X}} (A \ominus \Phi(A))$.

Then $A_0 \in \tau_{d_X}$ and $A_1 \in \tau_{d_X}$. Thus $A_1 = \sqcup_{\tau_{d_X}} \Phi = \bigvee_{A \in \tau_{d_X}} (A \ominus \Phi(A))$ from:

$$\begin{aligned} d_{\tau_{d_X}}(A_1, B) &= d_{\tau_{d_X}}(\bigvee_{A \in \tau_{d_X}} (A \ominus \Phi(A)), B) \\ &= \bigvee_{A \in \tau_{d_X}} d_{\tau_{d_X}}(A \ominus \Phi(A), B) \\ &= \bigvee_{A \in \tau_{d_X}} (d_{\tau_{d_X}}(A, B) \ominus \Phi(A)) \\ &= d_{\tau_{d_X}}(\sqcup_{\tau_{d_X}} \Phi, B). \end{aligned}$$

Moreover, $A_0 = \bigwedge_{A \in \tau_{d_X}} (A \oplus \Phi(A)) = \sqcap_{\tau_{d_X}} \Phi$ from:

$$\begin{aligned} d_{\tau_{d_X}}(B, A_0) &= d_{\tau_{d_X}}(B, \bigwedge_{A \in \tau_{d_X}} (\Phi(A) \oplus A)) \\ &= \bigvee_{A \in \tau_{d_X}} (d_{\tau_{d_X}}(B, A) \ominus \Phi(A)) \\ &= d_{\tau_{d_X}}(B, \sqcap_{\tau_{d_X}} \Phi). \end{aligned}$$

So $(\tau_{d_X}, d_{\tau_{d_X}})$ is a complete lattice.

(3) It is similarly proved as (1).

(4) $W(X) = \{\bigwedge_{x \in X} A(x) \oplus d_X(x, -) \mid A \in L^X\}$.
 Let $A \in \tau_{d_X}$. Then we have

$$\bigwedge_{x \in X} A(x) \oplus d_X(x, y) \geq A(y) \text{ and } \bigwedge_{x \in X} A(x) \oplus d_X(x, y) \leq A(y) \oplus d_X(y, y) = A(y).$$

Thus $A = \bigwedge_{x \in X} A(x) \oplus d_X(x, -) \in W(X)$.

Let $\bigwedge_{x \in X} (A(x) \oplus d_X(x, -)) \in W(X)$. Then we get

$$\bigwedge_{y \in X} \left(\bigwedge_{x \in X} A(x) \oplus d_X(x, y) \right) \oplus d_X(y, z) \geq \bigwedge_{x \in X} A(x) \oplus d_X(x, z).$$

Thus $\bigwedge_{x \in X} (A(x) \oplus d_X(x, -)) \in \tau_{d_X}$.

(5) It is similarly proved as (4). □

Definition 3.7. Let (X, d_X) and (Y, d_Y) be distance spaces and $f : X \rightarrow Y$ be a map. Define $f^* : L^X \rightarrow L^Y$ as

$$f^*(A)(y) = \begin{cases} \top & \text{if } f^{-1}(\{y\}) = \emptyset \\ \bigwedge A(x) & \text{if } x \in f^{-1}(\{y\}). \end{cases}$$

(i) f is called a *join preserving map*, if $f(\sqcup_X A) = \sqcup_{L^Y} f^*(A)$ for each $A \in L^X$ with $\sqcup_X A$ exists.

(ii) f is called a *meet preserving map*, if $f(\prod_X A) = \prod_{L^Y} f^*(A)$ for each $A \in L^X$ with $\prod_X A$ exists.

(iii) f is called a *join-meet preserving map*, if $f(\sqcup_X A) = \prod_{L^Y} f^*(A)$ for each $A \in L^X$ with $\sqcup_X A$ exists.

