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Abstract. In this paper, we introduce a new class of topology called an *LM-fuzzy bitopological spaces*. Also we introduces the closure operator $\tau_1\tau_2 - Cl(A, r)$, and interior operator $\tau_1\tau_2 - Int(A, r)$ in LM-fuzzy bitopological spaces as an extension of them in mated fuzzy bitopological spaces and establishes the relationship between $\tau_1\tau_2 - Cl(A, r)$ and the smooth closure $\tau_1\tau_2 - cl(A, r)$. Furthermore, we introduce different concepts of closed sets and open sets in double fuzzy bitopological spaces are extended to LM-fuzzy bitopological spaces and the algebraic structures associated with the families of these sets are investigated. As a result, certain monoids (denoted by $\tau_1\tau_2 - O_r$, $\tau_1\tau_2 - C_r$, $\tau_1\tau_2 - L_r$, $\tau_1\tau_2 - RO_r$, $\tau_1\tau_2 - RC_r$, $\tau_1\tau_2 - SO_r$,$\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$, $\tau_1\tau_2 - bC_r$, $\tau_1\tau_2 - GO_r$, $\tau_1\tau_2 - GC_r$, $\tau_1\tau_2 - RGO_r$, $\tau_1\tau_2 - RGC_r$, $\tau_1\tau_2 - GSO_r$, $\tau_1\tau_2 - GSC_r$ $\tau_1\tau_2 - GbO_r$ and $\tau_1\tau_2 - GbC_r$) contained in L^X are identified. Finally, the study associates a lattice of monoids to each element of M, which is associative, complemented but not modular.

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Corresponding Author: Fadhil Abbas (fadhilhaman@gmail.com)

1. Introduction

The concept of fuzzy sets was first presented in 1965 by Zadeh [1]. After that, Chang in 1968 [2] introduced the concept of fuzzy topology. The concept of LM-fuzzy topology was introduced by Kubiak [3] and Sostak [4], since that many authors have contributed to the development of the theory of LM-fuzzy topological spaces. Kelly first proposed the concept of a bitopological space in 1963 [5]. In an ideal topological space, he also introduced local function. In 1995, Kandil et al. [6] introduced a concept of fuzzy bitopological spaces.

In this paper, we introduce a new class of topology called an *LM-fuzzy bitopological spaces*. Also we introduces the closure operator $\tau_1\tau_2 - Cl(A, r)$, and interior operator $\tau_1\tau_2 - Int(A, r)$ in LM-fuzzy bitopological spaces as an extension of them in mated fuzzy bitopological spaces and establishes the relationship between $\tau_1\tau_2 - Cl(A, r)$ and the smooth closure $\tau_1\tau_2 - cl(A, r)$. Furthermore, we introduce different concepts of closed sets and open sets in double fuzzy bitopological spaces are extended to LM-fuzzy bitopological spaces and the algebraic structures associated with the families of these sets are investigated. As a result, certain monoids (denoted by $\tau_1\tau_2 - O_r$, $\tau_1\tau_2 - C_r$, $\tau_1\tau_2 - L_r$, $\tau_1\tau_2 - RO_r$, $\tau_1\tau_2 - RC_r$, $\tau_1\tau_2 - SO_r$,$\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$, $\tau_1\tau_2 - bC_r$, $\tau_1\tau_2 - GO_r$, $\tau_1\tau_2 - GC_r$, $\tau_1\tau_2 - RGO_r$, $\tau_1\tau_2 - RGC_r$, $\tau_1\tau_2 - GSO_r$, $\tau_1\tau_2 - GSC_r$ $\tau_1\tau_2 - GbO_r$ and $\tau_1\tau_2 - GbC_r$) contained in L^X are identified. Finally, the study associates a lattice of monoids to each element of M, which is associative, complemented but not modular.

2. Preliminaries

Throughout this paper, X denotes a non-empty set, $I = [0, 1]$. The constant L-set having the value α is denoted by $\underline{\alpha}$.

Definition 2.1 ([7])**.** Let L be a poset. Then L is called:
(i) a *join-semilattice*, if $a \vee b \in L$ for every $a, b \in L$,
(ii) a *meet-semilattice*, if $a \wedge b \in L$ for every $a, b \in L$.
L is called a *lattice*, if it is both a join-semilattice and a meet-semilattice.

Definition 2.2 ([8])**.** Let L be a lattice. Then $a \in L$ is called:
(i) a *minimal element* of L, if $\nexists b \in L$ such that $b \leqslant a$ and $b \neq a$,
(ii) a *maximal element* of L, if $\nexists b \in L$ such that $b \geq a$ and $b \neq a$,
(iii) an *atom* of L, if a is a minimal element in $L \backslash \{0\}$,
(iv) a *dual atom* of L, if a is a maximal element in $L \backslash \{1\}$.

Definition 2.3 ([8])**.** A completely distributive lattice L is a called a *F -lattice*, if L has an order reversing involution $' : L \longrightarrow L$.

Definition 2.4 ([8])**.** Let L be a poset. Then L is called:
(i) a *complete join-semilattice*, if every join for an arbitrary subset of L exists,
(ii) a *complete meet-semilattice*, if every meet for an arbitrary subset of L exists,
(iii) a *complete lattice*, if it is both a complete join-semilattice and a complete meet-semilattice.

Definition 2.5. A *monoid* is a set X with a binary operation $* : X \times X \longrightarrow X$ which is associative and has an identity element.

Definition 2.6 ([7])**.** A *DeMorgan algebra* is a structure $A = (A, \vee, \wedge, 0, 1,')$ such that
(i) $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice,
(ii) $'$ is a De Morgan involution: $(a \wedge b)' = a' \vee b'$ and $(a')' = a$.

Definition 2.7 ([8])**.** For every $A \in L^X, A'$ is defined by $A'(x) = (A(x))'$ for every $x \in X$.

Definition 2.8 ([8]). An *L-fuzzy point* x_α is an L-fuzzy set $A \in L^X$ such that $A(x) = \alpha \neq 0$ and $A(y) = 0$ for $x \neq y$. The set of all L-fuzzy points x_α is denoted by $pt(L^X)$.

