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Abstract. Our aim of the research is to study two aspects: First, we define new concept (called an *interval-valued soft set*) which combines an interval-valued set with a soft set, and discuss with its algebraic structures and give some examples. Second, we investigate basic topological structures based on interval-valued soft set, for example, subspace, base and subbase, neighborhood, closure and interior, and give some examples.

2020 AMS Classification: 54A40

Keywords: Interval-valued soft set, Interval-valued soft point, Interval-valued soft topological space, Interval-valued soft base and subbase, Interval-valued soft subspace, Interval-valued soft neighborhood, Interval-valued soft closure and interior.

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1. Introduction

In the real world, there are many complicated problems in dealing with economics, engineering, medical science, social science, etc., being highly dependent on the task of modeling uncertain data. To solve successfully undefinable or complex problems, some researchers had proposed various concepts, for example, probabilities, fuzzy sets [1], interval-valued fuzzy sets [2, 3], rough sets [4], intuitionistic fuzzy sets [5], interval-valued intuitionistic fuzzy sets [6] and vague sets [7]. However, to overcome the inherent difficulties of each of these concepts, Molodtsov [8] introduced the notion of soft sets which has rich potential for practical applications in several domains as a tool for dealing with uncertainties. After that time, Maji et al. [9] proposed some basic operations on soft sets and studied some of their properties (See [10, 11, 12] for the further researches). Aktaş and Çağman [13], Feng et al. [14], U. Acar et al. [15], and Sun et al. [16] applied soft sets to group theory, semiring theory, ring theory and module theory, respectively. Jun [17] dealt with soft BCK/BCI-algebras (Refer to [18, 19] for the more researches). Majumdar and Samanta [20] defined similarity measure based on soft sets and found some of its properties. Çağman and Enginoglu [21] proposed a *uni-int* decision making method. Also They [22] dealt with the soft *max-min* decision making method. On the other hand, Many researchers [23, 24, 25, 26, 27, 28, 29, 30, 31] introduced and studied topological structures via soft sets over a universe set with a fixed set of parameters. Recently, Debnath and Tripathy [32] introduced the notion of soft bitopological spaces and dealt with separation axioms in a soft bitopological space. Also, few researchers [33, 34, 35, 36, 37, 38] investigated soft topological groups, rings and modules.

Topology is an important area of mathematics with many applications in the domains of computer and physical science. Recently, Kim et al. [39] studied topological structures based on interval-valued sets as the generalization of classical sets and the special case of interval-valued fuzzy sets introduced by Zadeh [2].

We intend to study in the following two aspects: First, as a new tool to solve complex problems, we define an interval-valued soft set that combines a soft set and an interval-valued set, and study their algebraic structures. Second, we study topological structures based on interval-valued soft sets. In order to accomplish our aim, this paper is composed of five sections. In Section 2, we recall some definitions of interval-valued sets introduced by Yao [40] and three results obtained by Kim et al. [39]. Also, we recall some operations on soft sets. In Section 3, we define an interval-valued soft set and obtain its several properties. In Section 4, we introduce the concept of interval-valued soft topological spaces and find some of their properties, and give some examples. In Section 5, we define an interval-valued soft neighborhood of two types and interval-valued soft closure (interior), and deal with some of their properties.

2. Preliminaries

In this section, we recall basic concepts and three results related to interval-valued sets introduced by Yao [40] and Kim et al. [39]. Also, we recall operations for soft sets in [8, 9]. Throughout this section and the next sections, let $X,\ Y,\ Z,\cdots$ be non-empty universe sets, let $E,\ E',\ E'',\cdots$ be non-empty sets of parameters and let 2^X be the power set of X.

Definition 2.1 ([39, 40])**.** The form

$$[A^-, A^+] = \{B \subset X : A^- \subset B \subset A^+\}$$

is called an *interval-valued set* (briefly, IVS) or *interval set* in X, if $A^-,\ A^+ \subset X$ and $A^- \subset A^+$. In this case, A^- [resp. A^+] represents the set of minimum [resp. maximum] memberships of elements of X to A. In fact, A^- [resp. A^+] is a minimum [resp. maximum] subset of X agreeing or approving for a certain opinion, view, suggestion or policy. $[\varnothing, \varnothing]$ [resp. $[X, X]$] is called the *interval-valued empty* [resp. *whole*] *set* in X and denoted by $\widetilde{\varnothing}$ [resp. $\widetilde{X}$]. We will denote the set of all IVSs in X as $IVS(X)$.

It is obvious that $[A, A] \in IVS(X)$ for a classical subset A of X. Then we can consider an IVS in X as the generalization of a classical subset of X. Furthermore, if $A = [A^-, A^+] \in IVS(X)$, then

$$\chi_A = [\chi_{A^-}, \chi_{A^+}]$$

is an interval-valued fuzzy set in X introduced by Zadeh [2], where χ_A denotes the characteristic function of A. Thus we can consider an interval-valued fuzzy set as the generalization of an IVS.

Definition 2.2 ([39, 40])**.** Let $A,\ B \in IVS(X)$. Then
(i) we say that A *contained in* B, denoted by $A \subset B$, if $A^- \subset B^-$ and $A^+ \subset B^+$,
(ii) we say that A *equals to* B, denoted by $A = B$, if $A \subset B$ and $B \subset A$,
(iii) the *complement* of A, denoted A^c, is an interval-valued set in X defined by:

$$A^c = [(A^+)^c, (A^-)^c],$$

(iv) the *union* of A and B, denoted by $A \cup B$, is an interval-valued set in X defined by:

$$A \cup B = [A^- \cup B^-, A^+ \cup B^+],$$

(v) the *intersection* of A and B, denoted by $A \cap B$, is an interval-valued set in X defined by:

$$A \cap B = [A^- \cap B^-, A^+ \cap B^+].$$

The followings are (i1), (i2), (i3), (k1), (k2) and (k3) in [40].

Result 2.3. *Let $A,\ B,\ C \in IVS(X)$. Then*
(1) $\widetilde{\varnothing} \subset A \subset \widetilde{X}$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(4) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

The followings are (I1)–(I8) in [40].

Result 2.4. *Let $A,\ B,\ C \in IVS(X)$. Then*
(1) (Idempotent laws) $A \cup A = A,\ A \cap A = A$,
(2) (Commutative laws) $A \cup B = B \cup A,\ A \cap B = B \cap A$,
(3) (Associative laws) $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws) $A \cup (A \cap B) = A,\ A \cap (A \cup B) = A$,
(6) (DeMorgan's laws) $(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8_a) $A \cup \widetilde{\varnothing} = A,\ A \cap \widetilde{\varnothing} = \widetilde{\varnothing}$,
(8_b) $A \cup \widetilde{X} = \widetilde{X},\ A \cap \widetilde{X} = A$,
(8_c) $\widetilde{X}^c = \widetilde{\varnothing},\ \widetilde{\varnothing}^c = \widetilde{X}$,
(8_d) $A \cup A^c \neq \widetilde{X},\ A \cap A^c \neq \widetilde{\varnothing}$ *in general* (See Example 3.7 in [39]).

Definition 2.5 ([39])**.** Let $(A_j)_{j \in J}$ be a family of members of $IVS(X)$. Then

(i) the *intersection* of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$, is an IVS in X defined by:

$$\bigcap_{j\in J} A_j = [\bigcap_{j\in J} A_j^-, \bigcap_{j\in J} A_j^+],$$

(ii) the *union* of $(A_j)_{j\in J}$, denoted by $\bigcup_{j\in J} \widetilde{A}_j$, is an IVS in X defined by:

$$\bigcup_{j\in J} A_j = [\bigcup_{j\in J} A_j^-, \bigcup_{j\in J} A_j^+].$$

Result 2.6 (Proposition 3.9, [39]). *Let $A \in IVS(X)$ and let $(A_j)_{j\in J}$ be a family of members of $IVS(X)$. Then*
(1) $(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c,\ (\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$,
(2) $A \cap (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}(A \cap A_j),\ A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$.

Definition 2.7 ([39]). Let $a \in X$ and let $A \in IVS(X)$. Then the form $[\{a\}, \{a\}]$ [resp. $[\varnothing, \{a\}]$] is called an *interval-valued* [resp. *vanishing*] *point* in X and denoted by $a_{_{IVP}}$ [resp. $a_{_{IVVP}}$]. We denote the set of all interval-valued points in X as $IV_P(X)$.
(i) We say that $a_{_{IVP}}$ *belongs to* A, denoted by $a_{_{IVP}} \in A$, if $a \in A^-$.
(ii) We say that $a_{_{IVVP}}$ *belongs to* A, denoted by $a_{_{IVVP}} \in A$, if $a \in A^+$.

Result 2.8 (Proposition 3.11, [39]). *Let $A \in IVS(X)$. Then*

$$A = A_{_{IVP}} \cup A_{_{IVVP}},$$

where $A_{_{IVP}} = \bigcup_{a_{_{IVP}}\in A} a_{_{IVP}}$ *and* $A_{_{IVVP}} = \bigcup_{a_{_{IVVP}}\in A} a_{_{IVVP}}$.
In fact, $A_{_{IVP}} = [A^-, A^-]$ *and* $A_{_{IVVP}} = [\varnothing, A^+]$

For a set X, let $IVS^*(X) = \{A \in IVS(X) : A^- = A^+\}$. Then from the above Result, $A = A_{_{IVP}}$ for each $A \in IVS^*(X)$.

Result 2.9 (Theorem 3.14, [39]). *Let $(A_j)_{j\in J} \subset IVS(X)$ and let $a \in X$.*
(1) $a_{_{IVP}} \in \bigcap A_j$ *[resp.* $a_{_{IVVP}} \in \bigcap A_j$*] if and only if* $a_{_{IVP}} \in A_j$ *[resp.* $a_{_{IVVP}} \in A_j$*], for each* $j \in J$.
(2) $a_{_{IVP}} \in \bigcup A_j$ *[resp.* $a_{_{IVVP}} \in \bigcup A_j$*] if and only if there exists* $j \in J$ *such that* $a_{_{IVP}} \in A_j$ *[resp.* $a_{_{IVVP}} \in A_j$.

Result 2.10 (Theorem 3.15, [39]). *Let $A,\ B \in IVS(X)$. Then*
(1) $A \subset B$ *if and only if* $a_{_{IVP}} \in A \Rightarrow a_{_{IVP}} \in B$ *[resp.* $a_{_{IVVP}} \in A \Rightarrow a_{_{IVVP}} \in B$*] for each* $a \in X$.
(2) $A = B$ *if and only if* $a_{_{IVP}} \in A \Leftrightarrow a_{_{IVP}} \in B$ *[resp.* $a_{_{IVVP}} \in A \Leftrightarrow a_{_{IVVP}} \in B$*] for each* $a \in X$.

Definition 2.11 ([39]). Let τ be a non-empty family of IVSs on X. Then τ is called an *interval-valued topology* (briefly, IVT) on X, if it satisfies the following axioms:
(IVO_1) $\widetilde{\varnothing},\ \widetilde{X} \in \tau$,
(IVO_2) $A \cap B \in \tau$ for any $A,\ B \in \tau$,
(IVO_3) $\bigcup_{j\in J} A_j \in \tau$ for any family $(A_j)_{j\in J}$ of members of τ.
In this case, the pair (X, τ) is called an *interval-valued topological space* (briefly, IVTS) and each member of τ is called an *interval-valued open set* (briefly, IVOS)

in X. An IVS A is called an *interval-valued closed set* (briefly, IVCS) in X, if $A^c \in \tau$.

It is obvious that $\{\widetilde{\varnothing}, \widetilde{X}\}$ is an IVT on X, and is called the interval-valued indiscrete topology on X and denoted by $\tau_{_{IV,0}}$. Also $IVS(X)$ is an IVT on X, and is called the *interval-valued discrete topology* on X and denoted by $\tau_{_{IV,1}}$. The pair $(X, \tau_{_{IV,0}})$ [resp. $(X, \tau_{_{IV,1}})$] is called the *interval-valued indiscrete* [resp. *discrete*] *space*.

We denote the set of all IVTs on X as $IVT(X)$. For an IVTS X, we denote the set of all IVOSs [resp. IVCSs] in X as $IVO(X)$ [resp. $IVC(X)$].

Definition 2.12 ([39])**.** Let $\tau_1,\ \tau_2 \in IVT(X)$. Then we say that τ_1 *is contained in* τ_2 or τ_1 *is coarser than* τ_2 or τ_2 *is finer than* τ_1, if $\tau_1 \subset \tau_2$, i.e., $A \in \tau_2$ for each $A \in \tau_1$.

It is obvious that $\tau_{_{IV,0}} \subset \tau \subset \tau_{_{IV,1}}$ for each $\tau \in IVT(X)$.

Definition 2.13 ([8, 24])**.** An F_A is called a *soft set* over X, if $F_A : A \to 2^X$ is a mapping such that $F_A(e) = \varnothing$ for each $e \notin A$, where $A \subset X$.

In other words, a soft set over X is a parametrized family of subsets of X. For each $e \in A$, $F_A(e)$ may be considered as the set of e-approximate elements of the soft set F_A. It is clear that a soft set is not a set. We will denote the set of all soft sets over X as $SS(X)$.

It was well-known [8] that every Zadeh's fuzzy set A may be considered as the soft set $F_{[0,1]}$.

Definition 2.14 ([9, 24])**.** Let $F_A,\ F_B) \in SS(X)$. Then we say that:

(i) F_A is a *soft subset* of F_B, denoted by $F_A \widetilde{\subset} F_B$, if $A \subset B$ and $F_A(e) \subset F_B(e)$ for each $e \in A$,

(ii) F_A is a *soft super set* of F_B, denoted by $F_A \widetilde{\supset} F_B$, if $F_B \widetilde{\subset} F_A$,

(iii) F_A and F_B are *soft equal*, if $F_A \widetilde{\subset} F_B$ and $F_A \widetilde{\supset} F_B$.

Definition 2.15 ([9])**.** Let $E = \{e_1, e_2, \cdots, e_n\}$ be a set of parameters. Then the *NOT* set of E, denoted by $\rceil E$, is defined by:

$$\rceil E = \{\urcorner e_1, \urcorner e_2, \cdots, \urcorner e_n\}\},$$

where $\urcorner e_i = not\ e_i$ for each i.

Result 2.16 (Proposition 2.1, [9])**.** *Let $A,\ B \subset E$. Then*

(1) $\rceil(\rceil A) = A$,

(2) $\rceil(A \cup B) = \rceil A \cup \rceil B$,

(3) $\rceil(A \cap B) = \rceil A \cap \rceil B$.

Definition 2.17 ([9])**.** Let $F_A \in SS(X)$. Then the *complement* of F_A, denoted by $F^{'}_A$, is defined by:

$$F^{'}_A = F^{'}_{\rceil A},$$

where $F^{'}_{\rceil A} : \rceil A \to 2^X$ is a mapping given by $F^{'}_{\rceil A}(\alpha) = X - F_A(\urcorner \alpha)$ for each $\alpha \in \rceil A$.

It is obvious that $(F^{'}_A)^{'} = F_A$.

Definition 2.18 ([9, 10])**.** Let $F_A \in SS(X)$. Then F_A is called:

(i) a *null soft set* or a *relative null soft set* (with respect to A), denoted by $\varnothing_A$, if $F_A(e) = \varnothing$ for each $e \in A$,

(ii) an *absolute soft set* or a *relative whole soft set* (with respect to A), denoted by X_A, if $F_A(e) = X$ for each $e \in A$.

3. Interval-valued soft sets

In this section, we define an interval-valued soft set and some operations between interval-valued soft sets, and deal with some of their properties. In this section, unless otherwise stated, $A,\ B,\ C,\ \cdots$ represent a subset of E.

Definition 3.1. An $\mathbf{F}_A = [F^-_A, F^+_A]$ is called an *interval-valued soft set* (briefly, IVSS) over X, if $\mathbf{F}_A : A \to IVS(X)$ is a mapping such that $\mathbf{F}_A(e) = \widetilde{\varnothing}$ for each $e \notin A$, i.e. , $F^-_A,\ F^+_A \in SS(X)$ such that $F^-_A(e) \subset F^+_A(e)$ for each $e \in A$.

In other words, an IVSS over X is a parametrized family of IVSs of X. For each $e \in A$, $\mathbf{F}_A(e) = [F^-_A(e), F^+_A(e)]$ may be considered as an interval-valued set of e-approximate elements of the IVSS $\mathbf{F}_A$. We denote the set of all IVSSs over X as $IVSS(X)$.

it is obvious that if $F_A \in SS(X)$, then $[F_A, F_A] \in IVSS(X)$. Then we can see that an IVSS is the generalization of a soft set. Moreover, if $\mathbf{F}_A \in IVSS(X)$, then clearly, $\chi_{\mathbf{F}_A}$ is an interval-valued fuzzy soft set (briefly, IVFSS) over X introduced by Yang et al. [41]. Thus an IVSS is the special case of an IVFSS.

Example 3.2. (1) Let X be the set of houses under consideration and let E be the set of parameters, where each parameter is a word or a sentence. Consider E given by:

$$E = \{\text{expensive, beautiful, wooden, cheap, in the surroundings, modern, in good repair, in bad repair}\}.$$

In this case, to define an IVSS $\mathbf{F}_A$ over X means to point out the IVSs composed of the minimal subset and the maximal subsets of *expensive* houses, *beautiful* houses, and so on. Then we can think that the IVSS $\mathbf{F}_A$ describes the IVS of the "attractiveness of the houses" which a newly married couple would like to buy.

Now consider the universe set X and the set of parameters E given by:

$$X = \{h_1, h_2, h_3, h_4, h_5, h_6\} \text{ and } E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\},$$

where

e_1 stands for the parameter *expensive*,
e_2 stands for the parameter *beautiful*,
e_3 stands for the parameter *wooden*,
e_4 stands for the parameter *cheap*,
e_5 stands for the parameter *in the surroundings*,
e_6 stands for the parameter *modern*,
e_7 stands for the parameter *in good repair*,
e_8 stands for the parameter *in bad repair*.

Let $A \subset E$ such that $A = \{e_1, e_2, e_3, e_4, e_5\}$ and let $\mathbf{F}_A : A \to IVS(X)$ be the mapping given by:

$$\mathbf{F}_A(e_1) = [\{h_2, h_4\}, \{h_2, h_4, h_5\}],$$

$$\mathbf{F}_A(e_2) = [\{h_1, h_3\}, \{h_1, h_3, h_4\}],$$
$$\mathbf{F}_A(e_3) = [\{h_3, h_4, h_5\}, \{h_3, h_4, h_5\}],$$
$$\mathbf{F}_A(e_4) = [\{h_1, h_3\}, \{h_1, h_3\}],$$
$$\mathbf{F}_A(e_5) = [\{h_1\}, \{h_1, h_2\}].$$

Then clearly, $\mathbf{F}_A$ is an IVSS over X. Moreover, we can see that the IVSS $\mathbf{F}_A$ is a parametrized family $\{\mathbf{F}_A(e_i),\ i = 1,\ 2,\ 3,\ 4,\ 5\}$ of IVSs of X and gives us a collection of interval-valued approximate description of an object. consider the mapping $\mathbf{F}_A$ which is "[houses (.),houses (.)]", where dot (.) is to be filled up by a parameter $e_i \in A$. Thus $\mathbf{F}_A(e_1)$ means "[houses (expensive),houses (expensive)]" whose functional-value is the IVS $[\{h_2, h_4\}, \{h_2, h_4, h_5\}]$. So we can consider the IVSS $\mathbf{F}_A$ as a collection of interval-valued approximations as below:

$$\begin{aligned}\mathbf{F}_A = \{&\text{expensive houses} = [\{h_2, h_4\}, \{h_2, h_4, h_5\}],\\ &\text{beautiful houses} = [\{h_1, h_3\}, \{h_1, h_3, h_4\}],\\ &\text{wooden houses} = [\{h_3, h_4, h_5\}, \{h_3, h_4, h_5\}],\\ &\text{cheap houses} = [\{h_1, h_3\}, \{h_1, h_3\}],\\ &\text{in the surroundings} = [\{h_1\}, \{h_1, h_2\}]\},\end{aligned}$$

where each interval-valued approximation is composed of two parts:

(i) a predicate p and

(ii) an approximate IVS v (or simply, to be called an IVS v).

For example, for the interval-valued approximation

$$\text{"expensive houses} = [\{h_2, h_4\}, \{h_2, h_4, h_5\}]\text{"},$$

(i) the predicate name is expensive houses and

(ii) an approximate IVS or IVS is $[\{h_2, h_4\}, \{h_2, h_4, h_5\}]$.

(2) Let (X, τ) be an IVTS proposed by Kim et al. [39]. Then for each $x \in X$, we have two the families $T(x)$ and $T_V(x)$ of open neighborhoods and open vanishing neighborhoods of x (See [39] for the concept of an interval-valued neighborhood) given by:

$$T(x) = \{U = [U^-, U^+] \in \tau : x \in U^-\} \text{ and } T_V(x) = \{U = [U^-, U^+] \in \tau : x \in U^+\}.$$

Then for a fixed $x \in X$, we may consider $T(x)_\tau$ and $T_V(x)_\tau$ as IVSSs over τ, where $T(x)_\tau,\ T_V(x)_\tau : \tau \to IVS(X)$.