(iv) f is called a *meet-join preserving map*, if $f(\prod_X A) = \sqcup_{L^Y} f^*(A)$ for each $A \in L^X$ with $\prod_X A$ exists.

(v) f is called an *embedding map*, if f is injective and $d_X(x, y) = d_X(f(x), f(y))$ for each $x, y \in X$.

(vi) f is called a *dual embedding map*, if f is injective and $d_X(x, y) = d_X(f(y), f(x))$ for each $x, y \in X$.

Define $f^\oplus, f^{s\oplus} : L^X \rightarrow L^Y$ and $f_\oplus^\leftarrow, f_\oplus^{s\leftarrow} : L^X \rightarrow L^Y$ as

$$\begin{aligned} f^\oplus(A)(y) &= \bigwedge_{x \in X} (A(x) \oplus d_Y(f(x), y)), \\ f^{s\oplus}(A)(y) &= \bigwedge_{x \in X} (A(x) \oplus d_Y(y, f(x))), \end{aligned}$$

$$\begin{aligned} f_\oplus^\leftarrow(B)(x) &= \bigwedge_{z \in X} (B(f(z)) \oplus d_X(z, x)), \\ f_\oplus^{s\leftarrow}(B)(x) &= \bigwedge_{z \in X} (B(f(z)) \oplus d_X(x, z)). \end{aligned}$$

Theorem 3.8. Let (X, d_X) be a distance space.

(1) Define $f : (X, d_X) \rightarrow (\tau_{d_X}, d_{\tau_{d_X}})$ as $f(x) = (d_X)_x$. Then f is an embedding map. Moreover, if $\sqcup_X A$ exists, then

$$\begin{aligned} \sqcup_{\tau_{d_X}} f^*(A) &= \bigvee_{x \in X} (d_X(x, -) \oplus A(x)) = f(\sqcup_X A), \\ \prod_{\tau_{d_X}} f^*(A) &= \bigwedge_{z \in X} (A(z) \oplus d_X(z, -)). \end{aligned}$$

If $A \in \tau_{d_X}$, then $\prod_{\tau_{d_X}} f^*(A) = A$.

(2) Define $g : (X, d_X) \rightarrow (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ as $g(x) = (d_X)^x$. Then g is a dual embedding map. Moreover, if $\sqcup_X A$ exists, then

$$\begin{aligned} \sqcup_{\tau_{d_X^{-1}}} g^*(A) &= \bigvee_{x \in X} (d_X(-, x) \ominus A(x)) = g(\sqcup_X A), \\ \sqcap_{\tau_{d_X^{-1}}} g^*(A) &= \bigwedge_{z \in X} (A(z) \oplus d_X(-, z)). \end{aligned}$$

If $A \in \tau_{d_X^{-1}}$, then $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = A$.

Proof. (1) Since $(d_X)_x(z) \oplus d_X(z, w) \geq (d_X)_x(w)$, $f(x) \in \tau_{d_X}$. Then f is well-defined. If $f(x) = f(y) = (d_X)_x = (d_X)_y$, then $d_X(x, z) = d_X(y, z)$ for each $z \in X$ implies $x = y$. Thus f is injective. Moreover, for all $x, y \in X$,

$$\begin{aligned} d_{\tau_{d_X}}(f(x), f(y)) &= d_{\tau_{d_X}}((d_X)_x, (d_X)_y) \\ &= \bigvee_{z \in X} ((d_X)_x(z) \ominus (d_X)_y(z)) \\ &= \bigvee_{z \in X} (d_X(x, z) \ominus d_X(y, z)) \\ &= d_X(x, y). \end{aligned}$$