Definition 2.9 ([8]). Let $A, B \in L^X$. Then A *quasi coincides with* B *at* x, if $A(x) \not\leq B(x)$. Also, A is said to be *quasi coincident with* B, if A quasi coincides with B at some $x \in X$ and is denoted by $A\overline{q}B$.

Definition 2.10 ([3, 4]). Let L be an lattice and $M$is a completely distributive lattice. Then an *LM-fuzzy topology* on a set X is defined to be a mapping $\tau : L^X \longrightarrow M$ satisfying:
(i) $\tau(\underline{0}) = \tau(\underline{1}) = 1$,
(ii) $\tau(A \wedge B) \geq \tau(A) \wedge \tau(B)$ for every $A, B \in L^X$,
(iii) $\tau(\bigvee_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$ for every $\{A_j\}_{j\in J} \in L^X$.
The pair (X, τ) is called an *LM-fuzzy topological space* (LM-fts, for short). The elements of τ are called *LM-fuzzy open sets* and the complement of LM-fuzzy open sets are called *LM-fuzzy closed sets*.

Definition 2.11 ([9]). Let τ be a map from L^X to M. Then for $r \in M$, the *r-level decomposition* of τ is defined as $\{A \in L^X : \tau(A) \geq r\}$.

Definition 2.12 ([10]). Let (X, τ) be an LM-fts. The *closure* of A denoted by $cl(A)$ is defined as

$$cl(A) = \bigwedge\{F \in L^X : A \leqslant F, \tau(F^{'}) > 0\}.$$

Definition 2.13 ([10]). Let (X, τ) be an LM-fts. The *interior* of A denoted by $int(A)$ is defined as

$$int(A) = \bigvee\{U \in L^X : A \geq U, \tau(U) > 0\}.$$

3. LM-fuzzy bitopological spaces

In this section, we define the concept of an LM-fuzzy bitopological space on set X and we define closure and interior operators in LM-fuzzy topological spaces. Also we define a base and subbase of $\tau_i, i \in \{1, 2\}$.

Definition 3.1. Let (X, τ_1) and (X, τ_2) be a two LM-fuzzy topological spaces. Then the triple (X, τ_1, τ_2) is called an *LM-fuzzy bitopological space* on set X (LM-fbts, for short).

Definition 3.2. Let (X, τ_1, τ_2) be an LM-fbts and let $A \in L^X$. Then A is called $\tau_1\tau_2$-*LM-fuzzy open set*, if $A \subseteq U \cup V$ for every $U \in \tau_1$ and $V \in \tau_2$. The complement $\tau_1\tau_2$-LM-fuzzy open set is called $\tau_1\tau_2$-*LM-fuzzy closed set*.

Remark 3.3. (1) The union of any family of $\tau_1\tau_2$-LM-fuzzy open sets is a $\tau_1\tau_2$-LM-fuzzy open set.

(2) The intersection of any family of $\tau_1\tau_2$-LM-fuzzy open sets is a $\tau_1\tau_2$-LM-fuzzy open set.

Definition 3.4. Let (X, τ_1, τ_2) be an LM-fbts and let $A \in L^X$. Then $\tau_1\tau_2$-*closure* of A denoted by $\tau_1\tau_2 - cl(A)$ is defied as

$$\tau_1\tau_2 - cl(A) = \bigwedge\{F \in L^X : A \leqslant F, \tau_i(F^{'}) > 0, i \in \{1, 2\}\}.$$

Definition 3.5. Let (X, τ_1, τ_2) be an LM-fbts and let $A \in L^X$. Then $\tau_1\tau_2$-*interior* of A denoted by $\tau_1\tau_2 - int(A)$ is defied as

$$\tau_1\tau_2 - int(A) = \bigvee\{U \in L^X : A \geq U, \tau_i(U^{'}) > 0, i \in \{1, 2\}\}.$$

Definition 3.6. Let (X, τ_1, τ_2) be an LM-fbts. For $r \in M$ the operator $\tau_1\tau_2 - Cl$: $L^X \times M \longrightarrow L^X$ defined by

$$\tau_1\tau_2 - Cl(A, r) = \bigwedge\{F \in L^X : A \leqslant F, \tau_i(F^{'}) \geq r, i \in \{1, 2\}\}$$

is called the $\tau_1\tau_2$-*LM-fuzzy closure operator* on (X, τ_1, τ_2).

Definition 3.7. Let (X, τ_1, τ_2) be an LM-fbts. For $r \in M$ the operator $\tau_1\tau_2 - Int$: $L^X \times M \longrightarrow L^X$ defined by

$$\tau_1\tau_2 - Int(A, r) = \bigvee\{U \in L^X : A \geq U, \tau_i(U) \geq r, i \in \{1, 2\}\}$$

is called the $\tau_1\tau_2$-*LM-fuzzy interior operator* on (X, τ_1, τ_2).

Theorem 3.8. *Let (X, τ_1, τ_2) be an LM-fbts, let $A, B \in L^X$ and let $r, s \in M$. Then*
(1) $\tau_1\tau_2 - Cl(A, \underline{0}) = A = \tau_1\tau_2 - Int(A, \underline{0})$,
(2) $A \leqslant \tau_1\tau_2 - cl(A) \leqslant \tau_1\tau_2 - Cl(A, r)$,
(3) $\tau_1\tau_2 - Cl(A \vee B, r) = \tau_1\tau_2 - Cl(A, r) \vee \tau_1\tau_2 - Cl(B, r)$,
(4) $\tau_1\tau_2 - Cl(A, r) \leqslant \tau_1\tau_2 - Cl(A, s)$, *if* $r \leqslant s$,
(5) $\tau_1\tau_2 - Cl(\tau_1\tau_2 - Cl(A, r), r) = \tau_1\tau_2 - Cl(A, r)$,
(6) $\tau_1\tau_2 - Int(A^{'}, r) = (\tau_1\tau_2 - Cl(A, r))^{'}$,
(7) $\tau_1\tau_2 - Int(A, r) \leqslant \tau_1\tau_2 - int(A) \leqslant A$,
(8) $\tau_1\tau_2 - Int(A \wedge B, r) = \tau_1\tau_2 - Int(A, r) \wedge \tau_1\tau_2 - Int(B, r)$,
(9) $\tau_1\tau_2 - Int(A, r) \geq \tau_1\tau_2 - Int(A, s)$, *if* $r \leqslant s$,
(10) $\tau_1\tau_2 - Int(\tau_1\tau_2 - Int(A, r), r) = \tau_1\tau_2 - Int(A, r)$,
(11) *if* $A = \tau_1\tau_2 - Int(\tau_1\tau_2 - Cl(A, r), r)$, *then* $\tau_1\tau_2 - Cl(\tau_1\tau_2 - Int(A^{'}, r), r) = A^{'}$.