(3) Let $A = [A^-, A^+]$ be an interval-valued fuzzy set in X (See [2, 3]). Consider the family $\mathbf{F}_{[0,1]\times[0,1]}((\alpha, \beta))$ of $[\alpha, \beta]$-level sets for A defined as:

$$\mathbf{F}_{[0,1]\times[0,1]}([\alpha, \beta]) = \{[\{x \in X\}, \{x \in X\}] : A^-(x) \geq \alpha,\ A^+(x) \geq \beta\},$$

where $\alpha,\ \beta \in [0, 1]$ such that $\alpha \leq \beta$.

Then we can easily check that for each $x \in X$,

$$A(x) = \sup_{[\alpha,\beta]\in[0,1]\times[0,1],\ [\{x\},\{x\}]\in\mathbf{F}_{[0,1]\times[0,1]}([\alpha,\beta])}[\alpha, \beta].$$

Thus every interval-valued fuzzy set can be considered as the IVSS $\mathbf{F}_{[0,1]\times[0,1]}$.

Definition 3.3. Let $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$. Then we say that:

(i) $\mathbf{F}_A$ is an *interval-valued soft subset* of $\mathbf{F}_B$, denoted by $\mathbf{F}_A \subset \mathbf{F}_B$, if $A \subset B$ and $\mathbf{F}_A(e) \subset \mathbf{F}_B(e)$ for each $e \in A$,

(ii) $\mathbf{F}_A(e)$ is an *interval-valued soft super set* of $\mathbf{F}_B(e)$, denoted by $\mathbf{F}_A \supset \mathbf{F}_B$, if $\mathbf{F}_B \subset \mathbf{F}_A$,

(iii) $\mathbf{F}_A$ and $\mathbf{F}_B$ are *interval-valued soft equal*, if $\mathbf{F}_A \subset \mathbf{F}_B$ and $\mathbf{F}_A \supset \mathbf{F}_B$.

Example 3.4. Let $A = \{e_1, e_3, e_5\} \subset E$, $B = \{e_1, e_2, e_3, e_5\} \subset E$. Consider two IVSSs $\mathbf{F}_A$ and $\mathbf{F}_B$ over X given by:

$$\mathbf{F}_A(e_1) = [\{h_2, h_4\}, \{h_1, h_2, h_4\}],\ \mathbf{F}_A(e_3) = [\{h_3, h_4, h_5\}, \{h_3, h_4, h_5\}],$$
$$\mathbf{F}_A(e_5) = [\{h_1\}, \{h_1, h_4\}],$$
$$\mathbf{F}_B(e_1) = [\{h_2, h_4\}, \{h_1, h_2, h_4\}],\ \mathbf{F}_B(e_2) = [\{h_1, h_3\}, \{h_1, h_3, h_5\}],$$
$$\mathbf{F}_B(e_3) = [\{h_3, h_4, h_5\}, \{h_3, h_4, h_5\}],\ \mathbf{F}_B(e_5) = [\{h_1\}, \{h_1, h_4\}],$$

where $X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$.
Then clearly, $\mathbf{F}_A(e_i) \subset \mathbf{F}_B(e_i)$ for $i = 1,\ 2,\ 3,\ 4,\ 5,\ 6$. Thus $\mathbf{F}_A \subset \mathbf{F}_B$.

Definition 3.5. Let $\mathbf{F}_A \in IVSS(X)$. Then the *complement* of $\mathbf{F}_A$, denoted by $\mathbf{F}^{'}_A$, is the mapping $\mathbf{F}^{'}_A : \rceil A \to IVS(X)$ defined by: for each $\alpha \in \rceil A$,

$$\mathbf{F}^{'}_A(\alpha) = \widetilde{X} - \mathbf{F}_{\rceil A}(\neg\alpha) = [X - F^{+}_A(\neg\alpha), X - F^{-}_A(\neg\alpha)].$$

It is obvious that $(\mathbf{F}^{'}_A)^{'} = \mathbf{F}_A$. In fact, $\mathbf{F}^{'}_A = \mathbf{F}^{'}_{\rceil A}$.

Definition 3.6. Let $\mathbf{F}_A \in IVSS(X)$. Then $\mathbf{F}_A$ is called:

(i) a *relative null interval-valued soft set* (with respect to A), denoted by $\widetilde{\varnothing}_A$, if $\mathbf{F}_A(e) = \widetilde{\varnothing}$ for each $e \in A$,

(ii) a *relative whole interval-valued soft set* (with respect to A), denoted by $\widetilde{X}_A$, if $\mathbf{F}_A(e) = \widetilde{X}$ for each $e \in A$.

We denote the set of all IVSSs over X with respect to the fixed parameter set A as $IVSS_A(X)$.

Example 3.7. (1) Consider the IVSS $\mathbf{F}_A$ given in Example 3.2. Then

$\mathbf{F}^c_A = \{$not expensive houses $= [\{h_1, h_3, h_6\}, \{h_1, h_3, h_5, h_6\}]$,
not beautiful houses $= [\{h_2, h_5, h_6\}, \{h_2, h_4, h_5, h_6\}]$,
not wooden houses $= [\{h_1, h_2, h_6\}, \{h_1, h_2, h_6\}]$,
not cheap houses $= [\{h_2, h_4, h_5, h_6\}, \{h_2, h_4, h_5, h_6\}]$,
not in the surroundings $= [\{h_3, h_4, h_5, h_6\}, \{h_2, h_3, h_4, h_5, h_6\}]\}$.

(2) Let X be the universe set and let A be the set of parameters given by:

$$X = \{h_1, h_2, h_3, h_4, h_5\} \text{ and } A = \{\text{brick, muddy, steel, stone}\},$$

where X denotes the set of wooden houses under consideration.
Let $\mathbf{F}_A : A \to IVS(X)$ be the mapping defined as follows:

$\mathbf{F}_A$(brick)=the IVS of the brick built houses,
$\mathbf{F}_A$(muddy)=the IVS of the muddy built houses,
$\mathbf{F}_A$(steel)=the IVS of the steel built houses,
$\mathbf{F}_A$(stone)=the IVS of the stone built houses.

Then we can easily see that

$$\mathbf{F}_A(\text{brick}) = \mathbf{F}_A(\text{muddy}) = \mathbf{F}_A(\text{steel}) = \mathbf{F}_A(\text{stone}) = \widetilde{\varnothing}.$$

Thus $\mathbf{F}_A$ is a null interval-valued soft set.

(2) Let X and A be the universe set and the set of parameters given in (2), respectively and let $B = \rceil A$, i,e., $B = \{$not brick, not muddy, not steel, not stone$\}$. Consider the mapping $\mathbf{F}_B : B \to IVS(X)$ defined as follows:

$\mathbf{F}_B$(not brick)=the IVS of the houses not built by brick,
$\mathbf{F}_B$(not muddy)=the IVS of the not muddy built houses,
$\mathbf{F}_B$(not steel)=the IVS of the houses not built by steel,
$\mathbf{F}_B$(not stone)=the IVS of the houses not built by stone.
Then we can easily see that

$$\mathbf{F}_B(\text{not brick}) = \mathbf{F}_B(\text{not muddy}) = \mathbf{F}_B(\text{not steel}) = \mathbf{F}_B(\text{stone}) = \widetilde{X}.$$

Thus $\mathbf{F}_B$ is an absolute interval-valued soft set.

Definition 3.8. Let $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$. Then

(i) $\mathbf{F}_A$ AND $\mathbf{F}_B$, denoted by $\mathbf{F}_A \wedge \mathbf{F}_B$, is the mapping $\mathbf{F}_A \wedge \mathbf{F}_B : A \times B \to IVS(X)$ defined as follows: for each $(e, f) \in A \times B$,

$$(\mathbf{F}_A \wedge \mathbf{F}_B)(e, f) = \mathbf{F}_A(e) \cap \mathbf{F}_B(f),$$

(ii) $\mathbf{F}_A$ OR $\mathbf{F}_B$, denoted by $\mathbf{F}_A \vee \mathbf{F}_B$, is the mapping $\mathbf{F}_A \vee \mathbf{F}_B : A \times B \to IVS(X)$ defined as follows: for each $(e, f) \in A \times B$,

$$(\mathbf{F}_A \vee \mathbf{F}_B)(e, f) = \mathbf{F}_A(e) \cup \mathbf{F}_A(f).$$

Example 3.9. Let X be the universe set and let $A,\ B$ be the sets of parameters given by:
$X = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}$,
$A = \{\text{very costly, costly, cheap}\},\ B = \{\text{beautiful, in the surroundings, cheap}\}$.

Let us consider two mappings $\mathbf{F}_A : A \to IVS(X)$ and $\mathbf{F}_B : B \to IVS(X)$ defined as follows:
$\mathbf{F}_A(\text{very costly}) = [\{h_2, h_4, h_7\}, \{h_2, h_4, h_7, h_8\}]$,
$\mathbf{F}_A(\text{costly}) = [\{h_1, h_3\}, \{h_1, h_3, h_5\}]$,
$\mathbf{F}_A(\text{cheap}) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]$,
$\mathbf{F}_B(\text{beautiful}) = [\{h_2, h_3\}, \{h_2, h_3, h_7\}]$,
$\mathbf{F}_B(\text{in the surroundings}) = [\{h_5, h_6\}, \{h_5, h_6, h_8\}]$,
$\mathbf{F}_B(\text{cheap}) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]$.
Then we have

$$\begin{aligned} A \times B = \{&(\text{very costly}, \text{beautiful}), (\text{very costly}, \text{in the surroundings}),\\ &(\text{very costly}, \text{cheap}), (\text{costly}, \text{beautiful}),\\ &(\text{costly}, \text{in the surroundings}), (\text{costly}, \text{cheap}),\\ &(\text{cheap}, \text{beautiful}), (\text{cheap}, \text{in the surroundings}), (\text{cheap}, \text{cheap})\}. \end{aligned}$$

Thus we get

$$\mathbf{F}_A \wedge \mathbf{F}_B = \mathbf{H}_{A\times B},$$

where $\mathbf{H}_{A\times B}(\text{very costly}, \text{beautiful}) = [\{h_2\}, \{h_2, h_7\}]$,
$\mathbf{H}_{A\times B}(\text{very costly}, \text{in the surroundings}) = [\varnothing, \{h_8\}]$,
$\mathbf{H}_{A\times B}(\text{very costly}, \text{cheap}) = \widetilde{\varnothing}$,
$\mathbf{H}_{A\times B}(\text{costly}, \text{beautiful}) = [\{h_3\}, \{h_3\}]$,
$\mathbf{H}_{A\times B}(\text{costly}, \text{in the surroundings}) = [\varnothing, \{h_5\}]$,
$\mathbf{H}_{A\times B}(\text{costly}, \text{cheap}) = \widetilde{\varnothing}$,
$\mathbf{H}_{A\times B}(\text{cheap}, \text{beautiful}) = \widetilde{\varnothing}$,
$\mathbf{H}_{A\times B}(\text{cheap}, \text{in the surroundings}) = [\{h_6, \{h_6\}]$,
$\mathbf{H}_{A\times B}(\text{cheap}, \text{cheap}) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]$.

Also, we can check that

$$\mathbf{F}_A \vee \mathbf{F}_B = \mathbf{K}_{A\times B},$$

where $\mathbf{K}_{A\times B}(\text{very costly}, \text{beautiful}) = [\{h_2, h_3, h_4, h_7\}, \{h_2, h_3, h_4, h_7, h_8\}]$,
$\mathbf{K}_{A\times B}(\text{very costly}, \text{in the surroundings}) = [\{h_2, h_4, h_5, h_6, h_7\}, \{h_2, h_4, h_5, h_6, h_7, h_8\}]$,
$\mathbf{K}_{A\times B}(\text{very costly}, \text{cheap}) = [\{h_2, h_4, h_6, h_7, h_9\}, \{h_2, h_4, h_6, h_7, h_8, h_9, h_{10}\}]$,
$\mathbf{K}_{A\times B}(\text{costly}, \text{beautiful}) = [\{h_1, h_2, h_3\}, \{h_1, h_2, h_3, h_5, h_7\}]$,
$\mathbf{K}_{A\times B}(\text{costly}, \text{in the surroundings}) = [\{h_1, h_3, h_5, h_6\}, \{h_1, h_3, h_5, h_6, h_8\}]$,
$\mathbf{K}_{A\times B}(\text{costly}, \text{cheap}) = [\{h_1, h_3, h_6, h_9\}, \{h_1, h_3, h_5, h_6, h_9, h_{10}\}]$,
$\mathbf{K}_{A\times B}(\text{cheap}, \text{beautiful}) = [\{h_2, h_3, h_6, h_9\}, \{h_2, h_3, h_6, h_9, h_{10}\}]$,
$\mathbf{K}_{A\times B}(\text{cheap}, \text{in the surroundings}) = [\{h_5, h_6, h_9\}, \{h_5, h_6, h_8, h_9, h_{10}\}]$,
$\mathbf{K}_{A\times B}(\text{cheap}, \text{cheap}) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]$.

We obtain the similar result to Proposition 2.2 in [9].

Proposition 3.10. *Let* $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$. *Then*
(1) $(\mathbf{F}_A \vee \mathbf{F}_B)^{'} = \mathbf{F}^{'}_A \wedge \mathbf{F}^{'}_B$,
(2) $(\mathbf{F}_A \wedge \mathbf{F}_B)^{'} = \mathbf{F}^{'}_A \vee \mathbf{F}^{'}_B$.

Proof. (1) Let $\mathbf{F}_A \vee \mathbf{F}_B = \mathbf{K}_{A\times B}$. Then clearly, we have

$$(\mathbf{F}_A \vee \mathbf{F}_B)^{'} = \mathbf{K}^{'}_{A\times B} = \mathbf{K}^{'}_{\rceil(A\times B)}.$$

On the other hand,

$$\begin{aligned}\mathbf{F}^{'}_A \wedge \mathbf{F}^{'}_B &= \mathbf{F}^{'}_{\rceil A} \wedge \mathbf{F}^{'}_{\rceil B}\\ &= \mathbf{J}_{\rceil A\times \rceil B},\ [\text{where } \mathbf{J}(x,y) = \mathbf{F}^{'}_A(x) \cap \mathbf{F}^{'}_B(y)]\\ &= \mathbf{J}_{\rceil(A\times B)}.\end{aligned}$$

Now let $(\neg\alpha, \neg\beta) \in \rceil(A\times B)$. Then we get

$$\begin{aligned}\mathbf{K}^{'}_{\rceil(A\times B)}(\neg\alpha, \neg\beta) &= [X - K^{+}(\alpha,\beta), X - K^{-}(\alpha,\beta)]\\ &= [X - (F^{+}_A(\alpha) \cup F^{+}_B(\beta)), X - (F^{-}_A(\alpha) \cup F^{-}_B(\beta))]\\ &= [(X - F^{+}_A(\alpha)) \cap (X - F^{+}_B(\beta)), (X - F^{-}_A(\alpha)) \cap (X - F^{-}_B(\beta))]\\ &= \mathbf{F}^{'}_A(\neg\alpha) \cap \mathbf{F}^{'}_B(\neg\beta)\\ &= \mathbf{J}_{\rceil(A\times B)}(\neg\alpha, \neg\beta).\end{aligned}$$

Thus $\mathbf{K}^{'}_{\rceil(A\times B)}(\neg\alpha, \neg\beta) = \mathbf{J}_{\rceil(A\times B)}(\neg\alpha, \neg\beta)$. So the result holds.
(2) The proof is similar to (1). □

Definition 3.11 (See [9])**.** Let $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$. Then
(i) the *union* of $\mathbf{F}_A$ and $\mathbf{F}_B$, denoted by $\mathbf{F}_A \cup \mathbf{F}_B$, is the mapping $\mathbf{F}_A \cup \mathbf{F}_B : A\cup B \to IVS(X)$ defined as: for each $e \in A\cup B$,

$$(\mathbf{F}_A \cup \mathbf{F}_B)(e) = \begin{cases} \mathbf{F}_A(e) & \text{if } e \in A - B\\ \mathbf{F}_B(e) & \text{if } e \in B - A\\ \mathbf{F}_A(e) \cup \mathbf{F}_B(e) & \text{if } e \in A\cap B,\end{cases}$$

(ii) the *restricted union* of $\mathbf{F}_A$ and $\mathbf{F}_B$, denoted by $\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B$, is the mapping $\mathbf{F}_A \cup \mathbf{F}_B : A\cap B \to IVS(X)$ defined as: for each $e \in A\cap B$,

$$(\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B)(e) = \mathbf{F}_A(e) \cup \mathbf{F}_B(e),$$

(iii) the *intersection* of $\mathbf{F}_A$ and $\mathbf{F}_B$, denoted by $\mathbf{F}_A \cap \mathbf{F}_B$, is the mapping $\mathbf{F}_A \cap \mathbf{F}_B : A \cap B \to IVS(X)$ defined as: for each $e \in A \cap B$,

$$(\mathbf{F}_A \cap \mathbf{F}_B)(e) = \mathbf{F}_A(e) \text{ or } \mathbf{F}_B(e) \text{ (as both are same set)},$$

(iv) the *restricted intersection* of $\mathbf{F}_A$ and $\mathbf{F}_B$, denoted by $\mathbf{F}_A \cap_{\mathcal{R}} \mathbf{F}_B$, is the mapping $\mathbf{F}_A \cap_{\mathcal{R}} \mathbf{F}_B : A \cap B \to IVS(X)$ defined as: for each $e \in A \cap B$,

$$(\mathbf{F}_A \cap_{\mathcal{R}} \mathbf{F}_B)(e) = \mathbf{F}_A(e) \cap \mathbf{F}_B(e),$$

(v) the *extended intersection* of $\mathbf{F}_A$ and $\mathbf{F}_B$, denoted by $\mathbf{F}_A \cap_{\mathcal{E}} \mathbf{F}_B$, is the mapping $\mathbf{F}_A \cap_{\mathcal{E}} \mathbf{F}_B : A \cup B \to IVS(X)$ defined as: for each $e \in C = A \cup B$,

$$(\mathbf{F}_A \cap_{\mathcal{E}} \mathbf{F}_B)(e) = \begin{cases} \mathbf{F}_A(e) & \text{if } e \in A - B \\ \mathbf{F}_B(e) & \text{if } e \in B - A \\ \mathbf{F}_A(e) \cap \mathbf{F}_B(e) & \text{if } e \in A \cap B. \end{cases}$$

We write $\mathbf{F}_A \cup \mathbf{F}_B = \mathbf{F}_{A \cup B}$, $\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B = \mathbf{F}_{A \cup_{\mathcal{R}} B}$, $\mathbf{F}_A \cap \mathbf{F}_B = \mathbf{F}_{A \cap B}$, $\mathbf{F}_A \cap_{\mathcal{R}} \mathbf{F}_B = \mathbf{F}_{A \cap_{\mathcal{R}} B}$ and $\mathbf{F}_A \cap_{\mathcal{E}} \mathbf{F}_B = \mathbf{F}_{A \cap_{\mathcal{E}} B}$, respectively.

Definition 3.12. Let $\mathbf{F}_A \in IVSS(X)$ such that $A \cap B \neq \varnothing$. Then the *relative complement* of $\mathbf{F}_A$, denoted by $\mathbf{F}^r_A$, is the mapping $\mathbf{F}^r_A : A \to IVS(X)$ defined as: each $e \in A$,

$$\mathbf{F}^r_A(e) = (\mathbf{F}_A(e))^c = [F^-_A(e), F^+_A(e)]^c.$$

The following is the similar result to Proposition 2.3 in [9].

Proposition 3.13. *Let* $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$. *Then*
(1) $\mathbf{F}_A \cup \mathbf{F}_A = \mathbf{F}_A$, $\mathbf{F}_A \cap \mathbf{F}_A = \mathbf{F}_A$,
(2) $\mathbf{F}_A \cup \widetilde{\varnothing}_A = (\mathbf{F}, A)$, $\mathbf{F}_A \cap \widetilde{\varnothing}_A = \widetilde{\varnothing}_A$,
(3) $\mathbf{F}_A \cup \widetilde{X}_A = \widetilde{X}_A$, $\mathbf{F}_A \cap \widetilde{X}_A = \mathbf{F}_A$.

Proof. The proofs are straightforward. □

The following is the similar result to Theorem 4.1 in [10].

Proposition 3.14. *Let* $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$ *such that* $A \cap B \neq \varnothing$. *Then*
(1) $(\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B)^r = \mathbf{F}^r_A \cap_{\mathcal{R}} \mathbf{F}^r_B$,
(2) $(\mathbf{F}_A \cap_{\mathcal{R}} \mathbf{F}_B)^r = \mathbf{F}^r_A \cup_{\mathcal{R}} \mathbf{F}^r_B$.