So f is an embedding map. Since $\bigvee_{x \in X} (f(x) \ominus A(x)) \in \tau_{d_X}$, for $C \in \tau_{d_X}$,

$$\begin{aligned} d_{\tau_{d_X}}(\sqcup_{\tau_{d_X}} f^*(A), C) &= \bigvee_{D \in \tau_{d_X}} (d_{\tau_{d_X}}(D, C) \ominus f^*(A)(D)) \\ &= \bigvee_{D \in \tau_{d_X}} (d_{\tau_{d_X}}(D, C) \ominus \bigwedge_{x \in f^{-1}(D)} A(x)) \\ &= \bigvee_{D \in \tau_{d_X}} \bigvee_{x \in f^{-1}(D)} (d_{\tau_{d_X}}(f(x), C) \ominus A(x)) \\ &= d_{\tau_{d_X}}(\bigvee_{x \in X} (f(x) \ominus A(x)), C). \end{aligned}$$

Hence $\sqcup_{\tau_{d_X}} f^*(A) = \bigvee_{x \in X} (f(x) \ominus A(x)) = \bigvee_{x \in X} (d_X(x, -) \ominus A(x)) = d_X(\sqcup_X A, -)$.

If $\sqcup_X A$ exists, then $\sqcup_{\tau_{d_X}} f^*(A) = d_X(\sqcup_X A, -) = f(\sqcup_X A)$. Since $f(x) \in \tau_{d_X}$, by (A1) and (A2), $\bigwedge_{x \in X} (f(x) \oplus A(x)) \in \tau_{d_X}$. For $C \in \tau_{d_X}$,

$$\begin{aligned} d_{\tau_{d_X}}(C, \sqcap_{\tau_{d_X}} f^*(A)) &= \bigvee_{D \in \tau_{d_X}} (d_{\tau_{d_X}}(C, D) \ominus f^*(A)(D)) \\ &= \bigvee_{D \in \tau_{d_X}} (d_{\tau_{d_X}}(C, D) \ominus \bigwedge_{x \in f^{-1}(D)} A(x)) \\ &= \bigvee_{D \in \tau_{d_X}} \bigvee_{x \in f^{-1}(D)} (d_{\tau_{d_X}}(C, f(x)) \ominus A(x)) \\ &= \bigvee_{D \in \tau_{d_X}} \bigvee_{x \in f^{-1}(D)} d_{\tau_{d_X}}(C, f(x) \oplus A(x)) \\ &= d_{\tau_{d_X}}(C, \bigwedge_{x \in X} (f(x) \oplus A(x))). \end{aligned}$$

Then $\sqcap_{\tau_{d_X}} f^*(A) = \bigwedge_{x \in X} (f(x) \oplus A(x))$. If $A \in \tau_{d_X}$, $\sqcap_{\tau_{d_X}} f^*(A) = A$.

(2) Since $(d_X)^x(z) \oplus d_X(w, z) \geq (d_X)^x(w)$, $g(x) \in \tau_{d_X^{-1}}$. Then g is well-defined. If $g(x) = g(y) = (d_X)^x = (d_X)^y$, then $d_X(z, x) = d_X(z, y)$ for each $z \in X$ implies $x = y$. Thus g is injective. Moreover, for all $x, y \in X$,

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(g(x), g(y)) &= d_{\tau_{d_X^{-1}}}((d_X)^x, (d_X)^y) \\ &= \bigvee_{z \in X} ((d_X)^x(z) \ominus (d_X)^y(z)) \\ &= \bigvee_{z \in X} (d_X(z, x) \ominus d_X(z, y)) \\ &= d_X(y, x). \end{aligned}$$

Thus g is a dual embedding map. Since $\bigvee_{x \in X} (g(x) \ominus A(x)) \in \tau_{d_X^{-1}}$, for $C \in \tau_{d_X^{-1}}$,