Definition 3.9. Let (X, τ_1, τ_2) be an LM-fbts and let $\beta : L^X \longrightarrow M$ with $\beta \leqslant \tau_1 \vee \tau_2$. Then β is called a *base* of τ_i, if

$$\forall A \in L^X, \tau_i(A) = \bigvee_{\bigvee_{\alpha\in\Lambda} B_\alpha = A} \bigwedge_{\alpha\in\Lambda} \beta(B_\alpha),$$

where

$$\bigvee_{\bigvee_{\alpha\in\Lambda} B_\alpha = A} \bigwedge_{\alpha\in\Lambda} \beta(B_\alpha)$$

will be denoted by $\beta^{(\sqcup)}(A)$.

Definition 3.10. Let (X, τ_1, τ_2) be an LM-fbts and let $\phi : L^X \longrightarrow M$, with $\phi \leqslant \tau_1 \vee \tau_2$. Then ϕ is called a *subbase* of τ_i if $\phi^{(\sqcap)} : L^X \longrightarrow M$ is a base of τ_i where

$$\phi^{(\sqcap)}(A) = \bigvee_{(\sqcap)_{\alpha\in J} B_\alpha = A} \bigwedge_{\alpha\in J} \phi(B_\alpha)$$

for every $A \in L^X$ with $(\sqcap)$ standing for finite intersection.

Lemma 3.11. *Let (X, τ_1, τ_2) be an LM-fbts. Then $\phi : L^X \longrightarrow M$ with $\phi \leqslant \tau_1 \vee \tau_2$ is the base of τ_i iff $\phi^{(\sqcup)}(1_X) = 1$.*

4. Different types of $\tau_1\tau_2$-open sets and $\tau_1\tau_2$-closed sets

In this section, we introduces different types of closed and open sets in an LM-fuzzy bitopological space and studies the interrelations between them. Also, the algebraic structures associated with the collections of these sets are investigated.

Definition 4.1. Let (X, τ_1, τ_2) be an LM-fbts. For $r \in M$, a fuzzy set A is called
(i) $\tau_1\tau_2$-*r-fuzzy open*, if $\tau_i(A) \geq r, i \in \{1, 2\}$,
(ii) $\tau_1\tau_2$-*r-fuzzy closed*, if $\tau_i(A^{'}), i \in \{1, 2\}$. is $\tau_1\tau_2$-r-fuzzy open.
The family of all $\tau_1\tau_2$-r-fuzzy open sets are denoted by $\tau_1\tau_2 - O_r$ and the family of all $\tau_1\tau_2$-r-fuzzy closed sets are denoted by $\tau_1\tau_2 - C_r$.

Theorem 4.2. *Let $\tau_i : L^X \longrightarrow M, i \in \{1, 2\}$ be a function. Then τ_i is an LM-fuzzy bitopology on X iff $\tau_1\tau_2 - O_r$ is an L-bitopology on X for every $r \in M$.*

Remark 4.3. Clearly, $\tau_1\tau_2 - Cl(A, r)$ gives the $\tau_1\tau_2-$-closure of A in the L-bitopology $\tau_1\tau_2 - O_r$.

Theorem 4.4. *Let (X, τ_1, τ_2) be an LM-fbts. Then*
(1) $\tau_1\tau_2 - O_r$ and $\tau_1\tau_2 - C_r$ are lattices,
(2) $\tau_1\tau_2 - O_r$ is a complete join-semilattice and $\tau_1\tau_2 - C_r$ is a complete meet-semilattice,
(3) $\tau_1\tau_2 - O_r$ and $\tau_1\tau_2 - C_r$ are monoids.

Remark 4.5. However, $\tau_1\tau_2 - O_r$ need not be a complete meet-semilattice and $\tau_1\tau_2 - C_r$ need not be a complete join-semilattice. Let $X = L = I, M = I \times I$ with partial ordering defined by $r_1 \leq r_2$ and $s_1 \leq s_2$ and $\mathbb{V} = \{\underline{\alpha} : \alpha = \dfrac{1}{4} + \frac{1}{n}, n \in \mathbb{N}\backslash\{1\}\}$. Now, consider the LM-fuzzy bitopology τ_i defined by

$$\tau_1(B) = \tau_2(B) = \begin{cases} (1,1) & if \quad B \in \{\underline{0}, \underline{1}\} \\ (\alpha, \frac{1}{4}) & if \quad B = \underline{\alpha} \in \mathbb{V} \\ 0 & otherwise \end{cases}$$

Then for $r = (\frac{1}{4}, \frac{1}{4})$, $\tau_1\tau_2 - O_r = \mathbb{V} \cup \{\underline{0}, \underline{1}\}$ which is not closed under arbitrary meet since $\bigwedge_{B \in \mathbb{V}} B = (\underline{\frac{1}{4}}) \notin \tau_1\tau_2 - O_r$. Hence, $\tau_1\tau_2 - O_r$ is not a complete meet-semilattice and consequently, $\tau_1\tau_2 - C_r$ is not a complete join-semilattice.

For any $r \in M$, let $\tau_1\tau_2 - L_r = \tau_1\tau_2 - O_r \cap \tau_1\tau_2 - C_r$. Then, $\tau_1\tau_2 - L_r$ is a lattice containing $\underline{0}$ and $\underline{1}$ and if $A \in \tau_1\tau_2 - L_r$ then $A^{'} \in \tau_1\tau_2 - L_r$.