Proof. (1) Let $e \in A \cap B \neq \varnothing$. Then clearly, $(\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B)(e) = \mathbf{F}_A(e) \cup \mathbf{F}_B(e)$. Thus by Definition 3.12 (ii) and Result 2.4 (6), we have

$$(\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B)^r(e) = (\mathbf{F}_A(e) \cup \mathbf{F}_B(e))^c = (\mathbf{F}_A(e)))^c \cap (\mathbf{F}_B(e))^c = (\mathbf{F}^r_A \cap_{\mathcal{R}} \mathbf{F}^r_B)(e).$$

So $(\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B)^r(e) = (\mathbf{F}^r_A \cap_{\mathcal{R}} \mathbf{F}^r_B)(e)$. Hence $(\mathbf{F}_A \cup_{\mathcal{R}} \mathbf{F}_B)^r = \mathbf{F}^r_A \cap_{\mathcal{R}} \mathbf{F}^r_B$.
(2) The proof is similar to (1). □

Also we have the similar results to Propositions 2.5 and 2.6 in [9].

Proposition 3.15. *Let* $\mathbf{F}_A,\ \mathbf{F}_B,\ \mathbf{F}_C \in IVSS(X)$. *Then*
(1) $\mathbf{F}_A \cup (\mathbf{F}_B \cup \mathbf{F}_C) = (\mathbf{F}_A \cup \mathbf{F}_B) \cup \mathbf{F}_C$,
(2) $\mathbf{F}_A \cap (\mathbf{F}_B \cap \mathbf{F}_C) = (\mathbf{F}_A \cap \mathbf{F}_B) \cap \mathbf{F}_C$,
(3) $\mathbf{F}_A \cup (\mathbf{F}_B \cap \mathbf{F}_C) = (\mathbf{F}_A \cup \mathbf{F}_B) \cap (\mathbf{F}_A \cup \mathbf{F}_C)$,
(2) $\mathbf{F}_A \cap (\mathbf{F}_B \cup \mathbf{F}_C) = (\mathbf{F}_A \cap \mathbf{F}_B) \cup (\mathbf{F}_A \cap \mathbf{F}_C)$.

Proof. The proofs are straightforward. □

Proposition 3.16. *Let* $\mathbf{F}_A,\ \mathbf{F}_B,\ \mathbf{F}_C \in IVSS(X)$. *Then*
(1) $\mathbf{F}_A \vee (\mathbf{F}_B \vee \mathbf{F}_C) = (\mathbf{F}_A \vee \mathbf{F}_B) \vee \mathbf{F}_C$,
(2) $\mathbf{F}_A \wedge (\mathbf{F}_B \wedge \mathbf{F}_C) = (\mathbf{F}_A \wedge \mathbf{F}_B) \wedge \mathbf{F}_C$.

Proof. The proofs are straightforward. □

The following is the similar result to Theorem 4.2 in [10].

Proposition 3.17. *Let* $\mathbf{F}_A,\ \mathbf{F}_B \in IVSS(X)$ *such that* $A \cap B \neq \varnothing$. *Then*
(1) $(\mathbf{F}_A \cup \mathbf{F}_B)^{'} = \mathbf{F}^{'}_A \cap_{\mathcal{E}} \mathbf{F}^{'}_B$,
(2) $(\mathbf{F}_A \cap_{\mathcal{E}} \mathbf{F}_B)^{'} = \mathbf{F}^{'}_A \cup \mathbf{F}^{'}_B$.

Proof. (1) Let $\mathbf{F}_A \cup \mathbf{F}_B = \mathbf{F}_{A\cup B}$ and $e \in A \cup B$. Then clearly,

$$\mathbf{F}_{A\cup B}(e) = \begin{cases} \mathbf{F}_A(e) & \text{if } e \in A - B \\ \mathbf{F}_B(e) & \text{if } e \in B - A \\ \mathbf{F}_A(e) \cup \mathbf{F}_B(e) & \text{if } e \in A \cap B. \end{cases}$$

Thus by Result 2.16 (2) and Definition 3.5, $(\mathbf{F}_A \cup \mathbf{F}_B)^{'} = \mathbf{F}^{'}_{A\cup B}$ and $\mathbf{F}^{'}_{A\cup B} : \rceil A \cup \rceil B \to IVS(X)$ is the mapping defined by: for each $\neg e \in \rceil A \cap \rceil B$,

$$\begin{aligned} \mathbf{F}^{'}_{A\cup B}(\neg e) &= (\mathbf{F}_{A\cup B}(e))^c \\ &= (\mathbf{F}_A(e) \cup \mathbf{F}_B(e))^c \\ &= (\mathbf{F}_A(e))^c \cap (\mathbf{F}_B(e))^c \text{ [By Result 2.4 (6)]} \\ &= \mathbf{F}^{'}_A(\neg e) \cap \mathbf{F}^{'}_B(\neg e). \end{aligned}$$

So we get

$$\mathbf{F}^{'}_{A\cup B}(\neg e) = \begin{cases} \mathbf{F}^{'}_A(\neg e) & \text{if } \neg e \in \rceil A \cup \rceil B \\ \mathbf{F}^{'}_B(\neg e) & \text{if } \neg e \in \rceil B - \rceil A \\ \mathbf{F}^{'}_A(\neg e) \cap \mathbf{F}^{c}_B(\neg e) & \text{if } \neg e \in \rceil A \cap \rceil B. \end{cases}$$

On the other hand, by Result 2.16 (2) and Definitions 3.5 and 3.11 (v),
$\mathbf{F}^{'}_A \cap_{\mathcal{E}} \mathbf{F}^{'}_B : \rceil A \cup \rceil B \to IVS(X)$ is the mapping defined by: for each $\neg e \in \rceil A \cup \rceil B$,

$$(\mathbf{F}^{'}_A \cap_{\mathcal{E}} \mathbf{F}^{'}_B)(\neg e) = \begin{cases} \mathbf{F}^{'}_A(\neg e) & \text{if } \neg e \in \rceil A \cup \rceil B \\ \mathbf{F}^{'}_B(\neg e) & \text{if } \neg e \in \rceil B - \rceil A \\ \mathbf{F}^{'}_A(\neg e) \cap \mathbf{F}^{c}_B(\neg e) & \text{if } \neg e \in \rceil A \cap \rceil B. \end{cases}$$

Hence $\mathbf{F}^{'}_{A\cup B}(\neg e) = (\mathbf{F}^{'}_A \cap_{\mathcal{E}} \mathbf{F}^{'}_B)(\neg e)$. Therefore $(\mathbf{F}_A \cup \mathbf{F}_B)^{'} = \mathbf{F}^{'}_A \cap_{\mathcal{E}} \mathbf{F}^{'}_B$.
(2) The proof id similar to (1). □

Now let $IVSS_E(X)$ be the set of all IVSSs over X with respect to E. Then we will denote the members of $IVSS_E(X)$ as $\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \cdots$. In fact, $\mathbf{A},\ \mathbf{B},\ \mathbf{C} : E \to IVS(X)$. In particular, the interval-valued soft empty [resp. whole] set over X respect to E, denoted by $\widetilde{\varnothing}_E$ [resp. $\widetilde{X}_E$], is the IVS in X defined by $\widetilde{\varnothing}_E(e) = \widetilde{\varnothing}$ [resp. $\widetilde{X}_E(e) = \widetilde{X}$] for each $e \in E$.

Definition 3.18 (See Definitions 3.3 and 3.12 (ii))**.** Let $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. Then we say that
(i) $\mathbf{A}$ is an *interval-valued soft subset* of $\mathbf{B}$, denoted by $\mathbf{A} \subset \mathbf{B}$, if $\mathbf{A}(e) \subset \mathbf{B}(e)$ for each $e \in E$,

(ii) $\mathbf{A}$ and $\mathbf{B}$ are *interval-valued soft equal*, denoted by $\mathbf{A} = \mathbf{B}$, if $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \subset \mathbf{A}$,

(iii) the *interval-valued soft complement* of $\mathbf{A}$, denoted by $\mathbf{A}^c$, is the mapping $\mathbf{A}^c : E \to IVS(X)$ defined as: for each $e \in E$,

$$\mathbf{A}^c(e) = (\mathbf{A}(e))^c.$$

From the above definition, we can easily get the similar properties to Results 2.3 and 2.4.

Proposition 3.19. *Let* $\mathbf{A},\ \mathbf{B},\ \mathbf{C} \in IVSS_E(X)$. *Then*

(1) $\widetilde{\varnothing}_E \subset \mathbf{A} \subset \widetilde{X}_E$,
(2) *if* $\mathbf{A} \subset \mathbf{B}$ *and* $\mathbf{B} \subset \mathbf{C}$, *then* $\mathbf{A} \subset \mathbf{C}$,
(3) $\mathbf{A} \subset \mathbf{A} \cup \mathbf{B}$ *and* $\mathbf{B} \subset \mathbf{A} \cup \mathbf{B}$,
(4) $\mathbf{A} \cap \mathbf{B} \subset \mathbf{A}$ *and* $\mathbf{A} \cap \mathbf{B} \subset \mathbf{B}$,
(5) $\mathbf{A} \subset \mathbf{B}$ *if and only if* $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$,
(6) $\mathbf{A} \subset \mathbf{B}$ *if and only if* $\mathbf{A} \cup \mathbf{B} = \mathbf{B}$.

Proposition 3.20. *Let* $\mathbf{A},\ \mathbf{B},\ \mathbf{C} \in IVSS_E(X)$. *Then*

(1) (Idempotent laws) $\mathbf{A} \cup \mathbf{A} = \mathbf{A},\ \mathbf{A} \cap \mathbf{A} = \mathbf{A}$,
(2) (Commutative laws) $\mathbf{A} \cup \mathbf{B} = \mathbf{B} \cup \mathbf{A},\ \mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$,
(3) (Associative laws) $\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C},\ \mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$,
(4) (Distributive laws) $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C}) = (\mathbf{A} \cup \mathbf{B}) \cap (\mathbf{A} \cup \mathbf{C})$,
$\mathbf{A} \cap (\mathbf{B} \cup \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \cup (\mathbf{A} \cap \mathbf{C})$,
(5) (Absorption laws) $\mathbf{A} \cup (\mathbf{A} \cap \mathbf{B}) = \mathbf{A},\ \mathbf{A} \cap (\mathbf{A} \cup \mathbf{B}) = \mathbf{A}$,
(6) (DeMorgan's laws) $(\mathbf{A} \cup \mathbf{B})^c = \mathbf{A}^c \cap \mathbf{B}^c,\ (\mathbf{A} \cap \mathbf{B})^c = \mathbf{A}^c \cup \mathbf{B}^c$,
(7) $(\mathbf{A}^c)^c = \mathbf{A}$,
(8) (8_a) $\mathbf{A} \cup \widetilde{\varnothing}_E = \mathbf{A},\ \mathbf{A} \cap \widetilde{\varnothing}_E = \widetilde{\varnothing}_E$,
(8_b) $\mathbf{A} \cup \widetilde{X}_E = \widetilde{X}_E,\ A \cap \widetilde{X}_E = \mathbf{A}$,
(8_c) $\widetilde{X}_E^c = \widetilde{\varnothing}_E,\ \widetilde{\varnothing}_E^c = \widetilde{X}_E$,
(8_d) $\mathbf{A} \cup \mathbf{A}^c \neq \widetilde{X}_E,\ \mathbf{A} \cap \mathbf{A}^c \neq \widetilde{\varnothing}_E$ *in general* (See Example 3.21).

Example 3.21. Let the universe set X and the set of parameters E be given by:

$$X = \{h_1, h_2, h_3, h_4, h_5, h_6\} \text{ and } E = \{e_1, e_2, e_3\}.$$

Consider the IVSS $\mathbf{A}$ over X given by:

$$\mathbf{A}(e_1) = [\{h_1, h_2\}, \{h_1, h_2, h_3\}],\ \mathbf{A}(e_2) = [\{h_1\}, \{h_1, h_5, h_6\}],$$

$$\mathbf{A}(e_3) = [\{h_1, h_3, h_4\}, \{h_1, h_3, h_4\}].$$

Then clearly, we have

$$\mathbf{A}^c(e_1) = [\{h_4, h_5, h_6\}, \{h_3, h_4, h_5, h_6\}].$$

Thus we can easily check that

$$(\mathbf{A} \cup \mathbf{A}^c)(e_1) \neq \widetilde{X}_E(e_1) \text{ and } (\mathbf{A} \cap \mathbf{A}^c)(e_1) \neq \widetilde{\varnothing}_E(e_1).$$

Definition 3.22 (See Definition 3.11)**.** Let $(\mathbf{A}_j)_{j \in J} \subset IVSS_E(X)$, where J is an arbitrary index set. Then we say that

(i) the *interval-valued soft union* of $(\mathbf{A}_j)_{j\in J}$, denoted by $\bigcup_{j\in J}\mathbf{A}_j$, is the mapping $\bigcup_{j\in J}\mathbf{A}_j : E \to IVS(X)$ defined as: for each $e \in E$,

$$\left[\bigcup_{j\in J}\mathbf{A}_j\right](e) = \bigcup_{j\in J}\mathbf{A}_j(e),$$

(ii) the *interval-valued soft intersection* of $(\mathbf{A}_j)_{j\in J}$, denoted by $\bigcap_{j\in J}\mathbf{A}_j$, is the mapping $\bigcap_{j\in J}\mathbf{A}_j : E \to IVS(X)$ defined as: for each $e \in E$,

$$\left[\bigcap_{j\in J}\mathbf{A}_j\right](e) = \bigcap_{j\in J}\mathbf{A}_j(e).$$

Example 3.23. (1) Let $X = \mathbb{R}$, $E = \{0, 1\}$ and let $\mathbb{N}$ be the set of all positive integers. For each $n \in \mathbb{N}$, consider the mapping $\mathbf{A}_n : E \to IVS(X)$ defined by: for each $e \in E$,

$$\mathbf{A}_n(e) = \begin{cases} [(0,n)\,,(0,n+1)] & \text{if } e = 0 \\ [(-1-n,0)\,,(-n,0)] & \text{if } e = 1. \end{cases}$$

Then clearly, $\mathbf{A}_n \in IVSS_E(X)$ for each $n \in \mathbb{N}$. Moreover, we can easily check that $\bigcup_{n\in\mathbb{N}}\mathbf{A}_n$, where $\bigcup_{n\in\mathbb{N}}\mathbf{A}_n : E \to IVS(X)$ is the mapping defined as follows: for each $e \in E$,

$$\left(\bigcup_{n\in\mathbb{N}}\mathbf{A}_n\right)(e) = \begin{cases} [(0,\infty)\,,(0,\infty)] & \text{if } e = 0 \\ [(-\infty,0)\,,(-\infty,0)] & \text{if } e = 1. \end{cases}$$

(2) Let $X = \mathbb{R}$, $E = \{0, 1, 2\}$. For each $n \in \mathbb{N}$, consider the mapping $\mathbf{A}_n : E \to IVS(X)$ defined by: for each $e \in E$,

$$\mathbf{A}_n(e) = \begin{cases} \left[\left(-\frac{1}{n}, 1+\frac{1}{n}\right), \left[-\frac{1}{n}, 1+\frac{1}{n}\right)\right] & \text{if } e = 0 \\ \\ \left[\left(1-\frac{1}{n}, 2+\frac{1}{n}\right), \left[1-\frac{1}{n}, 2+\frac{1}{n}\right)\right] & \text{if } e = 1 \\ \\ \left[\left(2-\frac{1}{n}, 3+\frac{1}{n}\right), \left[2-\frac{1}{n}, 3+\frac{1}{n}\right)\right] & \text{if } e = 2. \end{cases}$$

Then clearly, $\mathbf{A}_n \in IVSS_E(X)$ for each $n \in \mathbb{N}$. Moreover, we can easily check that $\bigcap_{n\in\mathbb{N}}\mathbf{A}_n$, where $\bigcap_{n\in\mathbb{N}}\mathbf{A}_n : E \to IVS(X)$ is the mapping defined as follows: for each $e \in E$,

$$\left(\bigcap_{n\in\mathbb{N}}\mathbf{A}_n\right)(e) = \begin{cases} [(0,1)\,,[0,1)] & \text{if } e = 0 \\ \\ [(1,2)\,,[1,2)] & \text{if } e = 1 \\ \\ [(2,3)\,,[2,3)] & \text{if } e = 2. \end{cases}$$

Proposition 3.24. *Let $\mathbf{A} \in IVSS_E(X)$ and let $(\mathbf{A}_j)_{j\in J} \subset IVSS_E(X)$, where J is an arbitrary index set. Then*

(1) $\mathbf{A} \cap (\bigcup_{j\in J}\mathbf{A}_j) = \bigcup_{j\in J}(\mathbf{A} \cap \mathbf{A}_j)$, $\mathbf{A} \cup (\bigcap_{j\in J}\mathbf{A}_j) = \bigcap_{j\in J}(\mathbf{A} \cup \mathbf{A}_j)$,

(2) $(\bigcap_{j\in J}\mathbf{A}_j)^c = \bigcup_{j\in J}\mathbf{A}_j^c$, $(\bigcup_{j\in J}\mathbf{A}_j)^c = \bigcap_{j\in J}\mathbf{A}_j^c$.

Proof. The proofs are straightforward from Definitions 3.18 and 3.22. □

From Propositions 3.20 and 3.23, we can see that $(IVSS_E(X), \cup, \cap, ^c, \widetilde{\varnothing}_E, \widetilde{X}_E)$ forms a Boolean algebra except the property (8_d).

Definition 3.25. Let $\mathbf{A} \in IVSS_E(X)$. Then $\mathbf{A}$ is called an:

(i) *interval-valued soft point* (briefly, IVSP) *with the value* $a_{_{IVP}} = [\{a\}, \{a\}] \in IVS(X)$ and *the support* $e \in E$, denoted by $e_{a_{_{IVP}}}$, if for each $f \in E$,

$$e_{a_{_{IVP}}}(f) = \begin{cases} a_{_{IVP}} & \text{if } e = f \\ \widetilde{\varnothing} & \text{if } e \neq f. \end{cases}$$

(ii) *interval-valued soft vanishing point* (briefly, IVSVP) *with the value* $a_{_{IVVP}} = [\varnothing, \{a\}] \in IVS(X)$ and *the support* $e \in E$, denoted by $e_{a_{_{IVVP}}}$, if for each $f \in E$,

$$e_{a_{_{IVVP}}}(f) = \begin{cases} a_{_{IVVP}} & \text{if } e = f \\ \widetilde{\varnothing} & \text{if } e \neq f. \end{cases}$$

Definition 3.26. Let $\mathbf{A} \in IVSS_E(X)$.

(i) We say that $e_{a_{_{IVP}}}$ belongs to $\mathbf{A}$, denoted by $e_{a_{_{IVP}}} \in \mathbf{A}$, if $a_{_{IVP}} \in \mathbf{A}(e)$, i.e, $a \in A^-(e)$.

(ii) We say that $e_{a_{_{IVVP}}}$ belongs to $\mathbf{A}$, denoted by $e_{a_{_{IVVP}}} \in \mathbf{A}$, if $a_{_{IVVP}} \in \mathbf{A}(e)$, i.e, $a \in A^+(e)$.

Proposition 3.27. *Let* $\mathbf{A} \in IVSS_E(X)$*. Then*

$$\mathbf{A} = \mathbf{A}_{_{IVSP}} \cup \mathbf{A}_{_{IVSVP}},$$

where $\mathbf{A}_{_{IVSP}} = \bigcup_{e_{a_{_{IVVP}}} \in \mathbf{A}} e_{a_{_{IVVP}}}$ *and* $\mathbf{A}_{_{IVSVP}} = \bigcup_{e_{a_{_{IVSVP}}} \in \mathbf{A}} e_{a_{_{IVSVP}}}$.

In fact, $\mathbf{A}_{_{IVSP}}(e) = [A^-(e), A^-(e)]$ *and* $\mathbf{A}_{_{IVSVP}}(e) = [\varnothing, A^+(e)]$ *for each* $e \in E$.