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(\sqcup g^*(A), C) &= \bigvee_{D \in \tau_{d_X^{-1}}} (d_{\tau_{d_X^{-1}}}(D, C) \ominus g^*(A)(D)) \\ &= \bigvee_{D \in \tau_{d_X^{-1}}} (d_{\tau_{d_X^{-1}}}(D, C) \ominus \bigwedge_{x \in g^{-1}(D)} A(x)) \\ &= \bigvee_{D \in \tau_{d_X^{-1}}} \bigvee_{x \in g^{-1}(D)} (d_{\tau_{d_X^{-1}}}(g(x), C) \ominus A(x)) \\ &= d_{\tau_{d_X^{-1}}}(\bigvee_{x \in X} (g(x) \ominus A(x)), C). \end{aligned}$$

So $\sqcup_{\tau_{d_X^{-1}}} g^*(A) = \bigvee_{x \in X} (g(x) \ominus A(x)) = \bigvee_{x \in X} (d_X(-, x) \ominus A(x)) = d_X(-, \sqcup_X A)$.

If $\sqcap_X A$ exists, then $\sqcup_{\tau_{d_X^{-1}}} g^*(A) = d_X(-, \sqcap_X A) = g(\sqcap_X A)$. Since $g(x) \in \tau_{d_X^{-1}}$, by (A1) and (A2), $\bigwedge_{x \in X} (g(x) \ominus A(x)) \in \tau_{d_X^{-1}}$. For $C \in \tau_{d_X^{-1}}$,

$$\begin{aligned} d_{\tau_{d_X^{-1}}}(C, \sqcap_{\tau_{d_X^{-1}}} g^*(A)) &= \bigvee_{D \in \tau_{d_X^{-1}}} (d_{\tau_{d_X^{-1}}}(C, D) \ominus g^*(A)(D)) \\ &= \bigvee_{D \in \tau_{d_X^{-1}}} (d_{\tau_{d_X^{-1}}}(C, D) \ominus \bigwedge_{x \in g^{-1}(D)} A(x)) \\ &= \bigvee_{D \in \tau_{d_X^{-1}}} \bigvee_{x \in g^{-1}(D)} (d_{\tau_{d_X^{-1}}}(C, g(x)) \ominus A(x)) \\ &= \bigvee_{D \in \tau_{d_X^{-1}}} \bigvee_{x \in g^{-1}(D)} d_{\tau_{d_X^{-1}}}(C, g(x) \oplus A(x)) \\ &= d_{\tau_{d_X^{-1}}}(C, \bigwedge_{x \in X} (g(x) \oplus A(x))). \end{aligned}$$

Thus $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = \bigwedge_{x \in X} (g(x) \oplus A(x))$. If $A \in \tau_{d_X^{-1}}$, then $\sqcap_{\tau_{d_X^{-1}}} g^*(A) = A$. \square

Lemma 3.9. *Let $(L, \wedge, \vee, \oplus, \ominus, \perp, \top)$ be a complete co-residuated lattice. For each $x_i, y_i \in L$, we have the following properties.*

- (1) $(\bigvee_{i \in \Gamma} x_i) \ominus (\bigvee_{i \in \Gamma} y_i) \leq \bigvee_{i \in \Gamma} (x_i \ominus y_i)$.
- (2) $(\bigwedge_{i \in \Gamma} x_i) \ominus (\bigwedge_{i \in \Gamma} y_i) \leq \bigvee_{i \in \Gamma} (x_i \ominus y_i)$.

Proof. (1) Since $x_i \leq (x_i \ominus y_i) \oplus y_i \leq \bigvee_{i \in \Gamma} (x_i \ominus y_i) \oplus (\bigvee_{i \in \Gamma} y_i)$, we conclude that

$$\bigvee_{i \in \Gamma} x_i \leq \bigvee_{i \in \Gamma} (x_i \ominus y_i) \oplus (\bigvee_{i \in \Gamma} y_i).$$

Then $(\bigvee_{i \in \Gamma} x_i) \ominus (\bigvee_{i \in \Gamma} y_i) \leq \bigvee_{i \in \Gamma} (x_i \ominus y_i)$.