Theorem 4.6. *Let (X, τ_1, τ_2) be an LM-fbts. Then for any $r \in M$, $\tau_1\tau_2 - L_r = \tau_1\tau_2 - O_r \cap \tau_1\tau_2 - C_r$ is a De Morgan algebra.*

Theorem 4.7. *Let (X, τ_1, τ_2) be an LM-fbts where L is a chain. Then $\tau_1\tau_2 - L_r$ is a Boolean algebra iff $\tau_1\tau_2 - L_r \subseteq \{\chi_G : G \subseteq X\}$.*

Proof. Suppose $\tau_1\tau_2 - L_r$ is a Boolean algebra and let $A \in \tau_1\tau_2 - L_r$. If there exists $x \in X$ such that $0 < A(x) < 1$, then for every $B \in L^X$, either $(A \vee B) = \underline{1}$ or $(A \wedge B) = \underline{0}$ which contradicts the existence of complement for A in $\tau_1\tau_2 - L_r$. Consequently, $\tau_1\tau_2 - L_r \subseteq \{\chi_G : G \subseteq X\}$.

Conversely, suppose $\tau_1\tau_2 - L_r \subseteq \{\chi_G : G \subseteq X\}$ By Theorem 4.3, $\tau_1\tau_2 - L_r$ is a De-Morgan algebra with an order reversing involution $'$. Also, $\underline{0}, \underline{1} \in \tau_1\tau_2 - L_r$ with $A \wedge \underline{1} = A$ and $A \vee \underline{0} = A$ for every $A \in \tau_1\tau_2 - L_r$. □

Remark 4.8. Though $\tau_1\tau_2 - L_r$ is a lattice, it is is neither atomic nor dual atomic. Besides, it need not be a complete lattice.

Theorem 4.9. *Let (X, τ_1, τ_2) be an LM-fbts. If $\tau_1\tau_2 - O_r$ is closed under the order reversing involution defined in L^X then $\tau_1\tau_2 - L_r$ is a complete lattice and hence an L-bitopology.*

Remark 4.10. The converse of the above Theorem 4.5 is not true.

Theorem 4.11. *Let $G \subseteq L^X$ be a complete De Morgan algebra. Then for any non-trivial completely distributive lattice M, there exists an LM-fuzzy bitopology τ_i on X such that $G = \tau_1\tau_2 - L_r$ for some non-zero $r \in M$.*

Proof. Let M be any non-trivial completely distributive lattice and $0 \neq r \in M$. Then, define an LM-fuzzy bitopology τ_i on X as follows:

$$\tau_1(A) = \tau_2(A) = \begin{cases} 1 & if \quad A \in \{\underline{0}, \underline{1}\} \\ r & if \quad B \in A\backslash\{\underline{0}, \underline{1}\} \\ 0 & otherwise \end{cases}$$

Clearly, $\tau_1\tau_2 - O_r = \tau_1\tau_2 - C_r = \tau_1\tau_2 - L_r = A$. □

Definition 4.12. Let (X, τ_1, τ_2) be an LM-fbts, $A \in L^X$ and $r \in M$. Then A is called

(i) *$\tau_1\tau_2$-r-regular fuzzy open* (or *$\tau_1\tau_2$-r-rfo*) *set*, if $A = \tau_1 - Int(\tau_2 - Cl(A, r), r)$,

(ii) *$\tau_1\tau_2$-r-regular fuzzy closed* (or *$\tau_1\tau_2$-r-rfc*) *set*, if A' is $\tau_1\tau_2$-r-rfo,

(iii) *$\tau_1\tau_2$-r-semi fuzzy open* (or *$\tau_1\tau_2$-r-sfo*) *set*, if $A \leqslant \tau_1 - Cl(\tau_2 - Int(A, r), r)$,

(iv) *$\tau_1\tau_2$-r-semi fuzzy closed* (or *$\tau_1\tau_2$-r-sfc*) *set*, if A' is $\tau_1\tau_2$-r-sfo,

(v) *$\tau_1\tau_2$-r-fuzzy b-open* (or *$\tau_1\tau_2$-r-fbo*) *set*, if $A \leqslant (\tau_1 - Int(\tau_2 - Cl(A, r), r)) \vee (\tau_1 - Cl(\tau_2 - Int(A, r), r))$,

(vi) *$\tau_1\tau_2$-r-fuzzy b-closed* (or *$\tau_1\tau_2$-r-fbc*) *set*, if A' is $\tau_1\tau_2$-r-fbo set.

The family of all $\tau_1\tau_2$-r-regular fuzzy open (resp. $\tau_1\tau_2$-r-semi fuzzy open, $\tau_1\tau_2$-r-fuzzy b-open) are denoted by $\tau_1\tau_2 - RO_r$ (resp. $\tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - bO_r$) and the family of all $\tau_1\tau_2$-r-regular fuzzy closed (resp. $\tau_1\tau_2$-r-semi fuzzy closed, $\tau_1\tau_2$-r-fuzzy b-closed) are denoted by $\tau_1\tau_2 - RC_r$ (resp. $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bC_r$).

Theorem 4.13. *Let (X, τ_1, τ_2) be an LM-fbts, $A \in L^X$ and $r \in M$. Then the following statements hold:*

(1) *every $\tau_1\tau_2$-r-open is $\tau_1\tau_2$-r-semi fuzzy open,*

(2) *every $\tau_1\tau_2$-r-regular fuzzy closed is $\tau_1\tau_2$-r-semi fuzzy open,*

(3) *every $\tau_1\tau_2$-r-semi fuzzy open is $\tau_1\tau_2$-r-b fuzzy open.*

Proof. (1) Let A be an $\tau_1\tau_2$-r-open set. Then $A = \tau_1\tau_2 - Int(A) \leqslant \tau_1 - Cl(\tau_2 - Int(A, r), r)$. Thus A is $\tau_1\tau_2$-r-semi fuzzy open.

(2) Let A be an $\tau_1\tau_2$-r-regular fuzzy closed set. Then $A = \tau_1 - Cl(\tau_2 - Int(A, r), r)$, implies $A \leqslant \tau_1 - Cl(\tau_2 - Int(A, r), r)$. Thus A is $\tau_1\tau_2$-r-semi fuzzy open.