Proof. The proof is straightforward. □

Example 3.28. (1) Let $X = \{a, b, c\}$ and let $E = \{e, f\}$. Then we have the following IVSPs and IVSVPs in X:

$$e_{a_{_{IVP}}},\ e_{b_{_{IVP}}},\ e_{c_{_{IVP}}},\ f_{a_{_{IVP}}},\ f_{b_{_{IVP}}}, f_{c_{_{IVP}}}$$

and

$$e_{a_{_{IVVP}}},\ e_{b_{_{IVVP}}},\ e_{c_{_{IVVP}}},\ f_{a_{_{IVVP}}},\ f_{b_{_{IVVP}}}, f_{c_{_{IVVP}}}.$$

(2) Let $\mathbf{A}$ be the IVSS over X given in Example 3.21:

$$\mathbf{A}(e_1) = [\{h_1, h_2\}, \{h_1, h_2, h_3\}],\ \mathbf{A}(e_2) = [\{h_1\}, \{h_1, h_5, h_6\}],$$

$$\mathbf{A}(e_3) = [\{h_1, h_3, h_4\}, \{h_1, h_3, h_4\}].$$

Then clearly, we have

$\mathbf{A}_{_{IVSP}}(e_1) = [\{h_1, h_2\}, \{h_1, h_2\}],\ \mathbf{A}_{_{IVSVP}}(e_1) = [\varnothing, \{h_1, h_2, h_3\}]$,
$\mathbf{A}_{_{IVSP}}(e_2) = [\{h_1\}, \{h_1\}],\ \mathbf{A}_{_{IVSVP}}(e_1) = [\varnothing, \{h_1, h_5, h_6\}]$,
$\mathbf{A}_{_{IVSP}}(e_3) = [\{h_1, h_3, h_4\}, \{h_1, h_3, h_4\}],\ \mathbf{A}_{_{IVSVP}}(e_1) = [\varnothing, \{h_1, h_3, h_4\}]$.

Thus $\mathbf{A} = \mathbf{A}_{_{IVSP}} \cup \mathbf{A}_{_{IVSVP}}$.

Let $IVSS^*(X) = \{\mathbf{A} \in IVSS(X) : A^- = A^+\}$. Then from Proposition 3.27, $\mathbf{A} = \mathbf{A}_{_{IVP}}$ for each $A \in IVSS^*(X)$.

Theorem 3.29. *Let $(\mathbf{A}_j)_{j\in J} \subset IVSS_E(X)$ and let $a \in X$, $e \in E$.*

(1) $e_{a_{IVSP}} \in \bigcap_{j\in J} \mathbf{A}_j$ *[resp.* $e_{a_{IVSVP}} \in \bigcap_{j\in J} \mathbf{A}_j$*]*
if and only if $e_{a_{IVSP}} \in \mathbf{A}_j$ *[resp.* $e_{a_{IVSVP}} \in \mathbf{A}_j$*] for each* $j \in J$.

(2) $e_{a_{IVSP}} \in \bigcup_{j\in J} \mathbf{A}_j$ *[resp.* $e_{a_{IVSVP}} \in \bigcup_{j\in J} \mathbf{A}_j$*]*
if and only if there exists $j \in J$ *such that* $e_{a_{IVSP}} \in \mathbf{A}_j$ *[resp.* $e_{a_{IVSVP}} \in \mathbf{A}_j$*].*

Proof. The proof is straightforward. □

Theorem 3.30. *Let* $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. *Then*

(1) $\mathbf{A} \subset \mathbf{B}$ *if and only if* $e_{a_{IVSP}} \in \mathbf{A} \Rightarrow e_{a_{IVSP}} \in \mathbf{B}$
[resp. $e_{a_{IVSVP}} \in \mathbf{A} \Rightarrow e_{a_{IVSVP}} \in \mathbf{B}$*]* $\forall a \in X,\ \forall e \in E$.

(2) $\mathbf{A} = \mathbf{B}$ *if and only if* $e_{a_{IVSP}} \in \mathbf{A} \Leftrightarrow e_{a_{IVSP}} \in \mathbf{B}$
[resp. $e_{a_{IVSVP}} \in \mathbf{A} \Leftrightarrow e_{a_{IVSVP}} \in \mathbf{B}$*]* $\forall a \in X,\ \forall e \in E$.

Proof. (1) Suppose $\mathbf{A} \subset \mathbf{B}$ and let $e_{a_{IVSP}} \in \mathbf{A}$ for each $a \in X$ and $e \in E$. Then $a_{IVSP} \in \mathbf{A}(e)$, i.e., $a \in A^-(e)$. Since $\mathbf{A} \subset \mathbf{B}$, $\mathbf{A}(e) \subset \mathbf{B}(e)$. Thus $a \in B^-(e)$, i.e., $a_{IVSP} \in \mathbf{B}(e)$. So $e_{a_{IVSP}} \in \mathbf{B}$. Also the proof of the second part is similar. The proof of the converse is true.

(2) The proof is straightforward from Definition 3.3 and (1). □

Theorem 3.31. *Let* $\mathbf{A} \in IVSS_E(X)$. *Then* $e_{a_{IVSP}} \in \mathbf{A}$ *if and only if* $e_{a_{IVSP}} \notin \mathbf{A}^c$.

Proof. Suppose $e_{a_{IVSP}} \in \mathbf{A}$. Then clearly, $a \in A^-(e)$. Thus $a \notin A^-(e)^c$. Since $A^-(e) \subset A^+(e)$, $A^+(e)^c \subset A^-(e)^c$. So $a \notin A^+(e)^c = (A^c)^-(e)$. Hence $e_{a_{IVSP}} \notin \mathbf{A}^c$. The proof of the converse is similar. □

Proposition 3.32. *Let* $(\mathbf{A}_j)_{j\in J} \subset IVSS_E(X)$ *and let* $\mathbf{A} = \bigcup_{j\in J} \mathbf{A}_j$. *Then*

(1) $\mathbf{A}_{IVSP} = \bigcup_{j\in J} \mathbf{A}_{j_{IVSP}}$,

(2) $\mathbf{A}_{IVSVP} = \bigcup_{j\in J} \mathbf{A}_{j_{IVSVP}}$.

Proof. (1) For each $j \in J$, let $e \in E$. Then clearly, $\mathbf{A}_j(e) = [A_j(e)^-, A_j(e)^+]$. Thus we have we have

$$\mathbf{A}(e) = (\bigcup_{j\in J} \mathbf{A}_j)(e) = [\bigcup_{j\in J} A_j(e)^-, \bigcup_{j\in J} A_j(e)^+].$$

Now let $e_{a_{IVSP}} \in \mathbf{A}$. Then $e_{a_{IVSP}} \in \bigcup_{j\in J} \mathbf{A}_j$. Thus $a \in \bigcup_{j\in J} A_j(e)^-$. So there is $j_0 \in J$ such that $a \in A_{j_0}(e)^-$. Hence $e_{a_{IVSP}} \in \mathbf{A}_{j_0 IVSP}$, i.e., $\mathbf{A}_{IVSP} \subset \bigcup_{j\in J} \mathbf{A}_{j_{IVSP}}$.

Conversely, suppose $e_{a_{IVSP}} \in \bigcup_{j\in J} \mathbf{A}_{j_{IVSP}}$. Then there is $j_0 \in J$ such that $e_{a_{IVSP}} \in \mathbf{A}_{j_0 IVSP}$. Thus $a \in A_{j_0}(e)^-$. So $a \in \bigcup_{j\in J} A_j(e)^-$. Hence $e_{a_{IVSP}} \in A_{IVSP}$, i.e., $\bigcup_{j\in J} \mathbf{A}_{j_{IVSP}} \subset \mathbf{A}_{IVSP}$. Therefore $\mathbf{A}_{IVSP} = \bigcup_{j\in J} \mathbf{A}_{j_{IVSP}}$.

(2) The proof is similar to that of (1). □

4. Interval-valued soft topological spaces

In this section, we define an interval-valued soft topology and obtain some of its properties, and give some examples.

Definition 4.1. Let τ be a family of IVSSs over X with respect to E. Then τ is called an *interval-valued soft topology* (briefly, IVST) *on* X *with respect to* E, if it satisfies the following axioms:

[IVSO$_1$] $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau$,

[IVSO$_2$] $\mathbf{A} \cap \mathbf{B} \in \tau$ for any $\mathbf{A},\ \mathbf{B} \in \tau$,

[IVSO$_3$] $\bigcup_{j\in J} \mathbf{A}_j \in \tau$ for each $(\mathbf{A}_j)_{j\in J} \subset \tau$.

The triple (X, τ, E) is called an *interval-valued soft topological space* (briefly, IVSTS). Every member of τ is called an *interval-valued soft open set* (briefly, IVSOS) and the complement of an IVSOS is called an *interval-valued soft closed set* (briefly, IVSCS) in X, and the set of all IVSOSs [resp. IVSCSs] in X is denoted by $IVSO(X)$ [resp. $IVSC(X)$]. It is obvious that $\{\widetilde{\varnothing}_E, \widetilde{X}_E\},\ IVSS_E(X) \in IVST_E(X)$, where $IVST_E(X)$ denotes the set of all IVSTSs on X with respect to E. In this case, $\{\widetilde{\varnothing}_E, \widetilde{X}_E\}$ [resp. $IVSS_E(X)$] is called an *interval-valued soft indiscrete* [resp. *discrete*] *topology* on X and denoted by $\widetilde{\tau}_0$ [resp. $\widetilde{\tau}_1$].

It is obvious that if $\tau \in IVST_E(X)$, then $\chi_\tau = \{\chi_{\mathbf{U}} : \mathbf{U} \in \tau\}$ is an interval-valued fuzzy soft topology (briefly, IVFST) on X defined by Ali et al. [42]. Thus an IVFST is the generalization of an IVST.

Example 4.2. (1) Let $X = \mathbb{N}$, $E = \{0, 1\}$ and let τ be the collection of IVSSs over X given by:
$$\tau = \{\widetilde{\varnothing}_E, \widetilde{X}_E\} \cup \mathbf{A}_n : n \in \mathbb{N}\},$$
where $\mathbf{A}_n : E \to IVS(X)$ defined by: for each $e \in E$,
$$\mathbf{A}_n(e) = \begin{cases} [\{n+1, n+2, \cdots\}, \{n, n+1, n+2, \cdots\}] & \text{if } e = 0 \\ [\varnothing, \{n\}] & \text{if } e = 1. \end{cases}$$
Then we can easily see that (X, τ, E) is an IVSTS.

(2) Let (X, T) be a classical topological space and let E be a nonempty set of parameters. Consider the following family
$$\tau = \{\mathbf{A}_U \in IVS(X) : U \in T\},$$
where $\mathbf{A}_U : E \to IVS(X)$ defined as follows: for each $e \in E$,
$$\mathbf{A}_U(e) = [U, U].$$
Then clearly, $\tau \in IVST_E(X)$.

(3) Let (X, T) be an interval-valued topological space (briefly, IVTS) proposed by Kim et al. [39] and let E be a nonempty set of parameters. Consider the following family
$$\tau = \{\mathbf{A}_U \in IVS(X) : U \in T\},$$
where $\mathbf{A}_U : E \to IVS(X)$ defined as follows: for each $e \in E$,
$$\mathbf{A}_U(e) = \mathbf{U} = [U^-, U^+].$$
Then clearly, $\tau \in IVST_E(X)$.

(4) Let $X = \{h_1, h_2, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}$ be the universe set of houses and let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ be the set of parameters, where

e_1 stands for the parameter *verycostly*,

e_2 stands for the parameter *costly*,
e_3 stands for the parameter *cheap*,
e_4 stands for the parameter *beautiful*,
e_5 stands for the parameter *in the surroundings*,
e_6 stands for the parameter *wooden*,
e_7 stands for the parameter *modern*,
e_7 stands for the parameter *in good repair*,
e_8 stands for the parameter *in bad repair*.

Consider the IVSSs $\mathbf{A},\ \mathbf{B}, \mathbf{C},\ \mathbf{D}$ given by:

$\mathbf{A}(e_1) = [\{h_2, h_4\}, \{h_2, h_4, h_7, h_8\}]\,,\ \mathbf{A}(e_2) = [\{h_1, h_3\}, \{h_1, h_3, h_5\}]\,,$
$\mathbf{A}(e_3) = [\{h_6\}, \{h_6, h_9\}]\,,\ \mathbf{A}(e) = \widetilde{\varnothing}$ for each $e \in E \setminus \{e_1, e_2, e_3\}$,
$\mathbf{B}(e_3) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]\,,\ \mathbf{B}(e_4) = [\{h_2, h_3\}, \{h_2, h_3, h_7\}]\,,$
$\mathbf{B}(e_5) = [\{h_5, h_6\}, \{h_5, h_6, h_8\}]\,,\ \mathbf{B}(e) = \widetilde{\varnothing}$ for each $e \in E \setminus \{e_3, e_4, e_5\}$,
$\mathbf{B}(e_3) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]\,,\ \mathbf{B}(e_4) = [\{h_2, h_3\}, \{h_2, h_3, h_7\}]\,,$
$\mathbf{C}(e_3) = [\{h_6\}, \{h_6, h_9\}]\,,\ \mathbf{C}(e) = \widetilde{\varnothing}$ for each $e \in E \setminus \{e_3\}$,
$\mathbf{D}(e_1) = [\{h_2, h_4\}, \{h_2, h_4, h_7, h_8\}]\,,\ \mathbf{D}(e_2) = [\{h_1, h_3\}, \{h_1, h_3, h_5\}]\,,$
$\mathbf{D}(e_3) = [\{h_6, h_9\}, \{h_6, h_9, h_{10}\}]\,,\ \mathbf{D}(e_4) = [\{h_2, h_3\}, \{h_2, h_3, h_7\}]\,,$
$\mathbf{D}(e_5) = [\{h_5, h_6\}, \{h_5, h_6, h_8\}]\,,\ \mathbf{A}(e) = \widetilde{\varnothing}$ for each $e \in E \setminus \{e_1, e_2, e_3, e_4, e_5\}$.

Then we can check that $\tau = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \widetilde{X}_E\} \in IVST_E(X)$.

Remark 4.3. Let $\tau \in IVST_E(X)$. Then there are two soft topologies over X with respect to E given by:

$$\tau^- = \{U^- \in 2^X : \mathbf{U} \in \tau\},\ \tau^+ = \{U^+ \in 2^X : \mathbf{U} \in \tau\}.$$

Thus we can consider (X, τ^-, τ^+, E) as soft bi-topological space in the sense of Kelly [43] (Refer to [23, 24, 27, 30] for soft topological spaces).

From Definition 4.1 and Propositions 3.20 and 3.24, we get the following.the above comments, we have the following.

Proposition 4.4. *Let (X, τ, E) be an IVSTS and let*

$$\tau^c = \{\mathbf{U}^c \in IVSS(X) : \mathbf{U} \in \tau\}.$$

Then τ^c has the following properties:
(1) $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau^c$,
(2) $\mathbf{A} \cup \mathbf{B} \in \tau^c$ *for any* $\mathbf{A},\ \mathbf{B} \in \tau^c$,
(3) $\bigcap_{j \in J} \mathbf{A}_j \in \tau^c$ *for each* $(\mathbf{A}_j)_{j \in J} \subset \tau^c$.

Proposition 4.5. *Let (X, τ, E) be an IVSTS and for each $e \in E$, let*

$$\tau_e = \{\mathbf{U}(e) \in IVS(X) : \mathbf{U} \in \tau\}.$$

Then τ_e is an interval-valued topology (briefly, IVT) on X introduced by Kim et al. [39].

Proof. Since $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau$, $\widetilde{\varnothing}_E(e) = \widetilde{\varnothing},\ \widetilde{X}_E(e) = \widetilde{X}$. Then $\widetilde{\varnothing},\ \widetilde{X} \in \tau_e$. Suppose $\mathbf{A}(e),\ \mathbf{B}(e) \in \tau_e$. Then clearly, $(\mathbf{A} \cap \mathbf{B})(e) = \mathbf{A}(e) \cap \mathbf{B})(e)$ and $\mathbf{A} \cap \mathbf{B} \in \tau$. Thus $\mathbf{A}(e) \cap \mathbf{B})(e) \in \tau_e$. Finally, suppose $(\mathbf{A}_j(e))_{j \in J} \subset \tau_e$. Then we get $\bigcup_{j \in J} \mathbf{A}_j(e) = \left(\bigcup_{j \in J} \mathbf{A}_j\right)(e)$ and $\bigcup_{j \in J} \mathbf{A}_j \in \tau$. Thus $\bigcup_{j \in J} \mathbf{A}_j(e) \in \tau_e$. So τ_e is an IVT on X □

Remark 4.6. (1) From Proposition 4.5 and Remark 4.2 (1) in [39], the following two families of subsets of X:

$$\tau_e^- = \{A^- \in 2^X : \mathbf{A} \in \tau_e\} \text{ and } \tau_e^+ = \{A^+ \in 2^X : \mathbf{A} \in \tau_e\}$$

are classical topologies on X.

(2) The converse of Proposition 4.5 does not hold in general (See Example 4.7 (2)).

Proposition 4.5 shows that corresponding to each parameter $e \in E$, we get an IVT τ_e on X. Then an IVST on X with respect to E gives a parametrized family of IVTs on X.

Example 4.7. (1) Let $X = \{h_1, h_2, h_3\}$ and let $E = \{e_1, e_2\}$. Consider the family τ of IVSSs over X given by:

$$\tau = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\},$$

where $\mathbf{A}(e_1) = [\varnothing, \{h_2\}]$, $\mathbf{A}(e_2) = [\varnothing, \{h_1\}]$,
$\mathbf{B}(e_1) = [\{h_2\}, \{h_2, h_3\}]$, $\mathbf{B}(e_2) = [\{h_1\}, \{h_1, h_2\}]$,
$\mathbf{C}(e_1) = [\{h_1, h_2\}, X]$, $\mathbf{C}(e_2) = [\{h_1\}, X]$,
$\mathbf{D}(e_1) = [\{h_1\}, \{h_1, h_2\}]$, $\mathbf{D}(e_2) = [\{h_1\}, \{h_1, h_3\}]$.
Then clearly, (X, τ, E) is an IVSTS. Thus we can easily see that

$$\tau_{e_1} = \{\widetilde{\varnothing}, \widetilde{X}, [\varnothing, \{h_2\}], [\{h_2\}, \{h_2, h_3\}], [\{h_1, h_2\}, X], [\{h_1\}, \{h_1, h_2\}]\}$$

and

$$\tau_{e_2} = \{\widetilde{\varnothing}, \widetilde{X}, [\varnothing, \{h_1\}], [\{h_1\}, \{h_1, h_2\}], [\{h_1\}, X], [\{h_1\}, \{h_1, h_3\}]\}$$

are IVTs on X. Furthermore, we have four classical topologies on X from Remark 4.5 (1):

$$\tau_{e_1}^- = \{\varnothing, X, \{h_1\}, \{h_2\}, \{h_1, h_2\}\},\ \tau_{e_1}^+ = \{\varnothing, X, \{h_2\}, \{h_1, h_2\}, \{h_2, h_3\}\},$$
$$\tau_{e_2}^- = \{\varnothing, X, \{h_1\}\},\ \tau_{e_2}^+ = \{\varnothing, X, \{h_1\}, \{h_1, h_2\}, \{h_1, h_3\}\}.$$

(2) Let $X = \{h_1, h_2, h_3\}$ and let $E = \{e_1, e_2\}$. Consider the family τ of IVSSs over X given by:

$$\tau = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\},$$

where $\mathbf{A}(e_1) = [\{h_2\}, \{h_2\}]$, $\mathbf{A}(e_2) = [\{h_1\}, \{h_1\}]$,
$\mathbf{B}(e_1) = [\{h_2, h_3\}, \{h_2, h_3\}]$, $\mathbf{B}(e_2) = [\{h_1, h_2\}, \{h_1, h_2\}]$,
$\mathbf{C}(e_1) = [\{h_1, h_2\}, \{h_1, h_2\}]$, $\mathbf{C}(e_2) = [\{h_1, h_2\}, \{h_1, h_2\}]$,
$\mathbf{D}(e_1) = [\{h_2\}, \{h_2\}]$, $\mathbf{D}(e_2) = [\{h_1, h_3\}, \{h_1, h_3\}]$.
Then we have $(\mathbf{B} \cup \mathbf{C})(e_1) = \widetilde{X}$. Thus $\mathbf{B} \cup \mathbf{C} \notin \tau$. So $\tau \notin IVST_E(X)$. But we can easily check that the following two families:

$$\tau_{e_1} = \{\widetilde{\varnothing}, \widetilde{X}, [\{h_2\}, \{h_2\}], [\{h_1, h_2\}, \{h_1, h_2\}], [\{h_2, h_3\}, \{h_2, h_3\}],$$
$$\tau_{e_2} = \{\widetilde{\varnothing}, \widetilde{X}, [\{h_1\}, \{h_1\}], [\{h_1, h_2\}, \{h_1, h_2\}], [\{h_1, h_3\}, \{h_1, h_3\}]$$

are IVTs on X. Moreover, we get four classical topologies on X:

$$\tau_{e_1}^- = \tau_{e_1}^+ = \{\varnothing, X, \{h_2\}, \{h_1, h_2\}, \{h_2, h_3\}\},$$
$$\tau_{e_2}^- = \tau_{e_2}^+ = \{\varnothing, X, \{h_1\}, \{h_1, h_2\}, , \{h_1, h_3\}\}.$$

Proposition 4.8. *If* $\tau_1,\ \tau_2 \in IVST_E(X)$*, then* $\tau_1 \cap \tau_2 \in IVST_E(X)$*.*

Proof. Since $\tau_1,\ \tau_2 \in IVST_E(X)$, $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau_1 \cap \tau_2$. Then $\tau_1 \cap \tau_2$ satisfies the axiom [IVSO$_1$]. Let $\mathbf{A},\ \mathbf{B} \in \tau_1 \cap \tau_2$. Then clearly, $\mathbf{A},\ \mathbf{B} \in \tau_1$ and $\mathbf{A},\ \mathbf{B} \in \tau_2$. Thus $\mathbf{A} \cap \mathbf{B} \in \tau_1$ and $\mathbf{A} \cap \mathbf{B} \in \tau_2$. So $\mathbf{A} \cap \mathbf{B} \in \tau_1 \cap \tau_2$. Hence $\tau_1 \cap \tau_2$ satisfies the axiom [IVSO$_2$]. Finally, let $(\mathbf{A}_j)_{j\in J} \subset \tau_1 \cap \tau_2$. Then clearly, $\mathbf{A}_j \in \tau_1$ and $\mathbf{A}_j \in \tau_2$ for each $j \in J$. Thus $\bigcup_{j\in J} \mathbf{A}_j \in \tau_1$ and $\bigcup_{j\in J} \mathbf{A}_j \in \tau_2$. So $\bigcup_{j\in J} \mathbf{A}_j \in \tau_1 \cap \tau_2$. Hence $\tau_1 \cap \tau_2$ satisfies the axiom [IVSO$_3$]. Therefore $\tau_1 \cap \tau_2 \in IVST_E(X)$. □

Corollary 4.9. *$\bigcap_{j\in J} \tau_j \in IVST_E(X)$ for any $(\tau_j)_{j\in J} \subset IVST_E(X)$.*

Remark 4.10. The interval-valued soft union of two IVVSTs need not be an IVST (See Example 4.11).