- (2) Since $x_i \leq (x_i \ominus y_i) \oplus y_i$, we conclude that

$$\bigwedge_{i \in \Gamma} x_i \leq (x_i \ominus y_i) \oplus y_i \text{ iff } x_i \ominus y_i \geq \bigwedge_{i \in \Gamma} x_i \ominus y_i.$$

Then $\bigvee_{i \in \Gamma} (x_i \ominus y_i) \geq \bigvee_{i \in \Gamma} (\bigwedge_{i \in \Gamma} x_i \ominus y_i) = (\bigwedge_{i \in \Gamma} x_i) \ominus (\bigwedge_{i \in \Gamma} y_i)$. \square

Theorem 3.10. *Let (X, d_X) and (Y, d_Y) be distance spaces. Then the following properties hold.*

(1) *If $f : (X, d_X) \rightarrow (Y, d_Y)$ is a map with $d_X(x, y) \geq d_Y(f(x), f(y))$ for each $x, y \in X$, then $d_Y(\sqcup_Y f^{s\oplus}(A), f(\sqcup_X A)) = \perp$ and $d_Y(f(\sqcap_X A), \sqcap_Y f^\oplus(A)) = \perp$, for each $A \in L^X$.*

- (2) $d_{L^X}(B, A) \geq d_{L^Y}(f^\oplus(B), f^\oplus(A))$ and $d_{L^X}(B, A) \geq d_{L^Y}(f^{s\oplus}(B), f^{s\oplus}(A))$.
- (3) $d_{L^Y}(C, D) \geq d_{L^X}(f_\oplus^\leftarrow(C), f_\oplus^\leftarrow(E))$ and $d_{L^Y}(C, D) \geq d_{L^X}(f_\oplus^{s\leftarrow}(C), f_\oplus^{s\leftarrow}(E))$.
- (4) $f^\oplus(A) \in \tau_{d_Y}$ and $f^{s\oplus}(A) \in \tau_{d_Y^{-1}}$.
- (5) $f_\oplus^\leftarrow(A) \in \tau_{d_X}$ and $f_\oplus^{s\leftarrow}(A) \in \tau_{d_X^{-1}}$.

Proof. (1) For each $A \in L^X$,

$$\begin{aligned} d_Y(\sqcup_Y f^{s\oplus}(A), f(\sqcup_X A)) &= \bigvee_{y \in Y} (d_Y(y, f(\sqcup_X A)) \ominus f^{s\oplus}(A)(y)) \\ &= \bigvee_{y \in X} (d_Y(y, f(\sqcup_X A)) \ominus \bigwedge_{x \in X} (A(x) \oplus d_Y(y, f(x)))) \\ &= \bigvee_{y \in X} \bigvee_{x \in X} ((d_Y(y, f(\sqcup_X A)) \ominus d_Y(y, f(x))) \ominus A(x)) \\ &= \bigvee_{x \in X} (d_X(x, \sqcup_X A) \ominus A(x)) \\ &\leq \bigvee_{x \in X} (d_X(x, \sqcup_X A) \ominus A(x)) \\ &= d_X(\sqcup_X A, \sqcup_X A) = \perp. \end{aligned}$$

For $A \in L^X$,

$$\begin{aligned} d_Y(f(\sqcap_X A), \sqcap_Y f^\oplus(A)) &= \bigvee_{y \in Y} (d_Y(f(\sqcap_X A), y) \ominus f^\oplus(A)(y)) \\ &= \bigvee_{y \in X} (d_Y(f(\sqcap_X A), y) \ominus \bigwedge_{x \in X} (A(x) \oplus d_Y(f(x), y))) \end{aligned}$$

$$\begin{aligned}
 &= \bigvee_{y \in X} \bigvee_{x \in X} ((d_Y(f(\Pi_X A), y) \oplus d_Y(f(x), y)) \oplus A(x)) \\
 &= \bigvee_{x \in X} (d_Y(f(\Pi_X A), f(x)) \oplus A(x)) \\
 &\leq \bigvee_{x \in X} (d_X(\Pi_X A, x) \oplus A(x)) \\
 &= d_X(\Pi_X A, \Pi_X A) = \perp.
 \end{aligned}$$