(3) Let A be an $\tau_1\tau_2$-r-semi fuzzy open. Then $A \leqslant \tau_1 - Cl(\tau_2 - Int(A,r),r)$, implies $A \leqslant \tau_1 - Cl(\tau_2 - Int(A,r),r) \vee \tau_1 - Int(\tau_2 - Cl(A,r),r)$. Thus A is $\tau_1\tau_2$-r-b fuzzy open. □

Remark 4.14. $\tau_1\tau_2 - C_r \subseteq \tau_1\tau_2 - SC_r \subseteq \tau_1\tau_2 - bC_r$ and $\tau_1\tau_2 - O_r \subseteq \tau_1\tau_2 - SO_r \subseteq \tau_1\tau_2 - bO_r$.

Theorem 4.15. *Let (X, τ_1, τ_2) be an LM-fbts. Then*
(1) $\tau_1\tau_2 - RO_0 = \tau_1\tau_2 - RC_0 = L^X$,
(2) $\underline{0}, \underline{1} \in \tau_1\tau_2 - RO_r \cap \tau_1\tau_2 - RC_r$.

Theorem 4.16. *Let (X, τ_1, τ_2) be an LM-fbts.*
(1) *If $A \in \tau_1\tau_2 - C_r$, then $\tau_1\tau_2 - Int(A,r) \in \tau_1\tau_2 - RO_r$.*
(2) *If $A \in \tau_1\tau_2 - O_r$, then $\tau_1\tau_2 - Cl(A,r) \in \tau_1\tau_2 - RC_r$.*

Theorem 4.17. *Let (X, τ_1, τ_2) be an LM-fbts. Then $\tau_1\tau_2 - RO_r \cap \tau_1\tau_2 - RC_r = \tau_1\tau_2 - L_r$ for every $r \in M$.*

Proof. We have $\tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - O_r$ and $\tau_1\tau_2 - RC_r \subseteq \tau_i - C_r$. Then $\tau_1\tau_2 - RO_r \cap \tau_1\tau_2 - RC_r \subseteq \tau_1\tau_2 - L_r$. For the reverse implication, let $A \in \tau_1\tau_2 - L_r$. Then $\tau_1 - Int(\tau_2 - Cl(A,r),r) = \tau_1 - Int(A,r) = A$. Again, $\tau_1 - Cl(\tau_2 - Int(A,r),r) = \tau_1 - Cl(A,r) = A$. Thus $A \in \tau_1\tau_2 - RC_r$. □

Theorem 4.18. *Let (X, τ_1, τ_2) be an LM-fbts. Then*
(1) $\tau_1\tau_2 - RC_r \cup \tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - SC_r \cap \tau_1\tau_2 - SO_r$,
(2) $\tau_1\tau_2 - RC_r \cup \tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - bC_r \cap \tau_1\tau_2 - bO_r$,
(3) $\tau_1\tau_2 - SC_r \cup \tau_1\tau_2 - SO_r \subseteq \tau_1\tau_2 - bC_r \cap \tau_1\tau_2 - bO_r$.

Proof. (2) Let $A \in \tau_1\tau_2 - RO_r$. Then $A = \tau_1 - Int(\tau_2 - Cl(A,r),r) = A$ implies $(\tau_1 - Int(\tau_2 - Cl(A,r),r)) \wedge (\tau_1 - Cl(\tau_2 - Int(A,r),r)) \leqslant A$. Thus $A \in \tau_1\tau_2 - bO_r$. Again, if $A \in \tau_1\tau_2 - RO_r$, then $A' \in \tau_1\tau_2 - RC_r \subseteq \tau_1\tau_2 - C_r$. Thus $\tau_1\tau_2 - Cl(A', r) = A'$. Consequently, $\tau_1 - Int(\tau_2 - Cl(A', r), r) \leqslant A'$ and $(\tau_1 - Int(\tau_2 - Cl(A', r), r)) \wedge (\tau_1 - Cl(\tau_2 - Int(A', r), r)) \leqslant A'$. So $A' \in \tau_1\tau_2 - bC_r$ which implies $A \in \tau_1\tau_2 - bO_r$. Hence $\tau_1\tau_2 - RO_r \subseteq (\tau_1\tau_2 - bO_r \cap \tau_1\tau_2 - bC_r)$.

Similarly, $\tau_1\tau_2 - RC_r \subseteq (\tau_1\tau_2 - bO_r \cap \tau_1\tau_2 - bC_r)$.

The proofs of (1) and (3) are similar to (2). □

Theorem 4.19. *Let (X, τ_1, τ_2) be an LM-fbts. Then*
(1) *$\tau_1\tau_2 - RO_r, \tau_1\tau_2 - SC_r$ and $\tau_1\tau_2 - bC_r$ are meet-semilattices,*
(2) *$\tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$ and $\tau_1\tau_2 - bO_r$ are join-semilattices,*
(3) *$\tau_1\tau_2 - RO_r, \tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$ and $\tau_1\tau_2 - bC_r$ are monoids.*

Proof. (1) Let $A_1, A_2 \in \tau_1\tau_2 - RO_r$. Then $\tau_i(A_1), \tau_i(A_2) \geq r$. Thus $\tau_i(A_1 \wedge A_2) > r$. Also, $\tau_2 - Cl(A_1 \wedge A_2, r) \geq A_1 \wedge A_2$ and $\tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r)) \geq \tau_1 - Int(A_1 \wedge A_2, r) = A_1 \wedge A_2$.

Now $A_1 \geq A_1 \wedge A_2$ and $A_1 = \tau_1 - Int(\tau_2 - Cl(A_1, r), r) \geq \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$. Also $A_2 \geq A_1 \wedge A_2$ and $A_2 = \tau_1 - Int(\tau_2 - Cl(A_2, r), r) \geq \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$. So $A_1 \wedge A_2 \geq \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$. Hence $A_1 \wedge A_2 = \tau_1 - Int(\tau_2 - Cl(A_1 \wedge A_2, r), r)$ which shows that$\tau_1\tau_2 - RO_r$ is a meet-semilattice.

Similarly, $\tau_1\tau_2 - SC_r$ and $\tau_1\tau_2 - bC_r$ are meet-semilattices.