Example 4.11. Let $X = \{h_1, h_2, h_3\}$ and let $E = \{e_1, e_2\}$. Consider two family τ_1 and of τ_2 IVSSs over X given by:

$$\tau_1 = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\},$$

$$\tau_2 = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}\},$$

where $\mathbf{A}(e_1) = [\{h_1\}, \{h_1, h_2\}],\ \mathbf{A}(e_2) = [\{h_2\}, \{h_2, h_3\}]$,
$\mathbf{B}(e_1) = [\{h_2\}, \{h_2, h_3\}],\ \mathbf{B}(e_2) = [\{h_2\}, \{h_1, h_2\}]$,
$\mathbf{C}(e_1) = [\varnothing, \{h_2\}],\ \mathbf{C}(e_2) = [\{h_2\}, \{h_2\}]$,
$\mathbf{D}(e_1) = [\{h_1, h_2\}, X],\ \mathbf{D}(e_2) = [\{h_2\}, X]$,
$\mathbf{E}(e_1) = [\{h_1\}, \{h_1\}],\ \mathbf{E}(e_2) = [\{h_3\}, \{h_2, h_3\}]$,
$\mathbf{F}(e_1) = [\{h_2\}, \{h_1, h_2\}],\ \mathbf{F}(e_2) = [\{h_3\}, \{h_3\}]$,
$\mathbf{G}(e_1) = [\varnothing, \{h_1\}],\ \mathbf{G}(e_2) = [\{h_3\}, \{h_3\}]$,
$\mathbf{H}(e_1) = [\{h_1, h_2\}, \{h_1, h_2\}],\ \mathbf{H}(e_2) = [\{h_3\}, \{h_2, h_3\}]$.
Then clearly, $\tau_1 \cup \tau_2 = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{G}, \mathbf{H}\}$. Thus we have

$$(\mathbf{B} \cup \mathbf{G})(e_1) = [\{h_2\}, X].$$

So $\mathbf{B} \cup \mathbf{G} \notin \tau_1 \cup \tau_2$. Hence $\tau_1 \cup \tau_2 \notin IVST_E(X)$.

Definition 4.12. Let $\tau_1,\ \tau_2 \in IVST_E(X)$ Then we say that:
(i) τ_1 is *coarser than* τ_2 or τ_2 is *finer than* τ_1, if $\tau_1 \subset \tau_2$,
(ii) τ_1 is *strictly coarser than* τ_2 or τ_2 is *strictly finer than* τ_1, if $\tau_1 \subset \tau_2$ and $\tau_1 \neq \tau_2$,
(iii) τ_1 is *comparable with* τ_2, if either $\tau_1 \subset \tau_2$ or $\tau_2 \subset \tau_1$.

It is obvious that $\widetilde{\tau}_0 \subset \tau \subset \widetilde{\tau}_1$ for each $\tau \in IVST_E(X)$ and $(IVST_E(X), \subset)$ forms a meet lattice with the smallest element $\widetilde{\tau}_0$ and $\widetilde{\tau}_1$ from Corollary 4.9.

Definition 4.13. Let $\mathbf{A},\ \mathbf{B} \in IVSS_E(X)$. Then the *difference* of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} - \mathbf{B}$, is the mapping $\mathbf{A} - \mathbf{B} : E \to IVS(X)$ defined by: for each $e \in E$,

$$(\mathbf{A} - \mathbf{B})(e) = \mathbf{A}(e) - \mathbf{B}(e) = \mathbf{A}(e) \cap \mathbf{B}^c(e) = [A^-(e) \cap B^+(e), A^+(e) \cap B^-(e)].$$

Lemma 4.14. *Let $\mathbf{A},\ \mathbf{B},\ \mathbf{C} \in IVSS_E(X)$. If $\mathbf{A} - \mathbf{B} = \mathbf{A} \cap \mathbf{C}$, then $\mathbf{B} = \mathbf{A} \cap \mathbf{C}^c$.*

Proof. Suppose $\mathbf{A} - \mathbf{B} = \mathbf{A} \cap \mathbf{C}$ and let $e \in E$. Then we have

$$\mathbf{B} = \mathbf{A} - (\mathbf{A} - \mathbf{B}) = \mathbf{A} - (\mathbf{A} \cap \mathbf{C}).$$

Thus we get

$\mathbf{B}(e) = \mathbf{A}(e) \cap (\mathbf{A} \cap \mathbf{C})^c(e)$

$= [A^-(e), A^+(e)] \cap ([A^+(e), A^-(e)] \cup [C^+(e), C^-(e)]$
$= [A^-(e), A^+(e)] \cap [A^+(e) \cup C^+(e), A^-(e) \cup C^-(e)]$
$= [A^-(e) \cap (A^+(e) \cup C^+(e)), A^+(e) \cap (A^-(e) \cup C^-(e))]$
$= (\mathbf{A} \cap \mathbf{C}^c)(e).$

So $\mathbf{B} = \mathbf{A} \cap \mathbf{C}^c$. □

Proposition 4.15. *Let $\mathbf{A} \in IVSS_E(X)$ and let $\tau \in IVST_E(X)$. Then the following family*

$$\tau_{\mathbf{A}} = \{\mathbf{A} \cap \mathbf{U} : \mathbf{U} \in \tau\}$$

is an IVST on $\mathbf{A}$.

Proof. Clearly, $\widetilde{\varnothing}_E, \widetilde{X}_E \in \tau$. Then by Proposition 3.20 (8_a) and (8_b), $\mathbf{A} \cap \widetilde{\varnothing}_E = \widetilde{\varnothing}_E$ and $\mathbf{A} \cap \widetilde{X}_E = \mathbf{A}$. Thus $\widetilde{\varnothing}_E$, $\mathbf{A} \in \tau_{\mathbf{A}}$. So $\tau_{\mathbf{A}}$ satisfies the axiom [IVSO$_1$]. Let $\mathbf{B}$, $\mathbf{C} \in \tau_{\mathbf{A}}$. Then there are $\mathbf{U}$, $\mathbf{V} \in \tau$ such that $\mathbf{B} = \mathbf{A} \cap \mathbf{U}$ and $\mathbf{C} = \mathbf{A} \cap \mathbf{V}$. Thus by Proposition 3.20 (1) and (2), $\mathbf{B} \cap \mathbf{C} = \mathbf{A} \cap (\mathbf{U} \cap \mathbf{V})$ and $\mathbf{U} \cap \mathbf{V} \in \tau$. So $\mathbf{B} \cap \mathbf{C} \in \tau_{\mathbf{A}}$. Hence $\tau_{\mathbf{A}}$ satisfies the axiom [IVSO$_2$]. Now let $(\mathbf{A}_j)_{j\in J} \subset \tau_{\mathbf{A}}$. Then there is $\mathbf{U}_j \in \tau$ such that $\mathbf{A}_j = \mathbf{A} \cap \mathbf{U}_j$ for each $j \in J$. Thus by Proposition 3.24 (1), we have $\bigcup_{j\in J} \mathbf{A}_j = \mathbf{A} \cap (\bigcup_{j\in J} \mathbf{U}_j)$. So $\bigcup_{j\in J} \mathbf{A}_j \in \tau_{\mathbf{A}}$. Hence $\tau_{\mathbf{A}}$ satisfies the axiom [IVSO$_3$]. Therefore $\tau_{\mathbf{A}}$ is an IVST on $\mathbf{A}$. □

In Proposition 4.15, $\tau_{\mathbf{A}}$ is called an *interval-valued soft relative topology* (briefly, IVSRT) on $\mathbf{A}$ and the pair $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ called an *interval-valued soft subspace* of (X, τ, E). Every member of $\tau_{\mathbf{A}}$ is called an *interval-valued soft open set in* $\mathbf{A}$ and an IVSS $\mathbf{B}$ is called an *interval-valued soft closed set in* $\mathbf{A}$, if $\mathbf{A} - \mathbf{B} \in \tau_{\mathbf{A}}$, where $\mathbf{B} \subset \mathbf{A}$.

Example 4.16. (1) Let $X = \{h_1, h_2, h_3\}$ and let $E = \{e_1, e_2\}$. Consider the IVST τ given by:

$$\tau = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4\},$$

where $\mathbf{U}_1(e_1) = [\{h_1, h_2\}, X],\ \mathbf{U}_1(e_2) = [\{h_1\}, \{h_1, h_2\}]$,
$\mathbf{U}_2(e_1) = [\{h_2\}, \{h_2, h_3\}],\ \mathbf{U}_2(e_2) = [\{h_1, h_3\}, X]$,
$\mathbf{U}_3(e_1) = [\varnothing, \{h_2\}],\ \mathbf{U}_3(e_2) = [\{h_1\}, \{h_1, h_2\}]$,
$\mathbf{U}_4(e_1) = [\{\{h_1, h_2\}, X],\ \mathbf{U}_4(e_2) = [\{h_1, h_3\}, X\}]$.

Let $\mathbf{A}$ be an IVSS over X with respect to E given by:

$$\mathbf{A}(e_1) = [\{h_1, h_3\}, \{h_1, h_3\}],\ \mathbf{A}(e_2) = [\{h_1\}, \{h_1, h_3\}].$$

Then we have

$$\tau_{\mathbf{A}} = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{A} \cap \mathbf{U}_1, \mathbf{A} \cap \mathbf{U}_2, \mathbf{A} \cap \mathbf{U}_3, \mathbf{A} \cap \mathbf{U}_4\},$$

where $(\mathbf{A} \cap \mathbf{U}_1)(e_1) = [\{h_1\}, \{h_1, h_3\}],\ (\mathbf{A} \cap \mathbf{U}_1)(e_2) = [\{h_1\}, \{h_1\}]$,
$(\mathbf{A} \cap \mathbf{U}_2)(e_1) = [\varnothing, \{h_3\}],\ (\mathbf{A} \cap \mathbf{U}_2)(e_2) = [\{h_1\}, \{h_1, h_3\}]$,
$(\mathbf{A} \cap \mathbf{U}_3)(e_1) = \widetilde{\varnothing},\ (\mathbf{A} \cap \mathbf{U}_3)(e_2) = [\{h_1\}, \{h_1\}]$,
$(\mathbf{A} \cap \mathbf{U}_4)(e_1) = (\mathbf{A} \cap \mathbf{U}_4)(e_2) = [\{h_1\}, \{h_1, h_3\}]$.

(2) Every interval-valued soft subspace of an interval-valued soft discrete space is an interval-valued soft discrete space.

(3) Every interval-valued soft subspace of an interval-valued soft indiscrete space is an interval-valued soft indiscrete space.

Proposition 4.17. *Let (X, τ, E) be an IVSTS and let $\mathbf{A} \in IVSS_E(X)$. Then $(\mathbf{A}(e), \tau_{\mathbf{A}}(e))$ is an interval-valued subspace of (X, τ_e) for each $e \in E$ proposed by Lee et al.* [44].

Proof. From Propositions 4.5 and 4.15, it is clear that $\tau_{\mathbf{A}}(e)$ is an IVT on $\mathbf{A}(e)$ for each $e \in E$. Let $e \in E$. Then we have

$$\begin{aligned}\tau_{\mathbf{A}}(e) &= \{(\mathbf{A} \cap \mathbf{U})(e) : \mathbf{U} \in \tau\} = \{\mathbf{A}(e) \cap \mathbf{U}(e) : \mathbf{U} \in \tau\} \\ &= \{\mathbf{A}(e) \cap \mathbf{U}(e) : \mathbf{U}(e) \in \tau_e\}.\end{aligned}$$

Thus $(\mathbf{A}(e), \tau_{\mathbf{A}})$ is an interval-valued subspace of (X, τ_e) for each $e \in E$. □

Corollary 4.18. *Let (X, τ, E) be an IVSTS and let $\mathbf{A} \in IVSS_E(X)$. Then for each $e \in E$,*

$$(A^-(e), \tau_{\mathbf{A}}(e)^-),\ (A^+(e), \tau_{\mathbf{A}}(e)^+)$$

are classical subspaces of (X, τ_e^-) and (X, τ_e^+) respectively, where

$$\tau_{\mathbf{A}}(e)^- = \{A^-(e) \cap U^-(e) : U^-(e) \in \tau_e^-\},$$

$$\tau_{\mathbf{A}}(e)^+ = \{A^+(e) \cap U^+(e)^+ : U^+(e) \in \tau_e^+\}.$$

Proof. The proof is clear from Propositions 4.5 and 4.17, and Remark 4.6 (1). □

Example 4.19. Let $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ be the interval-valued subspace of the IVTS (X, τ, E) given in Example 4.16. Then we have two interval-valued relative topologies on $\mathbf{A}(e_1)$ and $\mathbf{A}(e_2)$, respectively:

$$\tau_{\mathbf{A}}(e_1) = \{\widetilde{\varnothing}, \widetilde{X}, [\{h_1\}, \{h_1, h_3\}], [\varnothing, \{h_3\}]\},$$

$$\tau_{\mathbf{A}}(e_2) = \{\widetilde{\varnothing}, \widetilde{X}, [\{h_1\}, \{h_1\}], [\{h_1\}, \{h_1, h_3\}]\}.$$

Moreover, we can check that

$$(\mathbf{A}(e_1)^-, \tau_{\mathbf{A}}(e_1)^-),\ (\mathbf{A}(e_1)^+, \tau_{\mathbf{A}}(e_1)^+),\ (\mathbf{A}(e_2)^-, \tau_{\mathbf{A}}(e_2)^-),\ (\mathbf{A}(e_2)^+, \tau_{\mathbf{A}}(e_2)^+)$$

are classical subspces of $(X, \tau_{e_1}^-)$, $(X, \tau_{e_1}^+)$, $(X, \tau_{e_2}^-)$, $(X, \tau_{e_2}^+)$ respectively, where

$$\tau_{\mathbf{A}}(e_1)^- = \{\varnothing, X, \{h_1\}\},\ \tau_{\mathbf{A}}(e_1)^+ = \{\varnothing, X, \{h_3, \{h_1, h_3\}\},$$

$$\tau_{\mathbf{A}}(e_2)^- = \{\varnothing, X, \{h_1\}\},\ \tau_{\mathbf{A}}(e_2)^+ = \{\varnothing, X, \{h_1, \{h_1, h_3\}\}.$$

Proposition 4.20. *Let $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ be an IVSTS (X, τ, E) and let $\mathbf{B} \in \tau_{\mathbf{A}}$. If $\mathbf{A} \in \tau$, then $\mathbf{B} \in \tau$.*

Proof. Let $\mathbf{B} \in \tau_{\mathbf{A}}$. Then clearly, there is $\mathbf{U} \in \tau$ such that $\mathbf{B} = \mathbf{A} \cap \mathbf{U}$. Since $\mathbf{A} \in \tau$, $\mathbf{A} \cap \mathbf{U} \in \tau$. Thus $\mathbf{B} \in \tau$. □

Theorem 4.21. *Let $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ be an IVSTS (X, τ, E) and let $\mathbf{B} \in IVSS(X)$. Then $\mathbf{B}$ is interval-valued soft closed in $\mathbf{A}$ if and only if there is an IVSCS $\mathbf{C}$ in X such that $\mathbf{B} = \mathbf{A} \cap \mathbf{C}$.*

Proof. Suppose $\mathbf{B}$ is an interval-valued soft closed in $\mathbf{A}$. Since $\mathbf{A} - \mathbf{B} \in \tau_{\mathbf{A}}$, there is $\mathbf{U} \in \tau$ such that $\mathbf{A} - \mathbf{B} = \mathbf{A} \cap \mathbf{U}$. Then by Lemma 4.15, $\mathbf{B} = \mathbf{A} \cap \mathbf{U}^c$ and $\mathbf{U}^c \in \tau^c$. Thus the necessary condition holds.

Conversely, suppose there is an IVSCS $\mathbf{C}$ in X such that $\mathbf{B} = \mathbf{A} \cap \mathbf{C}$ and let $e \in E$. Then clearly, $\mathbf{C}^c \in \tau$. Moreover, by Lemma 4.14, we have $\mathbf{A} - \mathbf{B} = \mathbf{A} \cap \mathbf{C}^c$. Thus $\mathbf{A} - \mathbf{B} \in \tau_{\mathbf{A}}$. So $\mathbf{B}$ is an interval-valued soft closed in $\mathbf{A}$. □

Definition 4.22. Let (X, τ, E) be an IVSTS and let $\beta,\ \sigma \subset \tau$. Then

(i) β is called an *interval-valued soft base* (briefly, IVSB) for τ, if $\mathbf{U} = \widetilde{\varnothing}_E$ or there is $\beta^{'} \subset \beta$ such that $\mathbf{U} = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta^{'}\}$ for any $\mathbf{U} \in \tau$.

(ii) σ is called an *interval-valued soft subbase* (briefly, IVSSB) for τ, if the family of all finite intersections of members of σ is an IVSB for τ.

Example 4.23. Let $X = \{a, b, c\}$ and let $E = \{e\}$. Consider the family β of IVSSs over X given by:

$$\beta = \{\widetilde{X}_E, \mathbf{A}, \mathbf{B}\},$$

where $\mathbf{A}(e) = [\{a, b\}, X],\ \mathbf{B}(e) = [\{b, c\}, X]$.
Assume that β is an IVSB for an IVST τ. Then clearly, $\beta \subset \tau$. Thus $\mathbf{A},\ \mathbf{B} \in \tau$. So $\mathbf{A} \cap \mathbf{B} \in \tau$ and $(\mathbf{A} \cap \mathbf{B})(e) = [\{a, b\}, X] \cap [\{b, c\}, X] = [\{b\}, X]$. But for any $\beta^{'} \subset \beta$, $[\{b\}, X] \neq (\bigcup \beta^{'})(e)$. Hence β is not an IVSB for τ.

Proposition 4.24. *Let β be an IVSB for an IVSTS (X, τ, E). Then for each $e \in E$, β_e is an IVB for the IVT τ_e defined by Kim et al.* [39], *where $\beta_e = \{\mathbf{B}(e) : \mathbf{B} \in \beta\}$.*

Proof. The proof is obvious from Proposition 4.5 and Definition 4.22. □

Theorem 4.25. *Let β be a family of IVSSs over X with respect to E. Then β is an IVSB for some IVST τ on X if and only if it satisfies the following conditions:*

(1) $\widetilde{X}_E = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta\}$,

(2) *if $\mathbf{B}_1,\ \mathbf{B}_2 \in \beta$ and $e_{a_{IVP}} \in \mathbf{B}_1 \cap \mathbf{B}_2$ [resp. $e_{a_{IVVP}} \in \mathbf{B}_1 \cap \mathbf{B}_2$], then there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$] and $\mathbf{B} \subset \mathbf{B}_1 \cap \mathbf{B}_2$.*

Proof. Suppose β is an IVSB for an IVST τ on X. Since $\widetilde{X}_E \in \tau$, $\widetilde{X}_E = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta\}$. Suppose $\mathbf{B}_1,\ \mathbf{B}_2 \in \beta$ and $e_{a_{IVP}} \in \mathbf{B}_1 \cap \mathbf{B}_2$. Then clearly, $\mathbf{B}_1,\ \mathbf{B}_2 \in \tau$. Thus $\mathbf{B}_1 \cap \mathbf{B}_2 \in \tau$. So there is a $\beta^{'} \subset \beta$ such that $\mathbf{B}_1 \cap \mathbf{B}_2 = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta^{'}\}$. Hence by Theorem 3.29 (2), there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ and $\mathbf{B} \subset \mathbf{B}_1 \cap \mathbf{B}_2$. The proof of the second part is similar.