(2) Since $d_Y(y, f(x)) \oplus A(x) \oplus (B(x) \oplus A(x)) \geq d_Y(y, f(x)) \oplus B(x)$, we conclude that

$$(B(x) \oplus A(x)) \geq (d_Y(y, f(x)) \oplus B(x)) \oplus (d_Y(y, f(x)) \oplus A(x)).$$

By Lemma 3.9,

$$\begin{aligned}
 d_{L^X}(B, A) &= \bigvee_{x \in X} (B(x) \oplus A(x)) \\
 &\geq \bigwedge_{x \in X} (d_Y(y, f(x)) \oplus B(x)) \oplus \bigwedge_{x \in X} (d_Y(y, f(x)) \oplus A(x)) \\
 &= d_{L^Y}(f^\oplus(B), f^\oplus(A)).
 \end{aligned}$$

(3) $d_{L^Y}(C, E) \geq d_{L^X} f_{\oplus}^{\leftarrow}(C), f_{\oplus}^{\leftarrow}(E)$ from:

$$\begin{aligned}
 f_{\oplus}^{\leftarrow}(C)(x) \oplus f_{\oplus}^{\leftarrow}(E)(x) &= \bigwedge_{z \in X} (d_X(x, z) \oplus C(f(x))) \oplus \bigwedge_{z \in X} (d_X(x, z) \oplus E(f(x))) \\
 &\leq \bigvee_{z \in X} (C(f(x)) \oplus E(f(x))) \\
 &\leq d_{L^Y}(C, E). \text{ [By Lemma 2.3 (8) and 3.9]}
 \end{aligned}$$

Similarly, $d_{L^Y}(C, E) \geq d_{L^X} f_{\oplus}^{s\leftarrow}(C), f_{\oplus}^{s\leftarrow}(E)$.

(4) By Theorem 3.6, $f^\oplus(A) \in \tau_{d_Y}$ from:

$$\begin{aligned}
 f^\oplus(A)(y) \oplus d_Y(y, w) &= \bigwedge_{y \in X} (A(x) \oplus d_Y(f(x), y) \oplus d_Y(y, w)) \\
 &\geq \bigwedge_{y \in X} (A(x) \oplus d_Y(f(x), w)) \\
 &= f^\oplus(A)(w).
 \end{aligned}$$

Also by Theorem 3.6, $f^{s\oplus}(A) \in \tau_{d_Y^{-1}}$ from:

$$\begin{aligned}
 f^{s\oplus}(A)(y) \oplus d_Y(w, y) &= \bigwedge_{y \in X} (A(x) \oplus d_Y(y, f(x)) \oplus d_Y(w, y)) \\
 &\geq \bigwedge_{y \in X} (A(x) \oplus d_Y(w, f(x))) \\
 &= f^{s\oplus}(A)(w). \quad \square
 \end{aligned}$$

Example 3.11. Let $([0, 1], \leq, \vee, \wedge, \oplus, \ominus, 0, 1)$ be a complete co-residuated lattice defined as $n(x) = 1 - x$,

$$x \oplus y = (x + y) \wedge 1, \quad x \ominus y = (x - y) \vee 0.$$

Let $X = \{a, b, c\}$ be a set and $A, B \in [0, 1]^X$ with

$$A(x) = 0.3, A(y) = 0.2, A(z) = 0.5, B(x) = 0.6, B(y) = 0.3, B(z) = 0.5.$$

Define $d_X \in L^{X \times X}$ as

$$d_X = \begin{pmatrix} 0 & 0.5 & 0.8 \\ 0.7 & 0 & 0.6 \\ 0.4 & 0.6 & 0 \end{pmatrix}$$