(2) Similarly, $\tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$ and $\tau_1\tau_2 - bO_r$ are join-semilattices.

(3) Associativity follows from (1) and (2). Again, $\underline{1} \in \tau_1\tau_2 - RO_r$ and $\underline{0} \in \tau_1\tau_2 - RC_r$ are the identity elements.

Similarly, for the rest. □

Remark 4.20. In general,

(1) $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - RC_r$, $\tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$ and $\tau_1\tau_2 - bC_r$ are not lattices and hence not L-bitopologies,

(2) $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - SC_r$ and $\tau_1\tau_2 - bC_r$ are not a complete meet-semilattices,

(3) $\tau_1\tau_2 - RC_r, \tau_1\tau_2 - SO_r$ and $\tau_1\tau_2 - bO_r$ are not a complete join-semilattices,

(4) The partial ordering in M does not induce any ordering in the collections of $\tau_1\tau_2 - RO_r, \tau_1\tau_2 - RC_r$, $\tau_1\tau_2 - SO_r$, $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - bO_r$ and $\tau_1\tau_2 - bC_r$.

Definition 4.21. Let (X, τ_1, τ_2) be an LM-fbts, $r \in M$ and $A \in L^X$. Then the *$\tau_1\tau_2$-LM-fuzzy semi closure operator* is defined by

$$\tau_1\tau_2 - SCl(A, r) = \bigwedge\{F \in L^X : A \leqslant F, F \in \tau_1\tau_2 - SC_r\}$$

and the *$\tau_1\tau_2$-LM-fuzzy semi interior operator* is defined by

$$\tau_1\tau_2 - SInt(A, r) = \bigvee\{U \in L^X : U \leqslant A, U \in \tau_1\tau_2 - SO_r\}.$$

Definition 4.22. Let (X, τ_1, τ_2) be an LM-fbts and let $r \in M$, $A \in L^X$. Then the *$\tau_1\tau_2$-LM-fuzzy b-closure operator* is defined by

$$\tau_1\tau_2 - bCl(A, r) = \bigwedge\{F \in L^X : A \leqslant F, F \in \tau_1\tau_2 - bC_r\}$$

and the *$\tau_1\tau_2$-LM-fuzzy b-interior operator* is defined by

$$\tau_1\tau_2 - bInt(A, r) = \bigvee\{U \in L^X : U \leqslant A, U \in \tau_1\tau_2 - bO_r\}.$$

Definition 4.23. Let (X, τ_1, τ_2) be an LM-fbts, $A, B \in L^X$ and $r \in M$. Then A is called

(i) *$\tau_1\tau_2$-r-generalized fuzzy closed* (or *$\tau_1\tau_2$-r-gfc*) *set*, if $\tau_2 - Cl(A, r) \leqslant B$ whenever $A \leqslant B$ and $B \in \tau_1 - O_r$,

(ii) *$\tau_1\tau_2$-r-generalized fuzzy open* (or *$\tau_1\tau_2$-r-gfo*) *set*, if A' is a $\tau_1\tau_2$-r-gfc set,

(iii) *$\tau_1\tau_2$-r-regular generalized fuzzy closed* (or *$\tau_1\tau_2$-r-rgfc*) *set*, if $\tau_2 - Cl(A, r) \leqslant B$ whenever $A \leqslant B$ and $B \in \tau_1 - RO_r$,

(iv) *$\tau_1\tau_2$-r-regular generalized fuzzy open* (or *$\tau_1\tau_2$-r-rgfo*) *set*, if A' is a $\tau_1\tau_2$-r-gfc set,

(v) *$\tau_1\tau_2$-r-generalized fuzzy semi-closed* (or *$\tau_1\tau_2$-r-gfsc*) *set*, if $\tau_2 - SCl(A, r) \leqslant B$ whenever $A \leqslant B$ and $B \in \tau_1 - O_r$,

(vi) *$\tau_1\tau_2$-r-generalized fuzzy semi-open* (or *$\tau_1\tau_2$-r-gfso*) *set*, if A' is a $\tau_1\tau_2$-r-gfsc set,

(vii) *$\tau_1\tau_2$-r-generalized fuzzy b-closed* (or *$\tau_1\tau_2$-r-gfbc*) *set*, if $\tau_2 - bCl(A, r) \leqslant B$ whenever $A \leqslant B$ and $B \in \tau_1 - O_r$,

(viii) *$\tau_1\tau_2$-r-generalized fuzzy b-open* (or *$\tau_1\tau_2$-r-gfbo*) *set*, if A' is a $\tau_1\tau_2$-r-gfbc set.

The family of all $\tau_1\tau_2$-r-generalized fuzzy closed (resp. $\tau_1\tau_2$-r-regular generalized fuzzy closed, $\tau_1\tau_2$-r-generalized fuzzy semi-closed, $\tau_1\tau_2$-r-generalized fuzzy b-closed) are denoted by $\tau_1\tau_2 - GC_r$ (resp. $\tau_1\tau_2 - RGC_r$, $\tau_1\tau_2 - GSC_r$, $\tau_1\tau_2 - GbC_r$) and the family of all $\tau_1\tau_2$-r-generalized fuzzy open (resp. $\tau_1\tau_2$-r-regular generalized fuzzy open, $\tau_1\tau_2$-r-generalized fuzzy semi-open, $\tau_1\tau_2$-r-generalized fuzzy b-open) are denoted by $\tau_1\tau_2 - GO_r$ (resp. $\tau_1\tau_2 - RGO_r$, $\tau_1\tau_2 - GSO_r$, $\tau_1\tau_2 - GbO_r$).

Theorem 4.24. *Let (X, τ_1, τ_2) be an LM-fbts. Then*
(1) $\tau_1\tau_2 - RC_r \subseteq \tau_1\tau_2 - C_r \subseteq \tau_1\tau_2 - GC_r$ *and* $\tau_1\tau_2 - RO_r \subseteq \tau_1\tau_2 - O_r \subseteq \tau_1\tau_2 - GO_r$,
(2) $\tau_1\tau_2 - GC_r \subseteq \tau_1\tau_2 - RGC_r \subseteq \tau_1\tau_2 - GSC_r \subseteq \tau_1\tau_2 - GbC_r$ *and* $\tau_1\tau_2 - GO_r \subseteq \tau_1\tau_2 - RGO_r \subseteq \tau_1\tau_2 - GSO_r \subseteq \tau_1\tau_2 - GbO_r$.

Theorem 4.25. *Let (X, τ_1, τ_2) be an LM-fbts. Then*
(1) $\tau_1\tau_2 - GO_r, \tau_1\tau_2 - RGO_r, \tau_1\tau_2 - GSC_r$ *and* $\tau_1\tau_2 - GbC_r$ *are meet-semilattices,*
(2) $\tau_1\tau_2 - GC_r, \tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSO_r$ *and* $\tau_1\tau_2 - GbO_r$ *are join-semilattices,*
(2) $\tau_1\tau_2 - GO_r, \tau_1\tau_2 - GC_r, \tau_1\tau_2 - RGO_r, \tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSO_r$, $\tau_1\tau_2 - SC_r$, $\tau_1\tau_2 - GbO_r$ *and* $\tau_1\tau_2 - GbC_r$ *are monoids.*

Proof. (1) Let $A_1, A_2 \in \tau_1\tau_2 - GO_r$. Then $A_1^{'}, A_2^{'} \in \tau_1\tau_2 - GC_r$. Thus $A_1^{'} \vee A_2^{'} \in \tau_1\tau_2 - GC_r$. So $(A_1 \wedge A_2)^{'} \in \tau_1\tau_2 - GC_r$. Hence $(A_1 \wedge A_2) \in \tau_1\tau_2 - GO_r$. Therefore $\tau_1\tau_2 - GO_r$ is a meet semi-lattice.

Similarly, $\tau_1\tau_2 - RGO_r, \tau_1\tau_2 - GSC_r$ and $\tau_1\tau_2 - GbC_r$ are meet-semilattices.

(2) Let $A_1, A_2 \in \tau_1\tau_2 - GC_r$ and $B \in \tau_1\tau_2 - O_r$ such that $A_1 \vee A_2 \leqslant B$. Since $A_1 \leqslant B$ and $A_1 \in \tau_1\tau_2 - GC_r, \tau_1\tau_2 - Cl(A_1, r) \leqslant B$. Similarly, $\tau_1\tau_2 - Cl(A_2, r) \leqslant B$. Then $\tau_1\tau_2 - Cl(A_1 \vee A_1, r) = \tau_1\tau_2 - Cl(A_1, r) \vee \tau_1\tau_2 - Cl(A_2, r) \leqslant B$. Thus $\tau_1\tau_2 - GC_r$ is a join semi-lattice.

Similarly, $\tau_1\tau_2 - RGC_r, \tau_1\tau_2 - GSO_r$ and $\tau_1\tau_2 - GbO_r$ are join-semilattices.

(3) Associativity follows from (1), (2) and $\underline{1} \in \tau_1\tau_2 - GC_r$ and $\underline{0} \in \tau_1\tau_2 - GO_r$ are the identity elements.

Similarly, for the rest. □

Theorem 4.26. *Let (X, τ_1, τ_2) be an LM-fbts.*
(1) *If $A \in \tau_1\tau_2 - GSC_r$, then $B \in \tau_1\tau_2 - GSC_r$ for all B such that $A \leqslant B \leqslant \tau_1\tau_2 - SC_r(A, r)$.*
(2) *If $A \in \tau_1\tau_2 - GbC_r$, then $B \in \tau_1\tau_2 - GbC_r$ for all B such that $A \leqslant B \leqslant \tau_1\tau_2 - bC_r(A, r)$.*

Proof. (2) Let A is an $\tau_1\tau_2$-r-gfbc and consider $B \in L^X$ such that $A \leqslant B \leqslant \tau_1\tau_2 - bC_r(A, r)$. Also, let C be a an $\tau_1\tau_2$-r-fo in L^X such that $B \leqslant C$. Then clearly, $A \leqslant C$ and $\tau_1\tau_2 - bC_r(A, r) \leqslant C$. Again, note that $\tau_1\tau_2 - bC_r(B, r) = \tau_1\tau_2 - bC_r(A, r)$. Thus $\tau_1\tau_2 - bC_r(B, r) \leqslant C$. So B is an $\tau_1\tau_2$-r-gfbc.

(i) The proof is similar to (1). □

Theorem 4.27. *Let (X, τ_1, τ_2) be an LM-fbts.*
(1) *If $A \in \tau_1\tau_2 - GSC_r$, then for every $B \in \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r(A, r)\overline{q}B$ iff $A\overline{q}B$.*
(2) *If $A \in \tau_1\tau_2 - GbC_r$, then for every $B \in \tau_1\tau_2 - bO_r, \tau_1\tau_2 - bC_r(A, r)\overline{q}B$ iff $A\overline{q}B$.*

Proof. (2) Let $B \in \tau_1\tau_2 - bO_r$ for some $r \in M$ and $A\overline{q}B$ for some $A \in LX$. Then $A \leqslant B^{'}$. Since $B^{'}$ is an $\tau_1\tau_2$-*r*-fbc set of L^X and A is an $\tau_1\tau_2$-*r*-gfbc set, $\tau_1\tau_2 - bC_r(A, r)\overline{q}B$.

Conversely, let B be a $\tau_1\tau_2$-r-fbc set of L^X such that $A \leqslant B, r \in M$. Then $A\overline{q}B^{'}$. But $\tau_i - bC_r(A,r)\overline{q}B^{'}$. Thus $\tau_1\tau_2 - bC_r(A,r) \leqslant B$. So A is an $\tau_1\tau_2$-r-gfbc.

(2) The proof is similar to (1). □

Remark 4.28. Let $\mathcal{O}$ be the collection of all $\tau_1\tau_2 - O_r$, $\mathcal{C}$ be the collection of all $\tau_1\tau_2 - C_r$ and $\mathcal{L}$ be the collection of all $\tau_1\tau_2 - L_r$ in an LM-fbts (X, τ_1, τ_2). Then $\mathcal{O}$, $\mathcal{C}$ and $\mathcal{L}$ are bounded lattices where the bounds for $\mathcal{O}$ are $\tau_1\tau_2 - O_0$ and $\tau_1\tau_2 - O_1$, the bounds of $\mathcal{C}$ are $\tau_1\tau_2 - C_0$ and $\tau_1\tau_2 - C_1$ and that of $\mathcal{L}$ are $\tau_1\tau_2 - L_0$ and $\tau_1\tau_2 - L_1$.

Remark 4.29. The lattices $\mathcal{O}$, $\mathcal{C}$ and $\mathcal{L}$ are neither atomic nor dual atomic. For example, let $X = \{a, b, c\}, L = M = I$ and define an LM-fuzzy bitopology on X as follows:

$$\tau_1(A) = \tau_2(A) = \begin{cases} 1 & if \quad A \in \{\underline{0}, \underline{1}\} \\ \alpha & if \quad A = \underline{\alpha}, \alpha \in I \backslash \{0,1\} \\ 0 & otherwise. \end{cases}$$

Clearly for $\alpha \in I$, $\tau_1\tau_2 - O_\alpha = \{\underline{0}\} \cup \{\beta : \beta \geq \alpha, \beta \in I\}$. Then it follows that $\mathcal{O} = \mathcal{C} = \mathcal{L} = \{\tau_1\tau_2 - O_\alpha : \alpha \in I\}$ which is neither atomic nor dual atomic.

Theorem 4.30. *Let (X, τ_1, τ_2) be an LM-fbts. Then*

(1) *$\mathcal{O}$, $\mathcal{C}$ and $\mathcal{L}$ are dual atomic, if M is atomic,*

(2) *$\mathcal{O}$, $\mathcal{C}$ and $\mathcal{L}$ are atomic, if M is dual atomic.*

Remark 4.31. Atoms and dual atoms may exist in $\mathcal{O}$, $\mathcal{C}$ and $\mathcal{L}$ without M being atomic or dual atomic. For example, let $X = \mathbb{R}$, the set of real numbers and $L = M = I$. Clearly, M is neither atomic nor dual atomic. Now, define an LM-fuzzy bitopology on X as follows:

$$\tau_1(A) = \tau_2(A) = \begin{cases} 1 & if \quad A \in \{\underline{0}, \underline{1}\} \\ \alpha & if \quad A = \underline{\alpha}, \alpha \in (\frac{1}{4}, \frac{2}{3}] \\ 0 & otherwise. \end{cases}$$

Then $\mathcal{O} = \{\tau_1\tau_2 - O_0, \tau_1\tau_2 - O_1\} \cup \{\tau_1\tau_2 - O_\alpha : \alpha \in [\frac{1}{4}, \frac{2}{3}]\}$ and $\mathcal{C} = \{\tau_1\tau_2 - C_0, \tau_1\tau_2 - C_1\} \cup \{\tau_1\tau_2 - C_\alpha : \alpha \in [\frac{1}{4}, \frac{2}{3}]\}$. Also $\mathcal{L} = \{\tau_1\tau_2 - L_0, \tau_1\tau_2 - L_1\} \cup \{\tau_i - L_\alpha : \alpha \in [\frac{1}{4}, \frac{2}{3}]\}$.

5. Conclusions

As a result of the study, we have identified certain monoids that are subsets of L^X. Some of them are distributive lattices too. Now, let $\tau_1\tau_2 - \Omega_r = \{\tau_1\tau_2 - L_r, \tau_1\tau_2 - O_r, \tau_1\tau_2 - C_r, \tau_1\tau_2 - GO_r, \tau_1\tau_2 - GC_r, \tau_1\tau_2 - RGO_r, \tau_1\tau_2 - RGC_r, \tau_1\tau_2 - SO_r, \tau_1\tau_2 - SC_r, \tau_1\tau_2 - bO_r, \tau_1\tau_2 - bC_r, L^X\}$. Then $\tau_1\tau_2 - \Omega_r$ is a lattice under set inclusion whose elements are all monoids. The Hasse diagram of this lattice is shown in Figure 1.

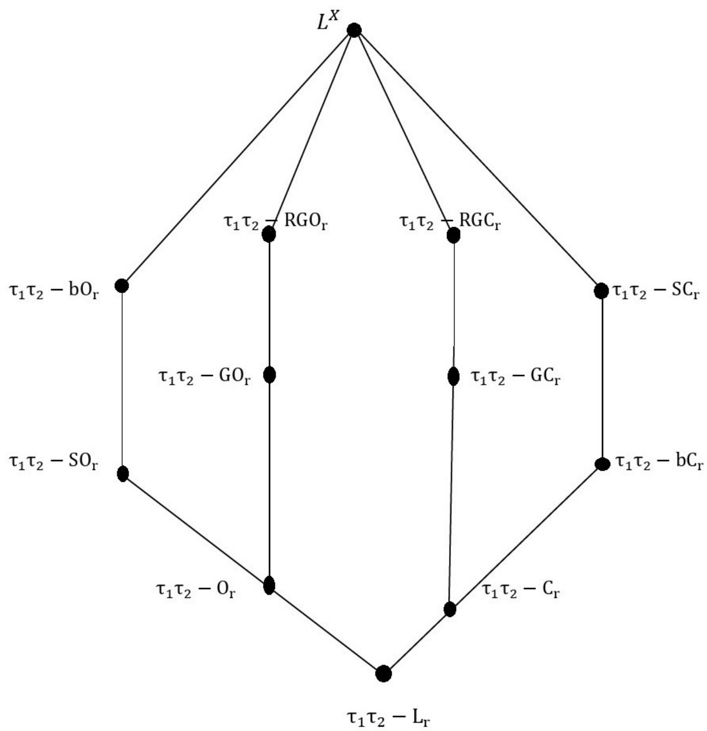

FIGURE 1. Hasse diagram of $\tau_1\tau_2 - \Omega_r$

It is clear that $\tau_1\tau_2 - \Omega_r$ is associative, complemented but not modular. Thus, by introducing various notions of openness and closedness in LM-fuzzy bitopological spaces, the study associates to each element of M, a lattice of monoids that are subsets of L^X.

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FADHIL ABBAS (fadhilhaman@gmail.com)
Salzburger Straβe 195, Linz, Austria