Conversely, suppose β is a family of IVSSs over X with respect to E satisfying the conditions (1)and (2). Let $\tau \subset IVSS_E(X)$ be given by:

$$\tau = \{\widetilde{\varnothing}_E\} \bigcup \{\mathbf{U} : \mathbf{U} = \bigcup_{\beta^{'} \subset \beta} \{\mathbf{B} : \mathbf{B} \in \beta^{'}\}\}.$$

Then clearly, $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau$. From the definition of τ, it is clear that $\bigcup_{j \in J} \mathbf{U}_j \in \tau$ for any $(\mathbf{U}_j)_{j \in J} \subset \tau$. Suppose $\mathbf{U}_1,\ \mathbf{U}_2 \in \tau$ and $e_{a_{IVP}} \in \mathbf{U}_1 \cap \mathbf{U}_2$ [resp. $e_{a_{IVVP}} \in \mathbf{U}_1 \cap \mathbf{U}_2$]. Then by the condition (2) and Theorem 3.29 (2), there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$] and $\mathbf{B} \subset \mathbf{B}_1 \cap \mathbf{B}_2$. Thus $\mathbf{U}_1 \cap \mathbf{U}_2$ can be expressed as the union of members of a subcollection of β. So $\mathbf{U}_1 \cap \mathbf{U}_2 \in \tau$. Hence $\tau \in IVST_E(X)$ and β is an IVSB for τ. This completes the proof. □

Example 4.26. (1) Let $X = \{a, b, c\}$ and let $E = \{e_1, e_2\}$. Consider the family β of IVSSs over X given by:

$$\beta = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}\},$$

where $\mathbf{A}(e_1) = [\{a\}, \{a, b\}],\ \mathbf{A}(e_{12}) = [\{b\}, \{b, c\}]$,
$\mathbf{B}(e_1) = [\{a, c\}, X],\ \mathbf{B}(e_{12}) = [\{b, c\}, \{b, c\}]$,

$\mathbf{C}(e_1) = [\{a, b\}, \{a, b\}],\ \mathbf{C}(e_{12}) = [\{b\}, \{b, c\}]$.
Then we can easily check that β satisfies the conditions of Theorem 4.25. Thus β is an IVSB for an IVST τ on X. In fact, $\tau = \{\widetilde{\varnothing}_E, \mathbf{A}, \mathbf{B}, \mathbf{C}, \widetilde{X}_E\}$.

The following provides a characterization for an IVST τ_2 to be finer than an IVST τ_1 in terms of IVSBs for τ_1 and τ_2.

Theorem 4.27. *Let (X, τ_1, E), (X, τ_2, E) be two IVSTSs and let β, $\beta^{'}$ be IVSBs for τ_1 and τ_2 respectively. Then τ_2 is finer than τ_1 if and only if for each $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$], there is $\mathbf{B}^{'} \in \beta^{'}$ such that $e_{a_{IVP}} \in \mathbf{B}^{'}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}^{'}$] and $\mathbf{B}^{'} \subset \mathbf{B}$.*

Proof. Suppose τ_2 is finer than τ_1 and let $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$]. Then clearly, $\mathbf{B} \in \tau_2$. Since $\beta^{'}$ is an IVSB for τ_2, by Theorem 3.29 (2), there is $\mathbf{B}^{'} \in \beta^{'}$ such that $e_{a_{IVP}} \in \mathbf{B}^{'}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}^{'}$] and $\mathbf{B}^{'} \subset \mathbf{B}$.

Conversely, suppose the necessary condition holds and let $\mathbf{U} \in \tau_1$ such that $e_{a_{IVP}} \in \mathbf{U}$ [resp. $e_{a_{IVVP}} \in \mathbf{U}$]. Since β is an IVSB for τ_1, there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$] and $\mathbf{B} \subset \mathbf{U}$. Then by the condition (2), there is $\mathbf{B}^{'} \in \beta^{'}$ such that $e_{a_{IVP}} \in \mathbf{B}^{'}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}^{'}$] and $\mathbf{B}^{'} \subset \mathbf{B}$. Thus $\mathbf{B}^{'} \subset \mathbf{U}$. So $\mathbf{U}$ is the union of members of a collection of $\mathbf{B}^{'}$. Hence $\mathbf{U} \in \tau_2$. Therefore τ_2 is finer than τ_1. □

Definition 4.28. Let (X, τ_1, E), (X, τ_2, E) be two IVSTSs and let β_1, β_2 be IVSBs for τ_1 and τ_2 respectively. Then β_1 and β_2 are said to be *equivalent*, if $\tau_1 = \tau_2$.

The following is an immediate consequence of Theorem 4.27.

Corollary 4.29. *Let (X, τ_1, E), (X, τ_2, E) be two IVSTSs and let β_1, β_2 be IVSBs for τ_1 and τ_2 respectively. Then β_1 and β_2 are equivalent if and only if the followings hold:*

(1) *for each $\mathbf{B}_1 \in \beta_1$ such that $e_{a_{IVP}} \in \mathbf{B}_1$ [resp. $e_{a_{IVVP}} \in \mathbf{B}_1$], there is $\mathbf{B}_2 \in \beta_2$ such that $e_{a_{IVP}} \in \mathbf{B}_2$ [resp. $e_{a_{IVVP}} \in \mathbf{B}_2$] and $\mathbf{B}_2 \subset \mathbf{B}_1$,*

(1) *for each $\mathbf{B}_2 \in \beta_2$ such that $e_{a_{IVP}} \in \mathbf{B}_2$ [resp. $e_{a_{IVVP}} \in \mathbf{B}_2$], there is $\mathbf{B}_1 \in \beta_1$ such that $e_{a_{IVP}} \in \mathbf{B}_1$ [resp. $e_{a_{IVVP}} \in \mathbf{B}_1$] and $\mathbf{B}_1 \subset \mathbf{B}_2$.*

Note that every IVST has an IVSB since the IVST itself forms an IVSB. The following gives a condition for one to check to see if a subcollection of an IVST τ is an IVSB for τ.

Proposition 4.30. *Let (X, τ_1, E) be an IVSTS. Suppose $\beta \subset \tau$ such that for each $\mathbf{U} \in \tau$ with $e_{a_{IVP}} \in \mathbf{U}$ [resp. $e_{a_{IVVP}} \in \mathbf{U}$], there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$] and $\mathbf{B} \subset \mathbf{U}$. Then β is an IVSB for τ.*

Proof. Let $e_{a_{IVP}} \in \widetilde{X}_E$ [resp. $e_{a_{IVVP}} \in \widetilde{X}_E$]. Since $\widetilde{X}_E \in \tau$, there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$] and $\mathbf{B} \subset \widetilde{X}_E$. Then $\widetilde{X}_E = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta\}$. Thus β satisfies the condition (1) of Theorem 4.25. Suppose $\mathbf{B}_1,\ \mathbf{B}_2 \in \beta$ and $e_{a_{IVP}} \in \mathbf{B}_1 \cap B_2$ [resp. $e_{a_{IVVP}} \in B_1 \cap B_2$]. Since $\mathbf{B}_1,\ \mathbf{B}_2 \in \tau$, $\mathbf{B}_1 \cap \mathbf{B}_2 \in \tau$. Then there is $\mathbf{B} \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}$ [resp. $e_{a_{IVVP}} \in \mathbf{B}$] and $\mathbf{B} \subset \mathbf{B}_1 \cap B_2$. Thus β satisfies the condition (2) of Theorem 4.25. So by Theorem 4.25, β is an IVSB for some IVST $\tau^{'}$ on X. From Theorem 4.27, it is clear that $\tau^{'}$ is finer than τ, i.e., $\tau \subset \tau^{'}$. Furthermore, it can be easily seen that $\tau^{'} \subset \tau$ holds. Hence $\tau = tau^{'}$. This completes the proof. $\square$

One advantage of the notion of an IVSSB is that we can define an IVST on X by simply choosing an arbitrary collection IVSSs in X whose union is $\widetilde{X}_E$.

Proposition 4.31. *Let $\sigma \subset IVSS_E(X)$ such that $\widetilde{X}_E = \bigcup\{\mathbf{S} : \mathbf{S} \in \sigma\}$. Then there is a unique IVST τ on X such that σ is an IVSSB for τ.*

Proof. Let $\beta = \{\mathbf{B} \in IVSS_E(X) : \mathbf{B} = \bigcup\{\mathbf{B} : \mathbf{B} \in \sigma_f\}, \sigma_f$ is a finite subset of $\sigma\}$. Let $\tau = \{\mathbf{U} \in IVSS_E(X) : \mathbf{U} = \widetilde{\varnothing}_E$ or $\exists\ \beta^{'} \subset \beta$ such that $\mathbf{U} = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta^{'}\}$. It is obvious that $\widetilde{\varnothing}_E,\ \widetilde{X}_E \in \tau$. Let $(\mathbf{U}_j)_{j\in J} \subset \tau$, where J is an index set. Then there is $j \in J$ such that $\beta_j \subset \beta$ and $\mathbf{U}_j = \bigcup\{\mathbf{B} : \mathbf{B} \in \beta_j\}$. Thus $\bigcup_{j\in J}\mathbf{U}_j = \bigcup_{j\in J}(\bigcup_{\mathbf{B}\in\beta_j}\mathbf{B})$. So $\bigcup_{j\in J}\mathbf{U}_j \in \tau$. Suppose $\mathbf{U}_1,\ \mathbf{U}_2 \in \tau$ such that $e_{a_{IVP}} \in \mathbf{U}_1 \cap \mathbf{U}_2$ [resp. $e_{a_{IVVP}} \in \mathbf{U}_1 \cap \mathbf{U}_2$]. Then there are $\mathbf{B}_1,\ \mathbf{B}_2 \in \beta$ such that $e_{a_{IVP}} \in \mathbf{B}_1 \cap \mathbf{B}_2$ [resp. $e_{a_{IVVP}} \in \mathbf{B}_1 \cap \mathbf{B}_2$], $\mathbf{B}_1 \subset \mathbf{U}_1$ and $\mathbf{B}_2 \subset \mathbf{U}_2$. Since each of $\mathbf{B}_1$ and $\mathbf{B}_2$ is the intersection of a finite number of members of σ, $\mathbf{B}_1 \cap \mathbf{B}_2 \in \beta$. Thus there is $\beta^{'} \subset \beta$ such that $\mathbf{U}_1 \cap \mathbf{U}_2 = \bigcup_{\mathbf{B}\in\beta'}\mathbf{B}$. So $\mathbf{U}_1 \cap \mathbf{U}_2 \in \tau$. Hence $\tau \in IVST_E(X)$. It is obvious that τ is the unique IVST having σ as an IVSSB. $\square$

Example 4.32. Let $X = \{a, b, c, d, e\}$ and let $E = \{e_1, e_2\}$. Consider the family σ of IVSSs over X given by:

$$\sigma = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4\},$$

where $\mathbf{S}_1(e_1) = [\{a\}, \{a\}],\ \mathbf{S}_1(e_2) = [\{b\}, \{b\}]$,
$\mathbf{S}_2(e_1) = [\{a, b, c\}, \{a, b, c\}],\ \mathbf{S}_2(e_2) = [\{b, c, d\}, \{b, c, d\}]$,
$\mathbf{S}_3(e_1) = [\{b, c, d\}, \{b, c, d\}],\ \mathbf{S}_3(e_2) = [\{c, d, e\}, \{c, d, e\}]$,
$\mathbf{S}_4(e_1) = [\{c, e\}, \{c, e\}],\ \mathbf{S}_4(e_2) = [\{a, d\}, \{a, d\}]$.
Then from Proposition 4.31, we can easily check that σ is an IVSSB for the unique IVST τ. Let β be the collection of all finite intersections of members of σ. Then we have

$$\beta = \{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \mathbf{B}_1, \mathbf{B}_2\},$$

where $\mathbf{B}_1(e_1) = [\{b, c\}, \{b, c\}],\ \mathbf{S}_1(e_2) = [\{c, d\}, \{c, d\}]$,
$\mathbf{B}_2(e_1) = [\{c\}, \{c\}],\ \mathbf{B}_2(e_2) = [\{d\}, \{d\}]$.
Thus we get

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4, \mathbf{U}_5, \mathbf{U}_6, \mathbf{U}_7, \mathbf{U}_8, \mathbf{U}_9, \mathbf{U}_{10}, \mathbf{U}_{11}, \mathbf{U}_{12}, \mathbf{U}_{13}, \widetilde{X}_E\},$$

where $\mathbf{U}_1 = \mathbf{S}_1,\ \mathbf{U}_2 = \mathbf{S}_2,\ \mathbf{U}_3 = \mathbf{S}_3,\ \mathbf{U}_4 = \mathbf{S}_4,\ \mathbf{U}_5 = \mathbf{B}_1,\ \mathbf{U}_6 = \mathbf{B}_2$,
$\mathbf{U}_7(e_1) = [\{a, b, c, d\}, \{a, b, c, d\}],\ \mathbf{U}_7(e_2) = [\{b, c, d, e\}, \{b, c, d, e\}]$,
$\mathbf{U}_8(e_1) = [\{a, c, e\}, \{a, c, e\}],\ \mathbf{U}_8(e_2) = [\{a, b, d\}, \{a, b, d\}]$,
$\mathbf{U}_9(e_1) = [\{a, b, c\}, \{a, b, c\}],\ \mathbf{U}_9(e_2) = [\{b, c, d\}, \{b, c, d\}]$,
$\mathbf{U}_{10}(e_1) = [\{a, b, c, e\}, \{a, b, c, e\}],\ \mathbf{U}_{10}(e_2) = [\{a, b, c, d\}, \{a, b, c, d\}]$,
$\mathbf{U}_{11}(e_1) = [\{a, c\}, \{a, c\}],\ \mathbf{U}_{11}(e_2) = [\{b, d\}, \{b, d\}]$,
$\mathbf{U}_{12}(e_1) = [\{b, c, d, e\}, \{b, c, d, e\}],\ \mathbf{U}_9(e_2) = [\{a, c, d, e\}, \{a, c, d, e\}]$,
$\mathbf{U}_{13}(e_1) = [\{b, c, e\}, \{b, c, e\}],\ \mathbf{U}_{13}(e_2) = [\{a, c, d\}, \{a, c, d\}]$.

5. Interval-valued soft neighborhoods, interval-valued soft closures and interiors

In this section, we introduce the concept of interval-valued soft neighborhoods of IVPs of two types and find their various properties and give some examples. Also, we define interval-valued soft closures and interiors, and deal with some of their properties. Moreover, we show that there is a unique IVST on X from the interval-valued soft closure [resp. interior] operator.

Definition 5.1. Let (X, τ, E) be an IVSTS and let $\mathbf{N} \in IVSS_E(X)$. Then

(i) $\mathbf{N}$ is called an *interval-valued soft neighborhood* (briefly, IVSN) of $e_{a_{IVP}} \in \widetilde{X}_E$, if there exists a $\mathbf{U} \in \tau$ such that

$$e_{a_{IVP}} \in \mathbf{U} \subset \mathbf{N}, \text{ i.e., } a \in U^-(e) \subset N^-(e),$$

(ii) $\mathbf{N}$ is called an *interval-valued soft vanishing neighborhood* (briefly, IVSVN) of $e_{a_{IVVP}} \in \widetilde{X}_E$, if there exists a $\mathbf{U} \in \tau$ such that

$$e_{a_{IVVP}} \in \mathbf{U} \subset \mathbf{N}, \text{ i.e., } a \in U^+(e) \subset N^+(e).$$

We will denote the set of all IVSNs [resp. IVSVNs] of $e_{a_{IVP}}$ [resp. $e_{a_{IVVP}}$] by $N(e_{a_{IVP}})$ [resp. $N(e_{a_{IVVP}})$].

For each $e \in E$, let $N_{IVSN,e} = N(e_{a_{IVP}}(e))$ [resp. $N_{IVSVN,e} = N(e_{a_{IVVP}}(e))$]. Then we can ewasily see that $N_{IVSN,e} = N(a_{IVP})$ [resp. $N_{IVSVN,e} = N(a_{IVVP})$] with respect to the IVT τ_e on X (See Proposition 4.5).

Example 5.2. Let $X = \{a, b, c, d\}$, let $E = \{e, f\}$. Consider IVST τ on X given by:

$$\tau = \{\widetilde{\varnothing}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \widetilde{X}_E\},$$

where $\mathbf{A}_1(e) = [\{b\}, \{b, d\}],\ \mathbf{A}_1(f) = [\{a\}, \{a, c\}]$,
$\mathbf{A}_2(e) = [\{a, c\}, \{a, b, c\}],\ \mathbf{A}_2(f) = [\{a, b\}, \{a, b, d\}]$,
$\mathbf{A}_3(e) = [\varnothing, \{b\}],\ \mathbf{A}_3(f) = [\{a\}, \{a\}]$,
$\mathbf{A}_4(e) = [\{a, b, c\}, X],\ \mathbf{A}_3(f) = [\{a, b\}, X]$.
Let $\mathbf{N} \in IVSS_E(X)$ given by:

$$\mathbf{N}(e) = [\{b\}, \{a, b, d\}],\ \mathbf{N}(f) = [\{a, c\}, \{a, c, d\}].$$

Then we can easily see that

$$\mathbf{N} \in N(e_{b_{IVP}}),\ \mathbf{N} \in N(e_{b_{IVVP}}),\ \mathbf{N} \in N(f_{a_{IVP}}).$$

From Proposition 4.5, we have two IVTs on X:

$$\tau_e = \{\widetilde{\varnothing}, \mathbf{A}_1(e), \mathbf{A}_2(e), \mathbf{A}_3(e), \mathbf{A}_4(e), \widetilde{X}\},$$

$$\tau_f = \{\widetilde{\varnothing}, \mathbf{A}_1(f), \mathbf{A}_2(f), \mathbf{A}_3(f), \mathbf{A}_4(f), \widetilde{X}\}.$$

Then we can see that $\mathbf{N}(e) \in N(b_{IVP}) \cap N(b_{IVVP})$ and $\mathbf{N}(f) \in N(a_{IVP})$.

Proposition 5.3. *Let (X, τ, E) be an IVSTS and let $e_{a_{IVP}} \in \widetilde{X}_E$.*

[IVSN$_1$] *If $\mathbf{N} \in N(e_{a_{IVP}})$, then $e_{b_{IVP}} \in \mathbf{N}$.*

[IVSN$_2$] *If $\mathbf{N} \in N(e_{a_{IVP}})$ and $\mathbf{N} \subset \mathbf{M}$, then $\mathbf{M} \in N(e_{a_{IVP}})$.*

[IVSN$_3$] *If* $\mathbf{N},\ \mathbf{M} \in N(e_{a_{IVP}})$, *then* $\mathbf{N} \cap \mathbf{M} \in N(e_{a_{IVP}})$.
[IVSN$_4$] *If* $\mathbf{N} \in N(e_{a_{IVP}})$, *then there exists* $\mathbf{M} \in N(e_{a_{IVP}})$ *such that* $\mathbf{N} \in N(e_{b_{IVP}})$ *for each* $e_{b_{IVP}} \in \mathbf{M}$.

Proof. The proofs of [IVSN$_1$] and [IVSN$_2$] are easy.
[IVSN$_3$] Suppose $\mathbf{N},\ \mathbf{M} \in N(e_{a_{IVP}})$. Then there are $\mathbf{U},\ \mathbf{V} \in \tau$ such that

$$e_{a_{IVP}} \in \mathbf{U} \subset \mathbf{N} \text{ and } e_{a_{IVP}} \in \mathbf{V} \subset \mathbf{M}.$$

Let $\mathbf{W} = \mathbf{U} \cap \mathbf{V}$. Then clearly, $\mathbf{W} \in \tau$ and $e_{a_{IVP}} \in \mathbf{W} \subset \mathbf{U} \cap \mathbf{V}$. Thus $\mathbf{N} \cap \mathbf{M} \in N(e_{a_{IVP}})$.
[IVSN$_4$] The proof is easy from Definition 5.1 and [IVSN$_2$]. $\square$

Proposition 5.4. *Let* (X, τ, E) *be an IVSTS and let* $e_{a_{IVP}} \in \widetilde{X}_E$.
[IVSVN$_1$] *If* $\mathbf{N} \in N(e_{a_{IVVP}})$, *then* $e_{a_{IVVP}} \in \mathbf{N}$.
[IVSVN$_2$] *If* $\mathbf{N} \in N(e_{a_{IVVP}})$ *and* $\mathbf{N} \subset \mathbf{M}$, *then* $\mathbf{M} \in N(e_{a_{IVVP}})$.
[IVSVN$_3$] *If* $\mathbf{N},\ \mathbf{M} \in N(e_{a_{IVVP}})$, *then* $\mathbf{N} \cap \mathbf{M} \in N(e_{a_{IVVP}})$.
[IVSVN$_4$] *If* $\mathbf{N} \in N(e_{a_{IVVP}})$, *then there exists* $\mathbf{M} \in N(e_{a_{IVVP}})$ *such that* $\mathbf{N} \in N(e_{b_{IVVP}})$ *for each* $e_{b_{IVVP}} \in M$.

Proof. The proof is similar to one of Proposition 5.3. $\square$

Proposition 5.5. *Let* (X, τ, E) *be an IVSTS and let us define two families:*

$$\tau_{IVSP} = \{\mathbf{U} \in IVSS_E(X) : \mathbf{U} \in N(e_{a_{IVP}}) \text{ for each } e_{a_{IVP}} \in \mathbf{U}\}$$

and

$$\tau_{IVSVP} = \{\mathbf{U} \in IVSS_E(X) : \mathbf{U} \in N(e_{a_{IVVP}}) \text{ for each } e_{a_{IVVP}} \in \mathbf{U}\}.$$

Then we have
(1) $\tau_{IVSP},\ \tau_{IVSVP} \in IVST_E(X)$,
(2) $\tau \subset \tau_{IVSP}$ *and* $\tau \subset \tau_{IVSVP}$.

Proof. (1) We only prove that $\tau_{IVSVP} \in IVST_E(X)$.
[IVSO$_1$] From the definition of τ_{IVSVP}, we have $\widetilde{\varnothing}_E,\ \widetilde{X}_e \in \tau_{IVSVP}$.
[IVSO$_2$] Let $\mathbf{U}\ ,\mathbf{V} \in IVSS_E(X)$ such that $U,\ V \in \tau_{IVSVP}$ and let $e_{a_{IVSVP}} \in \mathbf{U} \cap \mathbf{V}$. Then clearly, $\mathbf{U},\ \mathbf{V} \in N(e_{a_{IVSVP}})$. Thus by [IVSVN$_3$], $\mathbf{U} \cap \mathbf{V} \in N(e_{a_{IVSVP}})$. So $\mathbf{U} \cap \mathbf{V} \in \tau_{IVSVP}$.
[IVSO$_3$] Let $(\mathbf{U}_j)_{j \in J}$ be any family of IVSSs in τ_{IVSVP}, let $\mathbf{U} = \bigcup_{j \in J} \mathbf{U}_j$ and let $e_{a_{IVVP}} \in \mathbf{U}$. Then by Theorem 3.29 (2), there is $j_0 \in J$ such that $e_{a_{IVSVP}} \in \mathbf{U}_{j_0}$. Since $\mathbf{U}_{j_0} \in \tau_{IVSVP}$, $\mathbf{U}_{j_0} \in N(e_{a_{IVSVP}})$ by the definition of τ_{IVSVP}. Since $\mathbf{U}_{j_0} \subset \mathbf{U}$, $\mathbf{U} \in N(e_{a_{IVSVP}})$ by [IVSVN$_2$]. So by the definition of τ_{IVSVP}, $\mathbf{U} \in \tau_{IVSVP}$.
(2) Let $\mathbf{U} \in \tau$. Then clearly, $\mathbf{U} \in N(e_{a_{IVSP}})$ and $\mathbf{U} \in N(e_{a_{IVSVP}})$ for each $e_{a_{IVSP}} \in \mathbf{V}$ and $e_{a_{IVSVP}} \in \mathbf{V}$, respectively. Thus $\mathbf{U} \in \tau_{IVSP}$ and $\mathbf{U} \in \tau_{IVSVP}$. So the results hold. $\square$

Remark 5.6. (1) From the definitions of τ_{IVSP} and τ_{IVSVP}, we can easily have:

$$\tau_{IVSP} = \tau \cup \{[U^-, S] \in IVSS_E(X) : U^+ \subset S,\ \mathbf{U} \in \tau\}$$

and

$$\tau_{IVSVP} = \tau \cup \{\mathbf{S} \in IVSS_E(X) : \varnothing \neq S^- \subset X \setminus U^+,\ S^+ = S^- \cup U^+,\ \mathbf{U} = [\varnothing, U^+] \in \tau\}.$$

In fact, if $U^- \neq \varnothing$ for each $U \in \tau$, then $\tau_{IVSVP} = \tau$.

(2) From Proposition 4.5 and Proposition 5.5 in [39], we can easily see that for each $\tau \in IVST_E(X)$ and $e \in E$,

$$\tau_{IVSP,e} = \tau_{IVP},\ \tau_{IVSVP,e} = \tau_{IVVP},\ \text{where}$$
$$\tau_{IVSP,e} = \{\mathbf{U}(e) \in IVS(X) : \mathbf{U} \in \tau_{IVP}\},$$
$$\tau_{IVSVP,e} = \{\mathbf{U}(e) \in IVS(X) : \mathbf{U} \in \tau_{IVVP}\}.$$

Furthermore, from Remark 4.6 (1) and Remark 5.6 (1) in [39], we can have four ordinary topologies on X given by:

$$\tau^-_{IVSP,e} = \{U^- \in 2^X : \mathbf{U} \in \tau_{IVP}\},\ \tau^+_{IVSP,e} = \{U^+ \in 2^X : \mathbf{U} \in \tau_{IVP}\}$$

and

$$\tau^-_{IVSVP,e} = \{U^- \in 2^X : \mathbf{U} \in \tau_{IVVP}\},\ \tau^+_{IVSVP,e} = \{U^+ \in 2^X : \mathbf{U} \in \tau_{IVVP}\}.$$

Example 5.7. (1) Let $X = \{a, b, c, d\}$, $E = \{e\}$ and consider the family τ of IVSSs over X given by:

$$\tau = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5, \mathbf{A}_6, \mathbf{A}_7\},$$

where $\mathbf{A}_1(e) = [\{a, b\}, \{a, b, c\}],\ \mathbf{A}_2(e) = [\{c\}, \{b, c\}],\ \mathbf{A}_3(e) = [\varnothing, \{a, c\}],$
$\mathbf{A}_4(e) = [\{a, b, c\}, \{a, b, c\}],\ \mathbf{A}_5(e) = [\varnothing, \{b, c\}],\ \mathbf{A}_6(e) = [\varnothing, \{c\}],$
$\mathbf{A}_7(e) = [\{c\}, \{a, b, c\}].$

Then we can easily check that (X, τ, E) is an IVSTS. Thus from Remark 5.6 (1), we have:

$$\tau_{IVSP} = \tau \cup \{\mathbf{A}_8, \mathbf{A}_9, \mathbf{A}_{10}, \mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{13}, \mathbf{A}_{14}, \mathbf{A}_{15}, \mathbf{A}_{16}, \mathbf{A}_{17}\},$$
$$\tau_{IVSVP} = \tau \cup \{\mathbf{A}_{18}, \mathbf{A}_{19}, \mathbf{A}_{20}, \mathbf{A}_{21}, \mathbf{A}_{22}, \mathbf{A}_{23}, \mathbf{A}_{24}, \mathbf{A}_{25}, \mathbf{A}_{26}, \mathbf{A}_{27}, \mathbf{A}_{28}, \mathbf{A}_{29}\},$$

where $\mathbf{A}_8(e) = [\{a, b\}, X],\ \mathbf{A}_9(e) = [\{c\}, \{b, c, d\}], \mathbf{A}_{10}(e) = [\{c\}, X],$
$\mathbf{A}_{11}(e) = [\varnothing, \{a, b, c\}],\ \mathbf{A}_{12}(e) = [\varnothing, \{a, c, d\}], \mathbf{A}_{13}(e) = [\varnothing, X],$
$\mathbf{A}_{14}(e) = [\{a, b, c\}, X],\ \mathbf{A}_{15}(e) = [\varnothing, \{b, c, d\}],$
$\mathbf{A}_{16}(e) = [\varnothing, \{c, d\}],\ \mathbf{A}_{17}(e) = [\{c\}, X],$
$\mathbf{A}_{18}(e) = [\{b\}, \{a, b, c\}],\ \mathbf{A}_{19}(e) = [\{d\}, \{a, c, d\}],\ \mathbf{A}_{20}(e) = [\{b, d\}, X],$
$\mathbf{A}_{21}(e) = [\{a\}, \{a, b, c\}],\ \mathbf{A}_{22}(e) = [\{d\}, \{b, c, d\}],\ \mathbf{A}_{23}(e) = [\{b, d\}, X],$
$\mathbf{A}_{24}(e) = [\{a\}, \{a, c\}],\ \mathbf{A}_{25}(e) = [\{b\}, \{b, c\}],\ \mathbf{A}_{26}(e) = [\{d\}, \{c, d\}],$
$\mathbf{A}_{27}(e) = [\{a, d\}, \{a, c, d\}],\ \mathbf{A}_{28}(e) = [\{b, d\}, \{b, c, d\}],\ \mathbf{A}_{29}(e) = [\{a, b, d\}, X].$

So we can confirm that Proposition 5.5 holds.

Furthermore, we obtain six ordinary topologies on X for the IVT τ:

$$\tau^-_e = \{\varnothing, X, \{c\}, \{a, b\}, \{a, b, c\}\},$$
$$\tau^+_e = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$
$$\tau^-_{IVSP,e} = \{\varnothing, X, \{c\}, \{a, b\}, \{a, b, c\}\} = \tau^-_e,$$
$$\tau^+_{IVSP,e} = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\},$$
$$\tau^-_{IVSVP,e} = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}\},$$
$$\tau^+_{IVSVP,e} = \{\varnothing, X, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}.$$

(2) $X = \{a, b, c, d\}$, $E = \{e, f\}$ and τ be the IVST on X given by:

$$\tau = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\},$$

where $\mathbf{A}_1(e) = [\{b, c\}, \{b, c, d\}]$, $\mathbf{A}_2(e) = [\{a, b\}, \{a, b, c\}]$,
$\mathbf{A}_3(e) = [\{b\}, \{b, c\}]$, $\mathbf{A}_4(e) = [\{a, b, c\}, X]$,
$\mathbf{A}_1(f) = [\{a, c\}, \{a, c, d\}]$, $\mathbf{A}_2(f) = [\{a, b\}, \{a, b, c\}]$,
$\mathbf{A}_3(f) = [\{a\}, \{a, c\}]$, $\mathbf{A}_4(e) = [\{a, b, c\}, X]$.
Then we easily check that $\tau_{IVSVP} = \tau$.

The following is an immediate consequence of Propositions 4.4 and 5.5 (2).

Corollary 5.8. *Let (X, τ, E) be an IVSTS and let τ^c_{IVSP} [resp. τ^c_{IVSVP}] be the set of all IVSCSs w.r.t. τ_{IVSP} [resp. τ_{IVSVP}]. Then*

$$\tau^c \subset \tau^c_{IVSP}, \text{ and } \tau^c \subset \tau^c_{IVSVP}.$$

Example 5.9. Let (X, τ, E) be the IVSTS given in Example 5.7 (1). Then we have:

$$\tau^c = \{\widetilde{\varnothing}_E, \widetilde{X}_E, \mathbf{A}^c_1, \mathbf{A}^c_2, \mathbf{A}^c_3, \mathbf{A}^c_4, \mathbf{A}^c_5, \mathbf{A}^c_6, \mathbf{A}^c_7\},$$

$$\tau^c_{IVSP} = \tau^c \cup \{\mathbf{A}^c_8, \mathbf{A}^c_9, \mathbf{A}^c_{10}, \mathbf{A}^c_{11}, \mathbf{A}^c_{12}, \mathbf{A}^c_{13}, \mathbf{A}^c_{14}, \mathbf{A}^c_{15}, \mathbf{A}^c_{16}, \mathbf{A}^c_{17}\},$$

$$\tau^c_{IVSVP} = \tau^c \cup \{\mathbf{A}^c_{18}, \mathbf{A}^c_{19}, \mathbf{A}^c_{20}, \mathbf{A}^c_{21}, \mathbf{A}^c_{22}, \mathbf{A}^c_{23}, \mathbf{A}^c_{24}, \mathbf{A}^c_{25}, \mathbf{A}^c_{26}, \mathbf{A}^c_{27}, \mathbf{A}^c_{28}, \mathbf{A}^c_{29}\},$$

where $\mathbf{A}^c_1(e) = [\{d\}, \{c, d\}]$, $\mathbf{A}^c_2(e) = [\{a, d\}, \{a, b, d\}]$, $\mathbf{A}^c_3(e) = [\{b, d\}, X]$,
$\mathbf{A}^c_4(e) = [\{d\}, \{d\}]$, $\mathbf{A}^c_5(e) = [\{a, d\}, X]$, $\mathbf{A}^c_6(e) = [\{a, b, d\}, X]$,
$\mathbf{A}^c_7(e) = [\{d\}, \{a, b, d\}]$,
$\mathbf{A}^c_8(e) = [\varnothing, \{c, d\}]$, $\mathbf{A}^c_9(e) = [\{a\}, \{a, b, d\}]$,
$\mathbf{A}^c_{10}(e) = [\varnothing, \{a, b, d\}]$, $\mathbf{A}^c_{11}(e) = [\{d\}, X]$ $\mathbf{A}^c_{12}(e) = [\{b\}, X]$,
$\mathbf{A}^c_{13}(e) = [\varnothing, X]$, $\mathbf{A}^c_{14}(e) = [\varnothing, \{d\}]$, $\mathbf{A}^c_{15}(e) = [\{a\}, X]$,
$\mathbf{A}^c_{16}(e) = [\{a, b\}, X]$, $\mathbf{A}^c_{17}(e) = [\varnothing, \{a, b, d\}]$,
$\mathbf{A}^c_{18}(e) = [\{d\}, \{a, c, d\}]$, $\mathbf{A}^c_{19}(e) = [\{b\}, \{a, b, c\}]$, $\mathbf{A}^c_{20}(e) = [\varnothing, \{a, c\}]$,
$\mathbf{A}^c_{21}(e) = [\{d\}, \{b, c, d\}]$, $\mathbf{A}^c_{22}(e) = [\{a\}, \{a, b, c\}]$, $\mathbf{A}^c_{23}(e) = [\varnothing, \{a, c\}]$,
$\mathbf{A}^c_{24}(e) = [\{b, d\}, \{b, c, d\}]$, $\mathbf{A}^c_{25}(e) = [\{a, d\}, \{a, c, d\}]$, $\mathbf{A}^c_{26}(e) = [\{a, b\}, \{a, b, c\}]$,
$\mathbf{A}^c_{27}(e) = [\{b\}, \{b, c\}]$, $\mathbf{A}^c_{28}(e) = [\{a\}, \{a, c\}]$, $\mathbf{A}^c_{29}(e) = [\varnothing, \{c\}]$.
Thus we can confirm that Corollary 5.8 holds.

Now let us consider the converses of Propositions 5.3 and 5.4.

Proposition 5.10. *Suppose to each $e_{a_{IVVP}} \in \widetilde{X}_E$, there corresponds a family $N_*(e_{a_{IVVP}})$ of IVSSs over X satisfying the conditions [IVSVN$_1$], [IVSVN$_2$], [IVSVN$_3$] and [IVSVN$_4$] in Proposition 5.4. Then there is an IVST on X with respect to E such that $N_*(e_{a_{IVVP}})$ is the set of all IVSVNs of $e_{a_{IVVP}}$ in this IVST for each $e_{a_{IVVP}} \in \widetilde{X}_E$.*

Proof. Let

$$\tau_{IVSVP} = \{\mathbf{U} \in IVSS_E(X) : \mathbf{U} \in N(e_{a_{IVVP}}) \text{ for each } e_{a_{IVVP}} \in \mathbf{U}\},$$

where $N(e_{a_{IVVP}})$ denotes the set of all IVSVNs in τ.
Then clearly, $\tau_{IVSVP} \in IVST_E(X)$ by Proposition 5.5. We will prove that $N_*(e_{a_{IVVP}})$ is the set of all IVSVNs of $e_{a_{IVVP}}$ in τ_{IVSVP} for each $e_{a_{IVVP}} \in \widetilde{X}_E$.

Let $\mathbf{V} \in IVSS_E(X)$ such that $\mathbf{V} \in N_*(e_{a_{IVVP}})$ and let $\mathbf{U}$ be the union of all the IVSVPs $e_{b_{IVVP}}$ in X such that $\mathbf{U} \in N_*(e_{a_{IVVP}})$. If we can prove that

$$e_{a_{IVVP}} \in \mathbf{U} \subset \mathbf{V} \text{ and } \mathbf{U} \in \tau_{IVSVP},$$

then the proof will be complete.

Since $\mathbf{V} \in N_*(e_{a_{IVVP}})$, $e_{a_{IVVP}} \in \mathbf{U}$ by the definition of $\mathbf{U}$. Moreover, $\mathbf{U} \subset \mathbf{V}$. Suppose $e_{b_{IVVP}} \in \mathbf{U}$. Then by [IVSVN$_4$], there is an IVSS $\mathbf{W} \in N_*(e_{b_{IVVP}})$ such that $\mathbf{V} \in N_*(e_{c_{IVVP}})$ for each $e_{c_{IVVP}} \in \mathbf{W}$. Thus $e_{c_{IVVP}} \in \mathbf{U}$. By Proposition 3.30 (1), $\mathbf{W} \subset \mathbf{U}$. So by [IVSVN$_2$], $\mathbf{U} \in N_*(e_{b_{IVVP}})$ for each $e_{b_{IVVP}} \in \mathbf{U}$. Hence by the definition of τ_{IVSVP}, $\mathbf{U} \in \tau_{IVSVP}$. This completes the proof. □

Proposition 5.11. *Suppose to each $e_{a_{IVP}} \in \widetilde{X}_E$, there corresponds a set $N_*(e_{a_{IVP}})$ of IVSSs in X satisfying the conditions [IVSN$_1$], [IVSN$_2$], [IVSN$_3$] and [IVSN$_4$] in Proposition 5.3. Then there is an IVST over X such that $N_*(e_{a_{IVP}})$ is the set of all IVSNs of $e_{a_{IVP}}$ in this IVST for each $e_{a_{IVP}} \in \widetilde{X}_E$.*

Proof. The proof is similar to Proposition 5.10. □

Theorem 5.12. *Let (X, τ, E) be an IVTS and let $\mathbf{A} \in IVSS_E(X)$. Then $\mathbf{A} \in \tau$ if and only if $\mathbf{A} \in N(e_{a_{IVP}})$ and $\mathbf{A} \in N(e_{a_{IVVP}})$ for each $e_{a_{IVP}}$, $e_{a_{IVP}} \in \mathbf{A}$.*

Proof. Suppose $\mathbf{A} \in N(e_{a_{IVP}})$ and $\mathbf{A} \in N(e_{a_{IVVP}})$ for each $e_{a_{IVP}}$, $e_{a_{IVVP}} \in \mathbf{A}$. Then there are $\mathbf{U}_{e_{a_{IVP}}}$, $\mathbf{V}_{e_{a_{IVVP}}} \in \tau$ such that $e_{a_{IVP}} \in \mathbf{U}_{e_{a_{IVP}}} \subset \mathbf{A}$ and $e_{a_{IVVP}} \in \mathbf{V}_{e_{a_{IVVP}}} \subset \mathbf{A}$. Thus by Proposition 3.27, we get

$$\begin{aligned}\mathbf{A} &= (\textstyle\bigcup_{e_{a_{IVP}} \in \mathbf{A}} e_{a_{IVP}}) \cup (\bigcup_{e_{a_{IVVP}} \in \mathbf{A}} e_{a_{IVVP}})\\ &\subset (\textstyle\bigcup_{e_{a_{IVP}} \in \mathbf{A}} \mathbf{U}_{e_{a_{IVP}}}) \cup (\bigcup_{e_{a_{IVVP}} \in \mathbf{A}} \mathbf{V}_{e_{a_{IVP}}})\\ &\subset \mathbf{A}.\end{aligned}$$

So $\mathbf{A} = (\bigcup_{e_{a_{IVP}} \in \mathbf{A}} \mathbf{U}_{e_{a_{IVP}}}) \cup (\bigcup_{e_{a_{IVVP}} \in \mathbf{A}} \mathbf{V}_{e_{a_{IVP}}})$. Since $\mathbf{U}_{e_{a_{IVP}}}$, $\mathbf{V}_{e_{a_{IVVP}}} \in \tau$, $\mathbf{A} \in \tau$.

The proof of the necessary condition is easy. □

Now we provide the relationship among three IVSTs, τ, τ_{IVSP} and τ_{IVSVP}.

Proposition 5.13. $\tau = \tau_{IVSP} \cap \tau_{IVSVP}$.

Proof. From Proposition 5.5 (2), it is clear that $\tau \subset \tau_{IVSP} \cap \tau_{IVSVP}$.

Conversely, let $\mathbf{U} \in \tau_{IVSP} \cap \tau_{IVSVP}$. Then clearly, $\mathbf{U} \in \tau_{IVSP}$ and $\mathbf{U} \in \tau_{IVSVP}$ Thus $\mathbf{U}$ is an IVSN of each of its IVSP $e_{a_{IVP}}$ and an IVSVN of each of its IVVP $e_{a_{IVVP}}$. Thus there are $\mathbf{U}_{e_{a_{IVP}}}$, $\mathbf{U}_{e_{a_{IVVP}}} \in \tau$ such that $e_{a_{IVP}} \in \mathbf{U}_{e_{a_{IVP}}} \subset U$ and $e_{a_{IVVP}} \in \mathbf{U}_{e_{a_{IVVP}}} \subset \mathbf{U}$. So we have

$$\mathbf{U}_{IV} = \bigcup_{e_{a_{IVP}} \in \mathbf{U}} e_{a_{IVP}} \subset \bigcup_{e_{a_{IVVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVP}}} \subset \mathbf{U}$$

and

$$\mathbf{U}_{IVV} = \bigcup_{e_{a_{IVP}} \in \mathbf{U}} e_{a_{IVVP}} \subset \bigcup_{e_{a_{IVVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVVP}}} \subset \mathbf{U}.$$

By Proposition 3.27, we get

$$\mathbf{U} = \mathbf{U}_{IVSP} \cup \mathbf{U}_{IVSVP} \subset (\bigcup_{e_{a_{IVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVP}}}) \cup (\bigcup_{e_{a_{IVVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVVP}}}) \subset \mathbf{U}, \text{ i.e.,}$$

$$\mathbf{U} = (\bigcup_{e_{a_{IVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVP}}}) \cup (\bigcup_{e_{a_{IVVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVVP}}}).$$

It is obvious that $(\bigcup_{e_{a_{IVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVVP}}}) \cup (\bigcup_{e_{a_{IVVP}} \in \mathbf{U}} \mathbf{U}_{e_{a_{IVVP}}}) \in \tau$. Hence $\mathbf{U} \in \tau$. Therefore $\tau_{IVSP} \cap \tau_{IVSVP} \subset \tau$. This completes the proof. □

The following is an immediate consequence of Corollary 5.8 and Proposition 5.13.

Corollary 5.14. *Let (X, τ, E) be an IVSTS. Then*

$$\tau^c = \tau^c_{IVSP} \cap \tau^c_{IVSVP}.$$

Example 5.15. In Example 5.9, we can easily check that Corollary 5.14 holds.

Now we define interval-valued soft interiors and closures, and study some of their properties and give some examples.

Definition 5.16. Let (X, τ, E) be an IVSTS and let $\mathbf{A} \in IVS(X)_E$.

(i) The *interval-valued soft closure* of $\mathbf{A}$ w.r.t. τ, denoted by $IVScl(\mathbf{A})$, is an IVSS over X defined as:

$$IVScl(\mathbf{A}) = \bigcap \{\mathbf{K} \in \tau^c : \mathbf{A} \subset \mathbf{K}\}.$$

(ii) The *interval-valued soft interior* of $\mathbf{A}$ w.r.t. τ, denoted by $IVSint(\mathbf{A})$, is an IVSS over X defined as:

$$IVint(\mathbf{A}) = \bigcup \{\mathbf{U} : \mathbf{U} \in \tau \text{ and } \mathbf{U} \subset \mathbf{A}\}.$$

(iii) The *interval-valued soft closure* of $\mathbf{A}$ w.r.t. τ_{IVSP}, denoted by $cl_{IVSP}(\mathbf{A})$, is an IVSS over X defined as:

$$cl_{IVSP}(\mathbf{A}) = \bigcap \{\mathbf{K} \in \tau^c_{IVSP} : \mathbf{A} \subset \mathbf{K}\}.$$

(iv) The *interval-valued soft interior* of $\mathbf{A}$ w.r.t. τ_{IVSP}, denoted by $int_{IVSP}(\mathbf{A})$, is an IVSS over X defined as:

$$int_{IVSP}(\mathbf{A}) = \bigcup \{\mathbf{U} : \mathbf{U} \in \tau_{IVSP} \text{ and } \mathbf{U} \subset \mathbf{A}\}.$$

(v) The *interval-valued soft closure* of A w.r.t. τ_{IVSVP}, denoted by $cl_{IVSVP}(\mathbf{A})$, is an IVSS over X defined as:

$$cl_{IVSVP}(\mathbf{A}) = \bigcap \{\mathbf{K} \in \tau^c_{IVSVP} : \mathbf{A} \subset \mathbf{K}\}.$$

(vi) The *interval-valued soft interior* of $\mathbf{A}$ w.r.t. τ_{IVSVP}, denoted by $int_{IVSVP}(\mathbf{A})$, is an IVSS over X defined as:

$$int_{I_{IVSVP}}(\mathbf{A}) = \bigcup \{\mathbf{U} : \mathbf{U} \in \tau_{IVSVP} \text{ and } \mathbf{U} \subset \mathbf{A}\}.$$

Remark 5.17. (1) From the above definition, it is clear that the followings hold:

$$IVSint(\mathbf{A}) \subset int_{IVSP}(\mathbf{A}),\ IVSint(\mathbf{A}) \subset int_{IVSVP}(\mathbf{A})$$

and

$$cl_{I_{IVSP}}(\mathbf{A}) \subset IVScl(\mathbf{A}),\ cl_{IVSVP}(\mathbf{A}) \subset IVScl(\mathbf{A}).$$

(2) We can easily check that for each $e \in E$, the followings hold (See Definition 6.1 in [39]):

$$IVScl(\mathbf{A})(e) = IVcl(\mathbf{A}(e)),\ IVSint(\mathbf{A})(e) = IVint(\mathbf{A}(e)),$$
$$cl_{IVSP}(\mathbf{A})(e) = cl_{IVP}(\mathbf{A}(e)),\ int_{IVSP}(\mathbf{A})(e) = int_{IVP}(\mathbf{A}(e)),$$
$$cl_{IVSVP}(\mathbf{A})(e) = cl_{IVVP}(\mathbf{A}(e)),\ int_{IVSVP}(\mathbf{A})(e) = int_{IVVP}(\mathbf{A}(e)).$$

Example 5.18. Let (X, τ, E) be the IVTS given in Example 5.7. Consider two IVSSs $\mathbf{A}$, $\mathbf{B}$ over X such that $\mathbf{A}(e) = [\{a, c\}, \{a, b, c\}]$ and $\mathbf{B}(e) = [\{d\}, \{a, d\}]$. Then

$IVSint(\mathbf{A}) = \bigcup\{\mathbf{U} \in \tau : \mathbf{U} \subset \mathbf{A}\} = \mathbf{A}_2 \cup \mathbf{A}_3 \cup \mathbf{A}_5 \cup \mathbf{A}_5 \cup \mathbf{A}_6 \cup \mathbf{A}_7 = \mathbf{A}_7$,
$int_{IVSP}(\mathbf{A}) = \bigcup\{\mathbf{U} \in \tau_{IVSP} : \mathbf{U} \subset \mathbf{A}\} = \mathbf{A}_7 \cup \mathbf{A}_{11} = \mathbf{A}_7$,
$int_{IVSVP}(\mathbf{A}) = \bigcup\{\mathbf{U} \in \tau_{IVSVP} : \mathbf{U} \subset \mathbf{A}\} = \mathbf{A}_7 \cup \mathbf{A}_{21} \cup \mathbf{A}_{24} \cup \mathbf{A}_{25} = \mathbf{C}$,
where $\mathbf{C}(e) = [\{a, b, c\}, \{a, b, c\}]$
and
$IVScl(\mathbf{B}) = \bigcap\{\mathbf{K} \in \tau^c : \mathbf{B} \subset \mathbf{K}\} = \mathbf{A}_2^c \cap \mathbf{A}_3^c \cap \mathbf{A}_5^c \cap \mathbf{A}_6^c \cap \mathbf{A}_7^c = \mathbf{A}_7^c$,
$cl_{IVSP}(\mathbf{B}) = \bigcap\{\mathbf{K} \in \tau^c_{IVSP} : \mathbf{B} \subset \mathbf{K}\} = \mathbf{A}_7^c \cap \mathbf{A}_{11}^c = \mathbf{A}_7^c$,
$cl_{IVSVP}(\mathbf{B}) = \bigcap\{\mathbf{K} \in \tau^c_{IVSVP} : \mathbf{B} \subset \mathbf{K}\} = \mathbf{A}_7^c \cap \mathbf{A}_{18}^c \cap \mathbf{A}_{25}^c = \mathbf{B}$.
Moreover, we can confirm that Remark 5.17 holds.

Proposition 5.19. *Let (X, τ, E) be an IVSTS and let $\mathbf{A} \in IVSS_E(X)$. Then*

$$IVSint(\mathbf{A}^c) = (IVScl(\mathbf{A}))^c \text{ and } IVScl(\mathbf{A}^c) = (IVSint(\mathbf{A}))^c.$$

Proof. Let $e \in E$. Then we have

$$\begin{aligned} IVSint(\mathbf{A}^c)(e) &= \bigcup\{\mathbf{U}(e) \in \tau_e : \mathbf{U}(e) \subset \mathbf{A}^c(e)\} \\ &= \bigcup\{\mathbf{U}(e) \in \tau_e : U(e)^- \subset A(e)^{+c},\ U(e)^+ \subset A(e)^{-c}\} \\ &= \bigcup\{\mathbf{U}(e) \in \tau_e : A(e)^+ \subset U(e)^{-c},\ A(e)^- \subset U(e)^{+c}\} \\ &= \bigcap\{\mathbf{U}^c(e) \in \tau_e^c : \mathbf{A}(e) \subset \mathbf{U}^c(e)\} \\ &= IVScl(\mathbf{A}). \end{aligned}$$

Thus $IVSint(\mathbf{A}^c) = IVScl(\mathbf{A})$. Similarly, we can show that

$$IVScl(\mathbf{A}^c) = (IVSint(\mathbf{A}))^c.$$

□

Proposition 5.20. *Let (X, τ, E) be an IVTTS and let $\mathbf{A} \in IVSS_E(X)$. Then*

$$IVSint(\mathbf{A}) = int_{IVSP}(\mathbf{A}) \cap int_{IVSVP}(\mathbf{A}).$$

Proof. The proof is straightforward from Proposition 5.13 and Definition 5.16. □

The following is an immediate consequence of Definition 5.16, and Propositions 5.19 and 5.20.

Corollary 5.21. *Let (X, τ, E) be an IVSTS and let $\mathbf{A} \in IVSS_E(X)$. Then*

$$IVScl(\mathbf{A}) = cl_{IVSP}(\mathbf{A}) \cup cl_{IVSVP}(\mathbf{A}).$$

Example 5.22. Consider two IVSSs $\mathbf{A} = [\{a, c\}, \{a, b, c\}]$ and $\mathbf{B} = [\{d\}, \{a, d\}]$ in X given in Example 5.18. Then we have : for $e \in E$,

$$IVSint(\mathbf{A})(e) = [\{c\}, \{a, b, c\}] = int_{IVSP}(\mathbf{A})(e),\ int_{IVSVP}(\mathbf{A})(e) = [\{a, b, c\}, \{a, b, c\}]$$

and

$$IVScl(\mathbf{B})(e) = [\{a, d\}, \{a, b, d\} = cl_{IVSP}(\mathbf{B})(e),$$
$$cl_{IVSVP}(\mathbf{B})(e) = [\{d\}, \{a, d\}] = \mathbf{B}(e).$$

Thus we get

$$int_{IVSP}(\mathbf{A})(e) \cap int_{IVSVP}(\mathbf{A})(e) = [\{c\}, \{a, b, c\}] = IVSint(\mathbf{B})(e)$$

and

$$cl_{IVSP}(\mathbf{B})(e) \cup cl_{IVSVP}(\mathbf{B})(e) = [\{d\}, \{a, b, d\} = IVScl(\mathbf{B})(e).$$

So $IVSint(\mathbf{B}) = int_{IVSP}(\mathbf{A}) \cap int_{IVSVP}(\mathbf{A})$ and $IVScl(\mathbf{B}) = cl_{IVSP}(\mathbf{B}) \cup cl_{IVSVP}(\mathbf{B})$.

Theorem 5.23. *Let (X, τ, E) be an IVSTS and let $\mathbf{A} \in IVSS_E(X)$. Then*
(1) *$\mathbf{A} \in \tau^c$ if and only if $\mathbf{A} = IVScl(\mathbf{A})$,*
(2) *$\mathbf{A} \in \tau$ if and only if $\mathbf{A} = IVSint(\mathbf{A})$.*

Proof. Straightforward. □

Proposition 5.24 (Kuratowski Closure Axioms)**.** *Let (X, τ, E) be an IVSTS and let $\mathbf{A}, \mathbf{B} \in IVSS_E(X)$. Then*
[IVSK$_0$] *if $\mathbf{A} \subset \mathbf{B}$, then $IVScl(\mathbf{A}) \subset IVScl(\mathbf{B})$,*
[IVSK$_1$] *$IVScl(\widetilde{\varnothing}_E) = \widetilde{\varnothing}_E$,*
[IVSK$_2$] *$\mathbf{A} \subset IVScl(\mathbf{A})$,*
[IVSK$_3$] *$IVScl(IVScl(\mathbf{A})) = IVScl(\mathbf{A})$,*
[IVSK$_4$] *$IVScl(\mathbf{A} \cup \mathbf{B}) = IVScl(\mathbf{A}) \cup IVScl(\mathbf{A})$.*

Proof. Straightforward. □

Let $IVScl^* : IVSS_E(X) \to IVSS_E(X)$ be the mapping satisfying the properties [IVSK$_1$], [IVSK$_2$],[IVSK$_3$] and [IVSK$_4$]. Then we call the mapping $IVScl^*$ as the *interval-valued soft closure operator* (briefly, IVSCO) on X.

Proposition 5.25. *Let $IVScl^*$ be the IVSCO on X. Then there exists a unique IVST τ on X such that $IVScl^*(\mathbf{A}) = IVScl(\mathbf{A})$ for each $\mathbf{A} \in IVSS_E(X)$, where $IVScl(\mathbf{A})$ denotes the interval-valued soft closure of $\mathbf{A}$ in the IVSTS (X, τ, E). In fact,*

$$\tau = \{\mathbf{A}^c \in IVSS_E(X) : IVScl^*(\mathbf{A}) = \mathbf{A}\}.$$

Proof. The proof is almost similar to the case of ordinary topological spaces. □

Proposition 5.26. *Let (X, τ, E) be an IVSTS and let $\mathbf{A}, \mathbf{B} \in IVSS_E(X)$. Then*
[IVSI$_0$] *if $\mathbf{A} \subset \mathbf{B}$, then $IVSint(\mathbf{A}) \subset IVSint(\mathbf{B})$,*
[IVSI$_1$] *$IVSint(\widetilde{X}_E) = \widetilde{X}_E$,*
[IVSI$_2$] *$IVSint(\mathbf{A}) \subset \mathbf{A}$,*
[IVSI$_3$] *$IVSint(IVSint(\mathbf{A})) = IVSint(\mathbf{A})$,*
[IVSI$_4$] *$IVSint(\mathbf{A} \cap \mathbf{B}) = IVSint(\mathbf{A}) \cap IVSint(\mathbf{B})$.*

Proof. Straightforward. □

Let $IVSint^* : IVSS_E(X) \to IVSS_E(X)$ be the mapping satisfying the properties [IVSI$_1$], [IVSI2$_2$],[IVSI$_3$] and [IVSI4$_4$]. Then we call the mapping $IVSint^*$ as the *interval-valued soft interior operator* (briefly, IVSIO) on X.

Proposition 5.27. *Let $IVSint^*$ be the IVSIO on X. Then there exists a unique IVST τ on X such that $IVSint^*(\mathbf{A}) = IVSint(\mathbf{A})$ for each $\mathbf{A} \in IVSS_E(X)$, where $IVSint(\mathbf{A})$ denotes the interval-valued soft interior of $\mathbf{A}$ in the IVSTS (X, τ, E). In fact,*

$$\tau = \{\mathbf{A} \in IVSS_E(X) : IVSint^*(\mathbf{A}) = \mathbf{A}\}.$$

Proof. The proof is similar to one of Proposition 5.25. □

The following provides a criterion for an interval-valued soft closed set in an interval-valued soft subspace to be closed in the IVSTS.

Proposition 5.28. *Let (X, τ, E) be an IVSTS, and let $\mathbf{A} \in \tau^c$. If $\mathbf{C}$ is closed in $(\mathbf{A}, \tau_{\mathbf{A}}, E)$, then $\mathbf{C} \in \tau^c$.*

Proof. Suppose $\mathbf{C}$ is closed in $(\mathbf{A}, \tau_{\mathbf{A}}, E)$. Then by Theorem 4.21, there is $\mathbf{D} \in \tau^c$ such that $\mathbf{C} = \mathbf{A} \cap \mathbf{D}$. Since $\mathbf{A} \in \tau^c$ and $\mathbf{D} \in \tau^c$, $\mathbf{A} \cap \mathbf{D} \in \tau^c$. Thus $\mathbf{C} \in \tau^c$. □

When we deal with interval-valued soft subspaces of an IVSTS, we needs to exercise care in taking closures of n IVSS because the closure in the interval-valued soft subspace may be quite different from the closure in the IVSTS. The following gives a criterion for dealing with this situation.

Proposition 5.29. *Let $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ be an interval-valued soft subspace of an IVSTS (X, τ) and let $\mathbf{B} \subset \mathbf{A}$. Then $IVScl_{\tau_{\mathbf{A}}}(\mathbf{B}) = IVScl(\mathbf{B})$, where $IVScl_{\tau_{\mathbf{A}}}(\mathbf{B})$ denotes the interval-valued soft closure in $(\mathbf{A}, \tau_{\mathbf{A}}, E)$.*

Proof. Since $IVScl(\mathbf{B}) \in \tau^c$, by Theorem 4.21, $\mathbf{A} \cap IVScl(\mathbf{B})$ is closed in $(\mathbf{A}, \tau_{\mathbf{A}}, E)$. Since $\mathbf{B} \subset \mathbf{A}$ and $\mathbf{B} \subset IVScl(\mathbf{B})$, $\mathbf{B} \subset \mathbf{A} \cap IVScl(\mathbf{B})$. Then by the definition of $IVScl_{\tau_{\mathbf{A}}}(\mathbf{B})$, $IVScl_{\tau_{\mathbf{A}}}(\mathbf{B}) \subset \mathbf{A} \cap IVScl(\mathbf{B})$.

Since $IVScl_{\tau_{\mathbf{A}}}(\mathbf{B})$ is closed in $(\mathbf{A}, \tau_{\mathbf{A}}, E)$, by Theorem 4.21, there is $\mathbf{C} \in \tau^c$ such that $IVScl_{\tau_{\mathbf{A}}}(\mathbf{B}) = \mathbf{A} \cap \mathbf{C}$. □

Theorem 5.30. *Let $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ be an interval-valued soft subspace of an IVSTS (X, τ) and let $\mathbf{U} \subset \mathbf{A}$.*

(1) *$\mathbf{U}$ is an IVSN of $e_{a_{IVP}}$ with respect to $\tau_{\mathbf{A}}$ if and only if there is a $\mathbf{V} \in N(e_{a_{IVP}})$ such that $\mathbf{U} = \mathbf{A} \cap \mathbf{V}$.*

(2) *$\mathbf{U}$ is an IVSN of $e_{a_{IVVP}}$ with respect to $\tau_{\mathbf{A}}$ if and only if there is a $\mathbf{V} \in N(e_{a_{IVVP}})$ such that $\mathbf{U} = \mathbf{A} \cap \mathbf{V}$.*

Proof. (1) Suppose $\mathbf{U}$ is an IVSN of $e_{a_{IVP}}$ with respect to $\tau_{\mathbf{A}}$. Then there is an IVSOS $\mathbf{B}$ in $(\mathbf{A}, \tau_{\mathbf{A}}, E)$ such that $e_{a_{IVP}} \in \mathbf{B} \subset \mathbf{U}$. Thus by Proposition 4.15, there is $\mathbf{V} \in \tau$ such that $\mathbf{B} = \mathbf{A} \cap \mathbf{V}$. Since $e_{a_{IVP}} \in \mathbf{B}$, $e_{a_{IVP}} \in \mathbf{V}$. So by Theorem 5.12, $\mathbf{V} \in N(e_{a_{IVVP}})$. Hence the necessary condition holds.

The proof of the sufficient condition is easy.

(2) The proof is similar to (1). □

6. Conclusions

We introduced the new concept of interval-valued soft sets which are the generalization of soft sets and the special case of interval-valued fuzzy soft sets, and obtained its various properties. Next, we defined the notion of interval-valued soft topological spaces which are considered as a soft bi-topological space introduced by Kelly [43]. Moreover, we defined an interval-valued soft base and subbase and found the characterization of an interval-valued soft base. Also, we introduced the notion of interval-valued soft subspaces and found some of its properties. Finally, we introduced the concept of interval-valued soft neighborhoods of two types and obtained some similar properties to classical neighborhoods. Furthermore, we defined an interval-valued soft closure and interior and dealt with their some properties. In the future, we expect that one can apply the notion of interval-valued soft sets to group and ring theory, BCK-algebra, category theory and decision making problem, etc. Furthermore, we will study relation between interval-valued sets and rough sets and thus interval-valued soft sets and soft rough sets.

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