We easily show that d_X is a distance function. Moreover,

$$\begin{aligned}
 A &= \bigwedge_{x \in X} (A(x) \oplus d_X(x, -)) = \bigwedge_{x \in X} (A(x) \oplus d_X(-, x)) \\
 B &= \bigwedge_{x \in X} (B(x) \oplus d_X(x, -)) = \bigwedge_{x \in X} (B(x) \oplus d_X(-, x)).
 \end{aligned}$$

By Theorem 3.6, $A, B \in \tau_{d_X}, A, B \in \tau_{d_X^{-1}}$.

(1) Define $f : (X, d_X) \rightarrow (\tau_{d_X}, d_{\tau_{d_X}})$ as $f(x) = (d_X)_x$. Then $d_X(x, y) = d_{\tau_{d_X}}(f(x), f(y))$ for all $x, y \in X$. Since $\sqcup_X (d_X)^z = z$ from Example 3.3,

$$\begin{aligned} \sqcup f^*((d_X)^z) &= \bigvee_{x \in X} (d_X(x, -)) \ominus (d_X)^z(x) = d_X(z, -) \\ &= f(\sqcup_X (d_X)^z) = f(z) = (0.4, 0.6, 0), \\ \sqcap_{\tau_{d_X}} f^*((d_X)^z) &= \bigwedge_{y \in X} ((d_X)^z(y) \oplus d_X(y, -)) = (0.4, 0.6, 0). \end{aligned}$$

Since $\bigvee_{x \in X} (d_X(x, -)) \ominus A(x) \neq d_X(\sqcup_X A, -)$ for $A = (0.3, 0.2, 0.5)$, $\sqcup_X A$ does not exist.

$$\begin{aligned} \sqcup_{\tau_{d_X}} f^*(A) &= \bigvee_{x \in X} (d_X(x, -)) \ominus A(x) = (0.5, 0.2, 0.5), \\ \sqcap_{\tau_{d_X}} f^*(A) &= \bigwedge_{y \in X} ((d_X)^z(y) \oplus d_X(y, -)) = (0.3, 0.2, 0.5). \end{aligned}$$

(2) Define $g : (X, d_X) \rightarrow (\tau_{d_X^{-1}}, d_{\tau_{d_X^{-1}}})$ as $g(x) = (d_X)^x$. Then $d_X(x, y) = d_{\tau_{d_X^{-1}}}(g(y), g(x))$ for all $x, y \in X$. Since $\sqcap (d_X)_z = z$ from Example 3.3,

$$\begin{aligned} \sqcup_{\tau_{d_X^{-1}}} g^*((d_X)_z) &= \bigvee_{x \in X} (d_X(-, x)) \ominus (d_X)_z(x) = d_X(-, z) \\ &= d_X(-, \sqcap_X (d_X)_z) = g(\sqcap_X (d_X)_z) = (0.8, 0.6, 0), \\ \sqcap_{\tau_{d_X^{-1}}} g^*((d_X)_z) &= \bigwedge_{y \in X} ((d_X)_z(y) \oplus d_X(-, y)) = (0.4, 0.6, 0). \end{aligned}$$

Since $\bigvee_{x \in X} (d_X(-, x)) \ominus A(x) \neq d_X(-, \sqcap_X A)$ for $A = (0.3, 0.2, 0.5)$, $\sqcap_X A$ does not exist.

$$\begin{aligned} \sqcup_{\tau_{d_X^{-1}}} g^*(A) &= \bigvee_{x \in X} (d_X(-, x)) \ominus A(x) = (0.3, 0.4, 0.4), \\ \sqcap_{\tau_{d_X^{-1}}} g^*(A) &= \bigwedge_{z \in X} (A(z) \oplus d_X(-, z)) = (0.3, 0.2, 0.5). \end{aligned}$$

4. CONCLUSION

Using distance functions, we discuss the notions of fuzzy completeness and Alexandrov topologies on co-residuated lattices. In particular, we investigate join (meet) preserving maps between various operations based on co-residuated lattices.

In the future, by using the concepts of fuzzy completeness and various operators, information systems and decision rules are investigated in co-residuated lattices.

Funding: This work was supported by the Research Institute of Natural Science of Gangneung-Wonju National University.

REFERENCES

- [1] R. Bělohlávek, Fuzzy Relational Systems, Kluwer Academic Publishers, New York 2002.
- [2] R. Bělohlávek, Fuzzy Galois connection, Math. Log. Quart. 45 (2000) 497–504.
- [3] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht 1998.
- [4] Q. Junsheng and Hu. Bao Qing, On $(\odot, \&)$ -fuzzy rough sets based on residuated and co-residuated lattices, Fuzzy Sets and Systems 336 (2018) 54–86.
- [5] S. E. Rodabaugh and E. P. Klement, Topological and Algebraic Structures In Fuzzy Sets, The Handbook of Recent Developments in the Mathematics of Fuzzy Sets, Kluwer Academic Publishers, Boston, Dordrecht, London 2003.
- [6] E. Turunen, Mathematics Behind Fuzzy Logic, A Springer-Verlag Co. 1999.
- [7] M. Ward and R. P. Dilworth, Residuated lattices, Trans. Amer. Math. Soc. 45 (1939) 335–354.
- [8] Z. Pawlak, Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving, Kluwer Academic Publishers, Dordrecht, The Netherlands 1991.
- [9] Y. H. She and G. J. Wang, An axiomatic approach of fuzzy rough sets based on residuated lattices, Computers and Mathematics with Applications 58 (2009) 189–201.

- [10] Y.C. Kim, Categories of fuzzy preorders, approximation operators and Alexandrov topologies, *Journal of Intelligent and Fuzzy Systems* 31 (2016) 1787–1793.
- [11] Q. Y. Zhang and L. Fan, Continuity in quantitative domains, *Fuzzy Sets and Systems* 154 (2005) 118–131.
- [12] Q. Y. Zhang, Algebraic generations of some fuzzy powerset operators, *Iranian Journal of Fuzzy Systems* 8 (5) (2011) 31–58.
- [13] Q. Y. Zhang and W. X. Xie, Fuzzy complete lattices, *Fuzzy Sets and Systems* 160 (2009) 2275–2291.
- [14] G. Georgescu and A. Popescue, Non-commutative Galois connections, *Soft Computing* 7 (2003) 458–467.
- [15] G. Georgescu and A. Popescue, Non-dual fuzzy connections, *Arch. Math. Log.* 43 (2004) 1009–1039.
- [16] Y. C. Kim and J. M. Ko, Fuzzy complete lattices, Alexandrov L -fuzzy topologies and fuzzy rough sets, *Journal of Intelligent and Fuzzy Systems* 38 (2020) 3253–3266.
- [17] A. M. Radzikowska and E. E. Kerre, Characterisation of main classes of fuzzy relations using fuzzy modal operators, *Fuzzy Sets and Systems* 152 (2005) 223–247.
- [18] H. Lai and D. Zhang, Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory, *Int. J. Approx. Reasoning* 50 (2009) 695–707.
- [19] M. C. Zheng and G. J. Wang, Coresiduated lattice with applications, *Fuzzy systems and Mathematics* 19 (2005) 1–6.
- [20] Y. C. Kim and J. M. Ko, Preserving maps and approximation operators in complete co-residuated lattices, *Journal of the Korean Institute of Intelligent Systems* 30 (5) (2020) 389–398.

JU-MOK OH (jumokoh@gwnu.ac.kr)

Department of Mathematics, Faculty of Natural Science, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea

YONG CHAN KIM (yck@gwnu.ac.kr)

Department of Mathematics, Faculty of Natural Science, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea