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Abstract. We prove that commutative quantales can be embedded in the algebra structures on Alexandrov topologies. Moreover, commutative frames can be embedded in the frame structures on Alexandrov topologies. From these results, every commutative quantales and frames are subalgebras of algebraic structures on Alexandrov topologies. We study their properties and give their examples.

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Fuzzy partially ordered sets, Commutative quantale frames, Commutative quantales, Alexandrov topologies, (Frame) embedding maps.

Corresponding Author: Yong Chan Kim (yck@gwnu.ac.kr)

1. Introduction

Mulvey (See [1, 2]) introduced quantales which are certain partially ordered algebraic structures that generalize locales (point free topologies) as well as various multiplicative lattices of ideals from ring theory and functional analysis (C^*-algebras, von Neumann algebras). Höhle [3, 4] developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices.

Discrete duality is a duality between a class of algebras and an associated class of relational systems without a topology. Dualities are developed between algebras (resp. Boolean algebras, MV-algebra, Heyting algebra, BL-algebra, MTL-algebras) and logical relational systems (resp. classical propositional logic, Lukasiewicz logic, intuitionistic logic, basic fuzzy logics, Monoidal t-norm logics) (See [5, 6, 7, 8, 9, 10]). Oh and Kim (See [11, 12, 13]) introduced the notion of commutative quantales (resp. residuated connections) and commutative quantale frames (resp. residuated frames) in fuzzy logics as a duality between algebras and logical relational systems.

If map $f : (X, S_X) \to (Y, S_Y)$ is embedding, then some object X is said to be embedded in another object Y. The embedding is given by some injective and structure-preserving map $f : X \to Y$. We can consider that (X, S_X) is a subalgebraic structure of (Y, S_Y). The aim of this paper, we show that every commutative quantales and frames are subalgebras of algebraic structures on Alexandrov topologies.

In this paper, we introduce embedding maps on commutative quantales and commutative quantale frames. Orłowska and Rewitzky proved that every bounded distributive lattice with a necessity operator can be embedded in the complex algebra of its canonical frame Theorem 3.3 in [9]. As the extension of this result, In Theorem 3.2, let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale with r-fuzzy preorder e_X and $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A, B) \le \alpha(B)\}$. We show that a map $h : (X, \wedge, \vee, *, \nearrow, 0, 1, e_X) \to (\tau_{e_{\tau_{e_X}}}, \vee, \otimes, \Rightarrow, 0, 1, e_{\tau_{e_{\tau_{e_X}}}})$ defined as $h(x)(A) = \hat{x}(A) = A(x)$ is an embedding map.

Moreover, Theorem 3.4 in [9], every necessity frame can be embedded in the canonical frame of its complex algebra. As the extension of this result, in Theorem 3.5, let (X, e_X, P) be a cq-frame. We show that a map $k : X \to \tau_{e_{\tau_{e_X}}}$ as $k(x)(A) = \hat{x}(A) = A(x)$. Then $k : (X, e_X, P) \to (\tau_{e_{\tau_{e_X}}}, e_{\tau_{e_{\tau_{e_X}}}}, \hat{P})$ defined as $k(x)(A) = \hat{x}(A) = A(x)$ is a frame embedding map. Moreover, we give their examples.

2. Preliminaries

Definition 2.1 ([1])**.** A triple $(L, \le, \odot)$ is called a *commutative quantale* (*c-quantale*, for short), if it satisfies the following conditions:

(Q1) $L = (L, \le, \vee, \wedge, 0, 1)$ is a complete lattice, where 1 is the universal upper bound and 0 denotes the universal lower bound,

(Q2) $a \odot b = b \odot a$ and $a \odot (b \odot c) = (a \odot b) \odot c$ for all $a, b, c \in L$,

(Q3) $\odot$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i \in \Gamma} a_i) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

A c-quantale is *unital*, if $a = a \odot e$ for each $a \in L$. A unital c-quantale is called a *strictly two-sided, commutative quantale* (*sc-quantale*, for short), if $e = 1$.

In this paper, we assume $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a c-quantale. For $\alpha \in L, A, B \in L^X$, we denote $(\alpha \to A), (A \to B), (\alpha \odot A), \alpha_X \in L^X$ as $(\alpha \to A)(x) = \alpha \to A(x), (A \to B)(x) = (A(x) \to B(x)), (\alpha \odot A)(x) = \alpha \odot A(x), (A \odot B)(x) = A(x) \odot B(x)$, and $\alpha_X(x) = \alpha$.

Lemma 2.2 ([14])**.** *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \le z$, $(x \odot y) \le (x \odot z)$, $x \to y \le x \to z$ and $z \to x \le y \to x$.*
(2) *If L is an sc-quantale, then $x \odot y \le x \wedge y \le x \vee y$.*
(3) $x \to (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \to y_i)$ *and* $(\bigvee_{i \in \Gamma} x_i) \to y = \bigwedge_{i \in \Gamma} (x_i \to y)$.
(4) $x \to (\bigvee_{i \in \Gamma} y_i) \ge \bigvee_{i \in \Gamma} (x \to y_i)$
(5) $(\bigwedge_{i \in \Gamma} x_i) \to y \ge \bigvee_{i \in \Gamma} (x_i \to y)$.
(6) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.
(7) $x \odot (x \to y) \le y$ *and* $x \to y \le (y \to z) \to (x \to z)$.
(8) $y \odot z \le x \to (x \odot y \odot z)$ *and* $x \odot (x \odot y \to z) \le y \to z$.
(9) $x \to y \le (x \odot z) \to (y \odot z)$.

(10) $(x \to y) \odot (y \to z) \le x \to z$.
(11) *If L is an sc-quantale, then $x \to y = 1$ iff $x \le y$.*
(12) $x = \bigwedge_{y \in L}((x \to y) \to y)$.

Definition 2.3 ([15])**.** Let X be a set. A function $e_X : X \times X \to L$ is called:
(E1) reflexive, if $e_X(x,x) = 1$ for all $x \in X$,
(E2) transitive, if $e_X(x,y) \odot e_X(y,z) \le e_X(x,z)$ for all $x,y,z \in X$,
(E3) anti-symmetric, if $e_X(x,y) = e_X(y,x) = 1$ implies that $x = y$.
If e_X satisfies (E1) and (E2), (X, e_X) is a fuzzy preordered set. If e_X satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially ordered set (simply, fuzzy poset).

Definition 2.4 ([11])**.** Let (X, e_X) be a fuzzy poset.
(i) The triple (X, e_X, P_X) is called a *commutative quantale frame* (*cq-frame*, for short), if $P_X : X \times X \times X \to L$ satisfies the following conditions:
(P1) $P_X(x,y,z) \odot e_X(x',x) \odot e_X(y',y) \odot e_X(z,z') \le P_X(x',y',z')$,
(P2) $P_X(x,y,z) \le P_X(y,x,z)$,
(P3) $P_X(x,y,z) \odot P_X(z,y',z') \le \bigvee_{u \in X}(P_X(y,y',u) \odot P_X(x,u,z'))$,
(P4) $P_X(x,y,z) \odot P_X(x',z,z') \le \bigvee_{u \in X}(P_X(x',x,u) \odot P_X(u,y,z'))$.
(ii) A *cq*-frame (X, e_X, P_X) is called an *scq-frame*, if satisfies
(P5) $\bigvee_{u \in X} P_X(u,x,x) = 1$,
(P6) $P_X(x,y,z) \le e_X(x,z) \wedge e_X(y,z)$.

Definition 2.5 ([11])**.** Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a *c*-quantale. A fuzzy poset (X, e_X) is an *r-fuzzy poset*, if (R) $e_X(a * b, c) = e_X(a, b \nearrow c)$ for each $a,b,c \in X$.

Theorem 2.6 ([11])**.** *Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. A pair (X, e_X) is an r-fuzzy poset. We define*

$$P_X(x,y,z) = e_X(x * y, z).$$

Then the following properties hold.
(1) *(X, e_X, P_X) is a cq-frame.*
(2) *If it is an sc-quantale, $e_X(x * y, z) \le e_X(x,z) \wedge e_X(y,z)$ for each $x,y,z \in X$, then (X, e_X, P_X) is an scq-frame.*

Definition 2.7 ([3])**.** A subset $\tau_X \subset L^X$ is called an *Alexandrov topology* on X, if it satisfies the following conditions:
(O1) $\alpha_X \in \tau_X$,
(O2) if $A_i \in \tau_X$ for all $i \in I$, then $\bigvee_{i \in I} A_i, \bigwedge_{i \in I} A_i \in \tau_X$,
(O3) if $A \in \tau_X$ and $\alpha \in L$, then $\alpha \odot A, \alpha \to A \in \tau_X$.
The pair (X, τ_X) is called an *Alexandrov topological space*.

Remark 2.8. (1) Let $\tau_X \subset L^X$. Define $e_{\tau_X} : \tau_X \times \tau_X \to L$ as

$$e_{\tau_X}(A,B) = \bigwedge_{x \in X}(A(x) \to B(x)).$$

Then (τ_X, e_{τ_X}) is a fuzzy poset.
(2) If (X, e_X) is a fuzzy poset and we define a function $e_X^{-1}(x,y) = e_X(y,x)$, then (X, e_X^{-1}) is a fuzzy poset.

Theorem 2.9 ([11]). *Let* (X, e_X) *be a fuzzy poset. Define* $\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x, y) \leq A(y)\}$*. Then* τ_{e_X} *is an Alexandrov topology on* X *such that*

$$\tau_{e_X} = \{ \bigvee_{x \in X} (A(x) \odot e_X(x, -)) \mid A \in L^X\}.$$

Theorem 2.10 ([11]). *Let* (X, e_X, P_X) *be a cq-frame. For* $A, B \in \tau_{e_X}$*, we define*

$$(A \otimes B)(z) = \bigvee_{x,y \in X} (P_X(x, y, z) \odot A(x) \odot B(y)),$$
$$(A \Rightarrow B)(x) = \bigwedge_{y,z \in X} ((P_X(x, y, z) \odot A(y)) \to B(z)).$$

Then we have :

(1) $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ *is a c-quantale.*

(2) *If* (X, e_X, P_X) *be an scq-frame, then it is an sc-quantale.*

Theorem 2.11 ([11]). *Let* $(X, \wedge, \vee, *, \nearrow, 0, 1)$ *be a c-quantale with an r-fuzzy poset* (X, e_X)*. For* $A, B \in \tau_{e_X}$*, we define*

$$(A \otimes B)(z) = \bigvee_{x,y \in X} (e_X(x * y, z) \odot A(x) \odot B(y)),$$
$$(A \Rightarrow B)(x) = \bigwedge_{y,z \in X} ((e_X(x * y, z) \odot A(y)) \to B(z)).$$

Then we have the following properties:

(1) $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ *is a c-quantale, where*

$$(A \otimes B)(z) = \bigvee_{y \in X} A(y \nearrow z) \odot B(y) = \bigvee_{x \in X} A(x) \odot B(x \nearrow z),$$
$$(A \Rightarrow B)(x) = \bigwedge_{z \in X} (A(x \nearrow z) \to B(z)) = \bigwedge_{y \in X} (A(y) \to B(x * y)).$$

(2) *if it is an sc-quantale and* $e_X(x*y, z) \leq e_X(x, z) \wedge e_X(y, z)$ *for each* $x, y, z \in X$*, then* $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ *is an sc-quantale.*

Theorem 2.12 ([11]). *Let* $(X, \wedge, \vee, *, \nearrow, 0, 1)$ *be a c-quantale and* e_X *is an r-fuzzy poset on* X*. Define* $P_{L^X} : L^X \times L^X \times L^X \to L$ *as follows:*

$$P_{L^X}(A, B, C) = \bigwedge_{x \in X} ((A \otimes B)(x) \to C(x)),$$
$$(A \otimes B)(x) = \bigvee_{y,z \in X} (A(y) \odot B(z) \odot e_X(y * z, x)).$$

Then

(1) (L^X, e_{L^X}, P_{L^X}) *is a cq-frame,*

(2) *if* $(X, \wedge, \vee, *, \nearrow, 0, 1)$ *is a sc-quantale and* $P_{W(L^X)} : W(L^X) \times W(L^X) \times W(L^X) \to L$*, where* $W(L^X) = \{A \in L^X \mid A(1) = 1\}$*, then* $(W(L^X), e_{W(L^X)}, P)$ *is an scq-frame.*

(3) *if* $P_{\tau_{e_X}} : \tau_{e_X} \times \tau_{e_X} \times \tau_{e_X} \to L$*, then* $(\tau_{e_X}, e_{\tau_{e_X}}, P_{\tau_{e_X}})$ *is a cq-frame.*

3. Embedding maps on commutative quantales and commutative quantale frames

In this section, we assume $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a c-quantale.

Definition 3.1. (i) Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ and $(Y, \wedge, \vee, \star, \uparrow, 0, 1)$ be c-quantales with preorders e_X and e_Y, respectively. A map $k : (X, \wedge, \vee, *, \nearrow, 0, 1, e_X) \to (Y, \wedge, \vee, \star, \uparrow, 0, 1, e_Y)$ is called *homomorphism*, if

$$k(x * y) = k(x) \star k(y), k(x \nearrow y) = k(x) \uparrow k(y), \ e_X(x, y) \leq e_Y(k(x), k(y)).$$

If k is injective (resp. bijective), k is an *embedding* (resp. *isomorphism*) map.

(ii) Let (X, e_X, P_X) and (Y, e_Y, P_Y) be *cq*-frames. A map $k : (X, e_X, P_X) \to (Y, e_Y, P_Y)$ is called *frame homomorphism*, if

$$e_X(x, y) \leq e_Y(k(x), k(y)),\ P_X(x, y, z) = P_Y(k(x), k(y), k(z)).$$

If k is injective (resp. bijective), k is a *frame embedding* (resp. *frame isomorphism*) map.

Theorem 3.2. *Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale with r-fuzzy preorder e_X and $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A) \odot e_{\tau_{e_X}}(A, B) \leq \alpha(B)\}$. Define a map $h : X \to \tau_{e_{\tau_{e_X}}}$ as $h(x)(A) = \hat{x}(A) = A(x)$. Then $h : (X, \wedge, \vee, *, \nearrow, 0, 1, e_X) \to (\tau_{e_{\tau_{e_X}}}, \vee, \otimes, \Rightarrow, 0, 1, e_{\tau_{e_{\tau_{e_X}}}})$ is an embedding map.*

Proof. Let $P_X(x, y, z) = e_X(x * y, z)$ be given. By Theorem 2.6, (X, e_X, P_X) is a cq-frame. For $A, B \in \tau_{e_X}$, we define

$$(A \otimes B)(z) = \bigvee_{x,y \in X}(P_X(x, y, z) \odot A(x) \odot B(y)),$$
$$(A \Rightarrow B)(x) = \bigwedge_{y,z \in X}((P_X(x, y, z) \odot A(y)) \to B(z)).$$

By Theorem 2.11, $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a c-quantale. By Theorem 2.12 (3), $(\tau_{e_X}, e_{\tau_{e_X}}, P_{\tau_{e_X}})$ is a cq-frame, where $P_{\tau_{e_X}}(A, B, C) = \bigwedge_{x \in X}((A \otimes B)(x) \to C(x))$ with $(A \otimes B)(x) = \bigvee_{y,z}(A(y) \odot B(z) \odot e_X(y * z, x)$. By Theorem 2.10, $(\tau_{e_{\tau_{e_X}}}, \vee, \otimes, \Rightarrow, 0, 1)$ is a c-quantale.

Put $e_z(x) = e_X(x, z)$. Since, for all $z \in X$, $e_z(x) \odot e_X(x, y) \leq e_z(y)$, $e_z \in \tau_{e_X}$. If $h(x)(A) = h(y)(A)$ for each $A \in \tau_{e_X}$, then we get

$$e_X(x, x) = h(x)(e_x) = h(y)(e_x) = e_X(x, y) = 1 \text{ for } e_x \in \tau_{e_X}$$

and

$$e_X(y, x) = h(x)(e_y) = h(y)(e_y) = e_X(y, y) = 1 \text{ for } e_y \in \tau_{e_X}.$$

Thus By (E3), $x = y$. So h is injective.

We show $h(x) = \hat{x} \in \tau_{e_{\tau_{e_X}}}$ from

$$\begin{aligned}&\hat{x}(A) \odot e_{\tau_{e_X}}(A, B)\\ &= \hat{x}(A) \odot \bigwedge_{y \in X}(A(y) \to B(y))\\ &\leq A(x) \odot (A(x) \to B(x)) \leq B(x) = \hat{x}(B).\end{aligned}$$

For each $A \in \tau_{e_X}$, since $A(x) \odot e_X(x, y) \leq A(y)$, we have

$$\begin{aligned}&e_X(x, y) \leq \bigwedge_{A \in \tau_{e_X}}(A(x) \to A(y))\\ &= \bigwedge_{A \in \tau_{e_X}}(\hat{x}(A) \to \hat{y}(A)) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x}, \hat{y}).\end{aligned}$$

For $e_z \in \tau_{e_X}$ for all $z \in X$, $e_{\tau_{e_{\tau_{e_X}}}}(\hat{x}, \hat{y}) = \bigwedge_{A \in \tau_{e_X}}(A(x) \to A(y)) \leq \bigwedge_{e_z \in \tau_{e_X}}(e_z(x) \to e_z(y)) = \bigwedge_{z \in X}(e_X(z, x) \to e_X(z, y)) = e_X(x, y)$. Thus, $e_{\tau_{e_{\tau_{e_X}}}}(\hat{x}, \hat{y}) = e_X(x, y)$.

For each $x, y \in X, A \in \tau_{e_X}$, since

$$\begin{aligned}(h(x) \otimes h(y))(A) &= (\hat{x} \otimes \hat{y})(A)\\ &= \bigvee_{B,C \in \tau_{e_X}}\Big(P_{\tau_{e_X}}(B, C, A) \odot \hat{x}(B) \odot \hat{y}(C)\Big)\\ (\widehat{x * y})(A) &= A(x * y)\\ P_{\tau_{e_X}}(B, C, A) &= \bigwedge_{x \in X}((B \otimes C)(x) \to A(x)),\end{aligned}$$

$$\begin{aligned}
&(h(x)\otimes h(y))(A) = \bigvee_{B,C\in\tau_{e_X}} \Big(P_{\tau_{e_X}}(B,C,A)\odot \hat{x}(B)\odot \hat{y}(C)\Big)\\
&\le \bigvee_{B,C\in\tau_{e_X}} \Big((\bigwedge_{z\in X}((B\otimes C)(z)\to A(z)))\odot B(x)\odot C(y)\Big)\\
&\le \bigvee_{B,C\in\tau_{e_X}} \Big((B(x)\odot C(y)\odot e_X(x*y,x*y)\to A(x*y))\odot B(x)\odot C(y)\Big)\\
&\le A(x*y).
\end{aligned}$$

Put $B(z)=e_x(z)=e_X(x,z)$ and $C(z)=e_y(z)=e_X(y,z)$ for $x\in X$. Then $e_x, e_y\in\tau_{e_X}$ because $e_x(y)\odot e_X(y,z)\le e_x(z)$. Since

$$\begin{aligned}
&e_x(x')\odot e_y(y')\odot e_X(x'*y',z)\\
&= e_X(x,x')\odot e_X(y,y')\odot e_X(x',y'\nearrow z)\\
&\le e_X(x,y'\nearrow z)\odot e_X(y,y') = e_X(y',x\nearrow z)\odot e_X(y,y')\\
&\le e_X(y,x\nearrow z) = e_X(x*y,z),
\end{aligned}$$

we have

$$\begin{aligned}
&(h(x)\otimes h(y))(A) = \bigvee_{B,C\in\tau_{e_X}} \Big(P_{\tau_{e_X}}(B,C,A)\odot \hat{x}(B)\odot \hat{y}(C)\Big)\\
&\ge \bigwedge_{z\in X}\Big((e_x\otimes e_y)(z)\to A(z)\Big)\odot e_x(x)\odot e_y(y)\\
&= \bigwedge_{z\in X}\Big((\bigvee_{x',y'} e_x(x')\odot e_y(y')\odot e_X(x'*y',z))\to A(z)\Big)\odot e_x(x)\odot e_y(y)\\
&\ge \bigwedge_{z\in X}(e_X(x*y,z)\to A(z)) = A(x*y).
\end{aligned}$$

because $A(x*y)\odot e_X(x*y,z)\le A(z)$ implies $A(x*y)\le e_X(x*y,z)\to A(z)$ for $A\in\tau_{e_X}$. Hence $h(x*y)=h(x)\otimes h(y)$.

Since

$$\begin{aligned}
&P_{\tau_{e_X}}(A,B,C)\odot \hat{x}(B) \le \Big(\bigvee_{x'}(A(x')\odot B(x)\odot e_X(x'*x,y))\to C(y)\Big)\odot B(x)\\
&\le \bigwedge_{x'}\Big(B(x)\to (A(x')\odot e_X(x'*x,y))\to C(y)\Big)\odot B(x)\\
&\le \bigwedge_{x'}\Big((B(x)\to (A(x')\odot e_X(x'*x,y))\to C(y))\odot B(x)\Big)\\
&\le \bigwedge_{x'}(A(x')\odot e_X(x'*x,y)\to C(y)),
\end{aligned}$$

$$\begin{aligned}
&(h(x)\Rightarrow h(y))(A) = (\hat{x}\Rightarrow\hat{y})(A)\\
&= \bigwedge_{B,C\in\tau_{e_X}}\Big(P_{\tau_{e_X}}(A,B,C)\odot\hat{x}(B)\to\hat{y}(C)\Big)\\
&\ge \bigwedge_{C\in\tau_{e_X}}\Big(\bigwedge_{x'}(A(x')\odot e_X(x'*x,y)\to C(y))\to C(y)\Big)\\
&\ge \bigwedge_{C\in\tau_{e_X}}\bigvee_{x'}\Big((A(x')\odot e_X(x'*x,y)\to C(y))\to C(y)\Big)\\
&\ge \bigvee_{x'} A(x')\odot e_X(x'*x,y) = \bigvee_{x'} A(x')\odot e_X(x',x\nearrow y)\\
&= A(x\nearrow y) = \widehat{x\nearrow y}(A) = h(x\nearrow y).
\end{aligned}$$

Put $B=e_x$ and $C(z)=e_y^{-1\alpha}(z)=e_X(z,y)\to\alpha$. Since $(e_X(z,y)\to\alpha)\odot e_X(z,x)\odot e_X(x,y)\le\alpha$ implies $(e_X(z,y)\to\alpha)\odot e_X(z,x)\le e_X(x,y)\to\alpha$, then $e_y^{-1\alpha}\in\tau_{e_X}$. Since $\bigvee_{x^0}(e_X(x,x^0)\odot e_X(x'*x^0,y))=\bigvee_{x^0}(e_X(x,x^0)\odot e_X(x_0,x'\nearrow$

$y)) = e_X(x, x' \nearrow y) = e_X(x' * x, y)$, we have

$$\begin{aligned}
&(h(x) \Rightarrow h(y))(A) = \bigwedge_{B,C\in\tau_{e_X}} \Big(P_{\tau_{e_X}}(A,B,C) \odot \hat{x}(B) \to \hat{y}(C)\Big)\\
&\leq \bigwedge_{\alpha\in L} \Big(\Big(\bigwedge_{z\in X}((A\otimes e_x)(z) \to e_y^{-1\alpha}(z))\Big) \odot e_x(x) \to e_y^{-1\alpha}(y)\Big)\\
&\leq \bigwedge_{\alpha\in L} \Big(\Big(\bigwedge_{z\in X}(\bigvee_{x',x^0} A(x') \odot e_x(x^0) \odot e_X(x' * x^0, z) \to (e_X(z,y) \to \alpha))\Big) \to \alpha\Big)\\
&= \bigwedge_{\alpha\in L} \Big(((\bigvee_{x',x^0,z} A(x') \odot e_x(x^0) \odot e_X(x' * x^0, z) \odot e_X(z,y)) \to \alpha))\Big) \to \alpha\Big)\\
&= \bigwedge_{\alpha\in L} \Big(\Big(\bigvee_{x',x^0} A(x') \odot e_x(x^0) \odot e_X(x' * x^0, y) \to \alpha)\Big) \to \alpha\Big)\\
&= \bigwedge_{\alpha\in L} \Big(\Big(\bigvee_{x',x^0} A(x') \odot e_x(x^0) \odot e_X(x' * x^0, y) \to \alpha)\Big) \to \alpha\Big)\\
&= \bigwedge_{\alpha\in L} \Big(\Big(\bigvee_{x'} A(x') \odot e_X(x' * x, y) \to \alpha)\Big) \to \alpha\Big) \text{ (by Lemma 2.3 (12))}\\
&= \bigvee_{x'} A(x') \odot e_X(x' * x, y) = \bigvee_{x'} A(x') \odot e_X(x', x \nearrow y)\\
&= A(x \nearrow y) = \widehat{x \nearrow y}(A) = h(x \nearrow y)(A).
\end{aligned}$$

Therefore $h(x \nearrow y) = h(x) \Rightarrow h(y)$. □

Example 3.3. Let $(X, \wedge, \vee, *, \nearrow, 0, 1)$ be a c-quantale. Define $e_X : X \times X \to L$ as

$$e_X(x,y) = \begin{cases} 1, & \text{if } x \leq y,\\ 0, & \text{if } x \not\leq y.\end{cases}$$

Since $x * y \leq z$ iff $x \leq y \nearrow z$, $e_X(x * y, z) = e_X(x, y \nearrow z)$ for each $x, y, z \in X$. Then (X, e_X) is an r-fuzzy poset. Furthermore, define $P_X(x,y,z) = e_X(x * y, z)$. Then (X, e_X, P_X) is a cq-frame. Thus we have

$$\tau_{e_X} = \{A \in L^X \mid A(x) \odot e_X(x,y) \leq A(y)\} = \{A \in L^X \mid A(x) \leq A(y) \text{ if } x \leq y\}.$$

For $A, B \in \tau_{e_X}$, we define

$$(A \otimes B)(z) = \bigvee_{x,y\in X} (P(x,y,z) \odot A(x) \odot B(y)) = \bigvee_{x*y\leq z} (A(x) \odot B(y)).$$

$$(A \Rightarrow B)(x) = \bigwedge_{y,z\in X} (P(x,y,z) \odot A(y) \to B(z)) = \bigwedge_{x\leq y\nearrow z} (A(y) \to B(z)).$$

By Theorem 2.10, $(\tau_{e_X}, \leq, \vee, \wedge, \otimes, \Rightarrow, 1_X, 0_X)$ is a c-quantale. Define $P_{\tau_{e_X}} : \tau_{e_X} \times \tau_{e_X} \times \tau_{e_X} \to L$ as follows:

$$P_{\tau_{e_X}}(A,B,C) = \bigwedge_{x\in X} ((A\otimes B)(x) \to C(x)) = \bigwedge_{x\in X} \bigwedge_{y*z\leq x} ((A(y) \odot B(z)) \to C(x)).$$

Then $(\tau_{e_X}, e_{\tau_{e_X}}, P_{\tau_{e_X}})$ is a cq- frame.

For $x, y \in X$ and $A \in L^X$,

$$\begin{aligned}
&(h(x) \otimes h(y))(A) = \bigvee_{B,C\in\tau_{e_X}} \Big(P_{\tau_{e_X}}(B,C,A) \odot \hat{x}(B) \odot \hat{y}(C)\Big)\\
&= \bigvee_{B,C\in\tau_{e_X}} \Big((\bigwedge_{z\in X}(B\otimes C)(z) \to A(z))) \odot B(x) \odot C(y)\Big)\\
&\leq \bigvee_{B,C\in\tau_{e_X}} \Big((B\otimes C)(x*y) \to A(x*y))) \odot B(x) \odot C(y)\Big)\\
&\leq \bigvee_{B,C\in\tau_{e_X}} \Big((B(x) \odot C(y) \to A(x*y))) \odot B(x) \odot C(y)\Big)\\
&\leq A(x*y) = \widehat{x*y}(A) = h(x*y).
\end{aligned}$$

Since $x \le y$ and $y \le z$ implies $x \le z$, $e_x(y) \odot e_X(y,z) \le e_x(z)$. Thus $e_x \in \tau_{e_X}$. Since $(e_x \otimes e_y)(z) = \bigvee_{x' * y' \le z} e_x(x') \odot e_y(y') \le \bigvee_{x' * y' \le z} e_{x*y}(x' * y') = e_{x*y}(z)$, we have

$$\begin{aligned}&(h(x) \otimes h(y))(A) = \bigvee_{B,C \in \tau_{e_X}} \Big(P_{\tau_{e_X}}(B,C,A) \odot \hat{x}(B) \odot \hat{y}(C) \Big)\\ &\ge \bigwedge_{z \in X} \Big((e_x \otimes e_y)(z) \to A(z) \Big) \odot e_x(x) \odot e_y(y)\\ &\ge \bigwedge_{z \in X} (e_X(x*y,z) \to A(z)) = \bigwedge_{x*y \le z} A(z)\\ &= A(x*y) = \widehat{x*y}(A) = h(x*y).\end{aligned}$$

because $A(x) \le A(y)$ for $x \le y$ and $A \in \tau_{e_X}$.

$$\begin{aligned}(h(x) \Rightarrow h(y))(A) &= (\hat{x} \Rightarrow \hat{y})(A)\\ &= \bigwedge_{B,C \in \tau_{e_X}} \Big(P_{\tau_{e_X}}(A,B,C) \odot \hat{x}(B) \to \hat{y}(C) \Big)\end{aligned}$$

$$\begin{aligned}&P_{\tau_{e_X}}(A,B,C) \odot \hat{x}(B) \le \Big(\bigvee_{x' * x \le y} (A(x') \odot B(x)) \to C(y) \Big) \odot B(x)\\ &\le (\bigvee_{x' \le x \nearrow y} A(x')) \to C(y) = A(x \nearrow y) \to C(y)\end{aligned}$$

$$\begin{aligned}(h(x) \Rightarrow h(y))(A) &= \bigwedge_{B,C \in \tau_{e_X}} \Big(P_{\tau_{e_X}}(A,B,C) \odot \hat{x}(B) \to \hat{y}(C) \Big)\\ &\ge \bigwedge_{B,C \in \tau_{e_X}} \Big((A(x \nearrow y) \to C(y)) \to C(y) \Big)\\ &\ge A(x \nearrow y) = \widehat{x \nearrow y}(A) = h(x \nearrow y).\end{aligned}$$

Put $B = e_x$ and $C(z) = e_y^{-1\alpha}(z) = e_X(z,y) \to \alpha$ as

$$e_y^{-1\alpha}(z) = \begin{cases} \alpha, & \text{if } z \le y,\\ 1, & \text{if } z \not\le y.\end{cases}$$

If $z \le x \le y$, then $e_y^{-1\alpha}(x) = \alpha$ implies $e_y^{-1\alpha}(z) = \alpha$. Thus $(e_X(z,y) \to \alpha) \odot e_X(z,x) \le e_X(x,y) \to \alpha$; i.e. $e_y^{-1\alpha} \in \tau_{e_X}$. Since $\bigvee_{x^0} e_X(x,x^0) \odot e_X(x' * x^0, y) = e_X(x' * x, y)$ and $\bigvee_{x'} A(x') \odot e_X(x' * x, y) = A(x \nearrow y)$, we have

$$\begin{aligned}&(h(x) \Rightarrow h(y))(A) = \bigwedge_{B,C \in \tau_{e_X}} \Big(P_{\tau_{e_X}}(A,B,C) \odot \hat{x}(B) \to \hat{y}(C) \Big)\\ &\le \bigwedge_{\alpha \in L} \Big(\Big(\bigwedge_{z \in X} ((A \otimes e_x)(z) \to e_y^{-1\alpha}(z)) \Big) \odot e_x(x) \to e_y^{-1\alpha}(y) \Big)\\ &\le \bigwedge_{\alpha \in L} \Big(\Big(\bigvee_{x',x^0} A(x') \odot e_x(x^0) \odot e_X(x' * x^0, y) \to \alpha) \Big) \to \alpha \Big)\\ &= \bigwedge_{\alpha \in L} \Big(\Big(\bigvee_{x'} A(x') \odot e_X(x' * x, y) \to \alpha) \Big) \to \alpha \Big)\\ &= A(x \nearrow y) = \widehat{x \nearrow y}(A) = h(x \nearrow y).\end{aligned}$$

Example 3.4. Let $(X = [0,\infty], \le_{op}, \vee_{op}, +, \wedge_{op}, \infty, 0)$ be an *sc*-quantale, where $\le_{op} = \ge, \vee_{op} = \wedge, \wedge_{op} = \vee$ and

$$\begin{aligned}x \nearrow y &= \bigvee\nolimits_{op} \{z \in [0,\infty] \mid x + z \le_{op} y\}\\ &= \bigwedge \{z \in [0,\infty] \mid z \ge -x + y\} = (y - x) \vee 0,\end{aligned}$$

$$e_X^0(x,y) = \begin{cases} 0, & \text{if } x \ge y,\\ \infty, & \text{if } x < y.\end{cases}$$

Define $P_X(x,y,z) = e_X^0(x+y,z)$ for each $x,y,z \in [0,\infty]$. By Theorem 2.6, $([0,\infty], e_X^0, P_X)$ is a *cq*-frame. Then we get

$$\begin{aligned}\tau_{e_X^0} &= \{A \in [0,\infty]^{[0,\infty]} \mid A(x) + e_X^0(x,y) \le_{op} A(y)\}\\ &= \{A \in [0,\infty]^{[0,\infty]} \mid \bigwedge_{x \ge y} A(x) \le_{op} A(y)\}\end{aligned}$$

$= \{A \in [0,\infty]^{[0,\infty]} \mid A(x) \geq A(y), x \geq y\},$
because $A(x) \geq \bigwedge_{x\geq y} A(x) \geq A(y)$ for $y \leq x$.

For $A, B \in \tau_{e^0_X}$, we define

$$\begin{aligned}
&(A \otimes B)(z) = (\bigvee_{op})_{x,y\in X}(P(x,y,z) + A(x) + B(y)) = \bigwedge_{x+y\geq z}(A(x) + B(y)) \\
&= \bigwedge_{y\in X}(A((z-y)\vee 0) + B(y)) = \bigwedge_{x\in X}(A(x) + B((z-x)\vee 0)), \\
&(A \Rightarrow B)(x) = (\bigwedge_{op})_{y,z\in X}(P(x,y,z) + A(y)) \nearrow B(z)) \\
&= \bigvee_{x\geq(-y+z)\vee 0}((B(z) - A(y)) \vee 0) = \bigvee_{z\in X}((B(z) - A((z-x)\vee 0)) \vee 0) \\
&= \bigvee_{y\in X}((B(x+y) - A(y)) \vee 0).
\end{aligned}$$

Then $(\tau_{e^0_X}, \leq_{op}=\geq, \vee_{op}, \wedge_{op}, \otimes, \Rightarrow, \infty_X, 0_X)$ is a c-quantale, where $0_X(x) = 0$ and $\infty_X(x) = \infty$ for each $x \in [0,\infty]$.

Define $P_{\tau_{e^0_X}} : \tau_{e^0_X} \times \tau_{e^0_X} \times \tau_{e^0_X} \to [0,\infty]$ as follows:

$$P(A,B,C) = (\bigwedge_{op})_{z\in X}((A\otimes B)(z) \nearrow C(z)) = \bigvee_{z\in X}((C(z) - \bigwedge_{x+y\geq z}(A(x)+B(y)))\vee 0).$$

Then $(\tau_{e^0_X}, e_{\tau_{e^0_X}}, P_{\tau_{e^0_X}})$ is a cq-frame.

Since $e^0_X(x,u) + e^0_X(y,v) \geq e^0_X(x+y, u+v) = e^0_X(x+y,u+v) + e^0_X(u+v,z) \geq e^0_X(x+y,z)$ for each $u+v \geq z$,

$$\begin{aligned}
&(h(x)\otimes h(y))(A) = (\hat{x}\otimes\hat{y})(A) = (\bigvee_{op})_{B,C\in\tau_{e^0_X}}(P_{\tau_{e^0_X}}(B,C,A) + \hat{x}(B) + \hat{y}(C)) \\
&= \bigwedge_{B,C\in\tau_{e^0_X}}(\bigvee_{z\in X}((A(z) - \bigwedge_{u+v\geq z}(B(u)+C(v)))\vee 0) + B(x) + C(y)) \\
&\geq \bigwedge_{B,C\in\tau_{e^0_X}}((A(x+y) - (B(x)+C(y)))\vee 0) + B(x) + C(y)) \\
&= A(x+y) = \widehat{x+y}(A), \\
&(h(x)\otimes h(y))(A) \\
&= \bigwedge_{B,C\in\tau_{e^0_X}}(\bigvee_{z\in X}((A(z) - \bigwedge_{u+v\geq z}(B(u)+C(v)))\vee 0) + B(x) + C(y)) \\
&\leq \bigvee_{z\in X}((A(z) - \bigwedge_{u+v\geq z}((e^0_X)_x(u) + (e^0_X)_y(v)))\vee 0) + (e^0_X)_x(x) + (e^0_X)_y(y)) \\
&\leq \bigvee_{z\in X}((A(z) - \bigwedge_{u+v\geq z} e^0_X(x+y,u+v))\vee 0) \\
&\leq \bigvee_{z\in X}(A(z) - e^0_X(x+y,z))\vee 0 \leq A(x+y).
\end{aligned}$$

For $A, B, C \in \tau_{e^0_X}$,

$$\begin{aligned}
&P_{\tau_{e^0_X}}(A,B,C) + \hat{x}(B) \leq_{op} \Big(C(y) - (\bigvee_{x'+x\geq y})_{op}(A(x') + B(x))\Big) + B(x) \\
&\leq_{op} C(y) - \bigwedge_{x'\geq y-x} A(x') \leq_{op} C(y) - A(y-x),
\end{aligned}$$

$$\begin{aligned}
&(h(x)\Rightarrow h(y))(A) = (\hat{x}\Rightarrow\hat{y})(A) \\
&= (\bigwedge_{op})_{B,C\in\tau_{e^0_X}}\Big(P_{\tau_{e^0_X}}(A,B,C) + \hat{x}(B) \nearrow \hat{y}(C)\Big) \\
&= \bigvee_{B,C\in\tau_{e^0_X}}\Big((C(y) - (P_{\tau_{e^0_X}}(A,B,C) + \hat{x}(B))\vee 0\Big) \\
&\leq \bigvee_{B,C\in\tau_{e^0_X}}\Big(C(y) - (C(y) - A(y-x))\Big) = A(y-x) = h(x\nearrow y),
\end{aligned}$$

$$
\begin{aligned}
&(h(x) \Rightarrow h(y))(A) = (\hat{x} \Rightarrow \hat{y})(A) \\
&= (\bigwedge_{op})_{B,C\in\tau_{e^0_X}} \Big(P_{\tau_{e^0_X}}(A,B,C) + \hat{x}(B) \nearrow \hat{y}(C)\Big) \\
&\geq \bigvee_{C\in\tau_{e^0_X}} \Big(P_{\tau_{e^0_X}}(A,e_x,C) + \hat{x}(e_x) \nearrow \hat{y}(C)\Big) \\
&(C = A\otimes e_x) \\
&\geq (A\otimes e_x)(y) = \bigwedge_{z+w\geq y}(A(z)+e_x(w)) = A(y-x).
\end{aligned}
$$

Theorem 3.5. *Let (X, e_X, P) be a cq-frame and $\tau_{e_{\tau_{e_X}}} = \{\alpha \in L^{\tau_{e_X}} \mid \alpha(A)\odot e_{\tau_{e_X}}(A,B) \leq \alpha(B)\}$. Define a map $k : X \to \tau_{e_{\tau_{e_X}}}$ as $k(x)(A) = \hat{x}(A) = A(x)$. Then $k : (X, e_X, P) \to (\tau_{e_{\tau_{e_X}}}, e_{\tau_{e_{\tau_{e_X}}}}, \hat{P})$ is a frame embedding map with $e_X(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(k(x),k(y))$ and $\hat{P}(\hat{x},\hat{y},\hat{z}) = P(x,y,z)$, where*

$$\hat{P}(\hat{x},\hat{y},\hat{z}) = \bigwedge_{A\in\tau_{e_X}} ((\hat{x}\otimes\hat{y})(A)) \to \hat{z}(A))).$$

Proof. Let (X, e_X, P) be a *cq*-frame. By Theorem 2.10, $(\tau_{e_X}, \vee, \wedge, \otimes, \Rightarrow, 0_X, 1_X)$ is a *c*-quantale. By Theorem 2.12 (3), $(\tau_{e_{\tau_{e_X}}}, e_{\tau_{e_{\tau_{e_X}}}}, \hat{P})$ is a *cq*-frame, where

$$\hat{P}(\alpha,\beta,\gamma) = \bigwedge_{A\in\tau_{e_X}}((\alpha\otimes\beta)(A) \to \gamma(A)).$$

By Theorem 3.2, $e(x,y) = e_{\tau_{e_{\tau_{e_X}}}}(\hat{x},\hat{y})$.

Since $(B\otimes C)(z) = \bigvee_{x,y} B(x)\odot C(y)\odot P(x,y,z)$ and $(B\otimes C) \in \tau_{e_X}$ from Theorem 2.10, we have

$$
\begin{aligned}
\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_X}}(B\otimes C, A) &\leq B(x)\odot C(y)\odot \Big(B(x)\odot C(y)\odot P(x,y,z) \to A(z)\Big) \\
&\leq P(x,y,z) \to A(z),
\end{aligned}
$$

$$
\begin{aligned}
\hat{P}(\hat{x},\hat{y},\hat{z}) &= \bigwedge_{A\in\tau_{e_X}} (\hat{x}\otimes\hat{y})(A) \to \hat{z}(A)) \\
&= \bigwedge_{A\in\tau_{e_X}} (\bigvee_{B,C}(\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_X}}(B\otimes C, A)) \to \hat{z}(A))) \\
&\geq \bigwedge_{A\in\tau_{e_X}} ((P(x,y,z) \to A(z)) \to A(z)) \\
&\geq P(x,y,z).
\end{aligned}
$$

For $C^\alpha_{x,z}(y) = P(x,y,z) \to \alpha$ for each $y \in X$, since $P(x,y,z)\odot e_X(w,y) \leq P(x,w,z)$, we have

$$
\begin{aligned}
&P(x,y,z)\odot (P(x,w,z) \to \alpha)\odot e_X(w,y)) \\
&\leq P(x,w,z)\odot (P(x,w,z) \to \alpha) \leq \alpha.
\end{aligned}
$$

Then $C^\alpha_{x,z}(w)\odot e_X(w,y) \leq C^\alpha_{x,z}(y)$. Thus $C^\alpha_{x,z} \in \tau_{e_X}$. For $A = ((e_z^{-1}) \to \alpha) \in \tau_{e_X}$, $B = 1 \in \tau_{e_X}$ and $C^\alpha_{x,z} \in \tau_{e_X}$, $1(x)\odot C^\alpha_{x,z}(y) = P(x,y,z) \to \alpha$. Since

$$
\begin{aligned}
&e_{\tau_{e_X}}(1\otimes C^\alpha_{x,z}, (e_z^{-1}) \to \alpha)) \\
&= \bigwedge_{w\in X}(\bigvee_{x,y\in X}(1(x)\odot C^\alpha_{x,z}(y)\odot P(x,y,w) \to ((e_z^{-1})(w) \to \alpha)) \\
&= \bigwedge_{w\in X}\bigwedge_{x,y\in X}\Big(C^\alpha_{x,z}(y)\odot P(x,y,w) \to ((e_z^{-1})(w) \to \alpha)\Big) \\
&= \bigwedge_{w\in X}\bigwedge_{x,y\in X}\Big(C^\alpha_{x,z}(y) \to (P(x,y,w)\odot e_X(w,z) \to \alpha)\Big) \\
&= \bigwedge_{w\in X}\bigwedge_{x,y\in X}\Big((P(x,y,z) \to \alpha) \to (P(x,y,w)\odot e_X(w,z) \to \alpha)\Big) \text{ (P1)} \\
&\geq \bigwedge_{w\in X}\bigwedge_{x,y\in X}\Big((P(x,y,z) \to \alpha) \to (P(x,y,z) \to \alpha)\Big) = 1,
\end{aligned}
$$

we have

$$\begin{aligned}\hat{P}(\hat{x},\hat{y},\hat{z}) &= \bigwedge_{A\in\tau_{e_X}}(\bigvee_{B,C\in\tau_{e_X}}\hat{x}(B)\odot\hat{y}(C)\odot e_{L^X}(B\otimes C,A))\to\hat{z}(A)))\\ &\le (1(x)\odot C^{\alpha}_{x,z}(y)\odot e_{\tau_{e_X}}(1\otimes C^{\alpha}_{x,z},(e_z^{-1})\to\alpha))\to((e_z^{-1})(z)\to\alpha))\\ &\le (P(x,y,z)\to\alpha)\to\alpha.\end{aligned}$$

So $\hat{P}(\hat{x},\hat{y},\hat{z})\le\bigwedge_{\alpha\in L}((P(x,y,z)\to\alpha)\to\alpha)=P(x,y,z)$. □

Example 3.6. Let (X,e_X,P) be a *cq*-frame in Example 3.3. For $A,B\in\tau_{e_X}$, we define

$$(A\otimes B)(z)=\bigvee_{x,y\in X}(P(x,y,z)\odot A(x)\odot B(y))=\bigvee_{x*y\le z}(A(x)\odot B(y)),$$

$$(A\Rightarrow B)(x)=\bigwedge_{y,z\in X}(P(x,y,z)\odot A(y)\to B(z))=\bigwedge_{x\le y\nearrow z}(A(y)\to B(z)).$$

By Example 3.3, $(\tau_{e_X},\le,\vee,\wedge,\otimes,\Rightarrow,1_X,0_X,e_{\tau_{e_X}})$ is a *c*-quantale.

Let $\tau_{e_{\tau_{e_X}}}=\{\alpha\in L^{\tau_{e_X}}\mid \alpha(A)\odot e_{\tau_{e_X}}(A,B)\le\alpha(B)\}$ be given. By Example 3.3, $(\tau_{e_{\tau_{e_X}}},e_{\tau_{e_{\tau_{e_X}}}},\hat{P})$ is a *cq*-frame, where for $(\alpha\otimes\beta)(A)=\bigvee_{B,C\in\tau_{e_X}}\alpha(B)\odot\beta(C)\odot e_{\tau_{e_X}}(B\otimes C,A)$,

$$\hat{P}(\alpha,\beta,\gamma)=\bigwedge_{A\in\tau_{e_X}}((\alpha\otimes\beta)(A)\to\gamma(A)).$$

For each $\hat{x},\hat{y}\in\tau_{e_{\tau_{e_X}}}$, since $(B\otimes C)(z)=\bigvee_{x*y\le z}(B(x)\odot C(y))$ from Example 3.3, we have

$$\begin{aligned}&\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_X}}(B\otimes C,A)\\ &\le B(x)\odot C(y)\odot\Big(B(x)\odot C(y)\to A(x*y)\Big)\le A(x*y).\end{aligned}$$

Since $A(x*y)\odot e_X(x*y,z)\le A(z)$ iff $e_X(x*y,z)\le A(x*y)\to A(z)$,

$$\begin{aligned}&\hat{P}(\hat{x},\hat{y},\hat{z})=\bigwedge_{A\in\tau_{e_X}}(\hat{x}\otimes\hat{y})(A)\to\hat{z}(A))\\ &=\bigwedge_{A\in\tau_{e_X}}(\bigvee_{B,C}(\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_X}}(B\otimes C,A))\to\hat{z}(A)))\\ &\ge\bigwedge_{A\in\tau_{e_X}}(A(x*y)\to A(z))\ge e_X(x*y,z)=P(x,y,z).\end{aligned}$$

For $e^{-\alpha}_{x\nearrow z}(y)=e_X(y,x\nearrow z)\to\alpha$ for each $y\in X$, since $e_X(y,x\nearrow z)\odot e_X(w,y)\le e_X(w,x\nearrow z)$, we have

$$\begin{aligned}&e_X(y,x\nearrow z)\odot(e_X(w,x\nearrow z)\to\alpha)\odot e_X(w,y))\\ &\le e_X(w,x\nearrow z)\odot(e_X(w,x\nearrow z)\to\alpha)\le\alpha.\end{aligned}$$

Then $e^{-\alpha}_{x\nearrow z}(w)\odot e_X(w,y)\le e^{-\alpha}_{x\nearrow z}(y)$. Thus $e^{-\alpha}_{x\nearrow z}\in\tau_{e_X}$. For $A=((e_z^{-1})\to\alpha)\in\tau_{e_X}$, $B=1\in\tau_{e_X}$ and $e^{-\alpha}_{x\nearrow z}\in\tau_{e_X}$, $1(x)\odot e^{-\alpha}_{x\nearrow z}(y)=e_X(y,x\nearrow z)\to\alpha$. Since

$$\begin{aligned}e_{\tau_{e_X}}(1\otimes e^{-\alpha}_{x\nearrow z},(e_z^{-1})\to\alpha)) &= \bigwedge_{w\in X}(\bigvee_{x*y\le w}(1(x)\odot e^{-\alpha}_{x\nearrow z}(y))\to((e_z^{-1})(w)\to\alpha))\\ &=\bigwedge_{w\in X}\bigwedge_{x*y\le w}\Big(e^{-\alpha}_{x\nearrow z}(y)\odot(e_z^{-1})(w)\to\alpha\Big)\\ &=\bigwedge_{w\in X}\bigwedge_{x*y\le z}\bigwedge_{w\le z}(e^{-\alpha}_{x\nearrow z}(y)\to\alpha)\\ &=\bigwedge_{x*y\le z}(e_X(y,x\nearrow z)\to\alpha)\to\alpha)=1,\end{aligned}$$

we have

$$\begin{aligned}&\hat{P}(\hat{x},\hat{y},\hat{z}) = \bigwedge_{A\in\tau_{e_X}}(\bigvee_{B,C\in\tau_{e_X}} \hat{x}(B)\odot\hat{y}(C)\odot e_{L^X}(B\otimes C,A))\to\hat{z}(A)))\\ &\leq (1(x)\odot e^{-\alpha}_{x\nearrow z}(y)\odot e_{\tau_{e_X}}(1\otimes e^{-\alpha}_{x\nearrow z},(e_z^{-1})\to\alpha))\to((e_z^{-1})(z)\to\alpha))\\ &\leq (e_X(y,x\nearrow z)\to\alpha)\to\alpha.\end{aligned}$$

So $\hat{P}(\hat{x},\hat{y},\hat{z})\leq\bigwedge_{\alpha\in L}((e_X(y,x\nearrow z)\to\alpha)\to\alpha)=e_X(y,x\nearrow z)=P(x,y,z)$.

Example 3.7. Let (X,e^0_X,P_X) be a *cq*-frame in Example 3.4. For $A\in\tau_{e^0_X}$, $A(x)+e^0_X(x,y)\leq_{op}A(y)$ iff $A(x)\geq A(y)$ for each $x\geq y$. For $A,B\in\tau_{e^0_X}$, we define

$$(A\otimes B)(z)=(\bigvee_{op})_{x,y\in X}(P(x,y,z)\odot A(x)\odot B(y))=\bigwedge_{x+y\geq z}(A(x)+B(y)),$$

$$(A\Rightarrow B)(x)=(\bigwedge_{op})_{y,z\in X}(P(x,y,z)\odot A(y)\nearrow B(z))=\bigvee_{x\geq(-y+z)\vee 0}((B(z)-A(y))\vee 0).$$

By Example 3.4, $(\tau_{e^0_X},\leq_{op}=\geq,\vee_{op},\wedge_{op},\otimes,\Rightarrow,0_X,\infty_X,e_{\tau_{e^0_X}})$ is a *c*-quantale, where $0_X(x)=0$, $\infty_X(x)=\infty$ for each $x\in[0,\infty]$. Moreover, $(\tau_{e_{\tau_{e^0_X}}},e_{\tau_{e_{\tau_{e^0_X}}}},\hat{P})$ is a *cq*-frame, where

$$\hat{P}(\alpha,\beta,\gamma)=(\bigwedge_{op})_{A\in\tau_{e^0_X}}(\alpha\otimes\beta(A))\nearrow\gamma(A))).$$

From Theorem 3.5,

$$\begin{aligned}\hat{P}(\hat{x},\hat{y},\hat{z}) &= (\bigwedge_{op})_{A\in\tau_{e^0_X}}(\hat{x}\otimes\hat{y})(A)\nearrow\hat{z}(A))\\ &= (\bigwedge_{op})_{A\in\tau_{e^0_X}}((\bigvee_{op})_{B,C}(\hat{x}(B)+\hat{y}(C)+e_{\tau_{e^0_X}}(B\otimes C,A))\nearrow\hat{z}(A)))\\ &\geq_{op}(\bigwedge_{op})_{A\in\tau_{e^0_X}}((P(x,y,z)\nearrow A(z))\nearrow A(z))\\ &\geq_{op}P(x,y,z).\end{aligned}$$

Put $C^\alpha_{x,z}(y)=e^0_X(x+y,z)\nearrow\alpha$ and $(e^{0-1}_X)^\alpha_z(y)=e^0_X(y,z)\nearrow\alpha$ for each $y\in X$. Then

$$C^\alpha_{x,z}(y)=\begin{cases}\alpha, & \text{if } x+y\geq z,\\ 0, & \text{if } x+y<z,\end{cases}\quad (e^{0-1}_X)^\alpha_z(y)=\begin{cases}\alpha, & \text{if } y\geq z,\\ 0, & \text{if } y<z.\end{cases}$$

If $x+y\geq z$ and $w\geq y$, then $x+w\geq z$. Thus $C^\alpha_{x,z}(w)+e^0_X(w,y)\leq_{op}C^\alpha_{x,z}(y)$. Thus $C^\alpha_{x,z}\in\tau_{e^0_X}$. If $y\geq z$ and $w\geq y$, then $w\geq z$. Thus $(e^{0-1}_X)^\alpha_z(w)+e^0_X(w,y)\leq_{op}(e^{0-1}_X)^\alpha_z(y)$. So $(e^{0-1}_X)^\alpha_z\in\tau_{e^0_X}$. Moreover, we have

$$\begin{aligned}&e_{\tau_{e^0_X}}(0\otimes C^\alpha_{x,z},(e^{0-1}_X)^\alpha_z)\\ &=(\bigwedge_{op})_{w\in X}((\bigvee_{op})_{x,y\in X}(0(x)+C^\alpha_{x,z}(y)+P(x,y,w)\nearrow(e^{0-1}_X)^\alpha_z)\\ &=(\bigwedge_{op})_{w\in X}(\bigwedge_{op})_{x,y\in X}\Big(C^\alpha_{x,z}(y)+P(x,y,w)\nearrow(e^{0-1}_X)^\alpha_z(w)\Big)\\ &=0,\end{aligned}$$

because $w\geq z$, $x+y\geq w$ implies $x+y\geq z$. So we get

$$\begin{aligned}\hat{P}(\hat{x},\hat{y},\hat{z}) &= (\bigwedge_{op})_{A\in\tau_{e^0_X}}((\bigvee_{op})_{B,C\in\tau_{e^0_X}}\hat{x}(B)+\hat{y}(C)+e_{\tau_{e^0_X}}(B\otimes C,A))\nearrow\hat{z}(A)))\\ &\leq_{op}(0(x)+C^\alpha_{x,z}(y)+e_{\tau_{e^0_X}}(0\otimes C^\alpha_{x,z},(e^{0-1}_X)^\alpha_z)\nearrow(e^{0-1}_X)^\alpha_z(z))\\ &=(P(x,y,z)\nearrow\alpha)\nearrow\alpha.\end{aligned}$$

Hence $\hat{P}(\hat{x},\hat{y},\hat{z})\leq_{op}(\bigwedge_{op})_{\alpha\in[0,\infty]}((P(x,y,z)\nearrow\alpha)\nearrow\alpha)=P(x,y,z)$.

Since $P(x,y,z) \leq_{op} ((P(x,y,z) \nearrow \alpha) \nearrow \alpha)$, $P(x,y,z) \leq_{op} (\bigwedge_{op})_{\alpha\in L}((P(x,y,z) \nearrow \alpha) \nearrow \alpha)$. Put $P(x,y,z) = C^{\alpha}_{\alpha,\alpha}(y) = \alpha$ for each $y \in X$. Then $(\bigwedge_{op})_{\alpha\in L}((P(x,y,z) \nearrow \alpha) \nearrow \alpha) \leq_{op} (\alpha \nearrow \alpha) \nearrow \alpha) = \alpha = P(x,y,z)$. Thus $(\bigwedge_{op})_{\alpha\in[0,\infty]}((P(x,y,z) \nearrow \alpha) \nearrow \alpha) = P(x,y,z)$. So $\hat{P}(\hat{x},\hat{y},\hat{z}) = P(x,y,z)$.

Example 3.8. Let $L = \{0,x,y,1\}$ be such that $0 < x < y < 1$. Define $\odot$ and $\to$ as follows:

$\odot$	0	x	y	1
0	0	0	0	0
x	0	0	x	x
y	0	x	y	y
1	0	x	y	1

$\to$	0	x	y	1
0	1	1	1	1
x	x	1	1	1
y	0	x	1	1
1	0	x	y	1

Then $(L,\wedge,\vee,\odot,\to,0,1)$ is a c-quantale. Define $e_L(x,y) = x \to y$ and $P(x,y,z) = (x\odot y) \to z$ for each $x,y,z \in L$. Then (L,e_L,P) is a cq-frame. For $A = (A(0),A(x),A(y),A(1)) = (0,x,0,y)$, $B = (B(0),B(x),B(y),B(1)) = (y,y,x,x)$, $C = (C(0),C(x),C(y),C(1)) = (x,y,y,y)$, since $A(1)\odot e_L(1,y) = y\odot y = y \not\leq A(y) = 0$ and $B(x)\odot e_L(x,y) = y\odot 1 = y \not\leq B(y) = x$, $A \notin \tau_{e_L}, B \notin \tau_{e_L}$ and $C = \bigvee_{x\in L}(A(x)\odot e_L(x,-)) \in \tau_{e_L}$. $D = \bigvee_{x\in L}(A(x)\odot e_L(x,-)) = (0,x,y,y) \in \tau_{e_L}$ and $E = \bigvee_{x\in L}(B(x)\odot e_L(x,-)) = (y,y,y,y) \in \tau_{e_L}$. Since $C\otimes D(z) = \bigvee_{x,y\in L}(((x\odot y) \to z)\odot C(x)\odot D(y))$, $(C\otimes D) = (x,y,y,y) = C$.

Since $(C \Rightarrow D)(x) = \bigwedge_{y,z\in L}(x \to (y \to z)) \to (C(y) \to D(z))$, $(C \Rightarrow D) = (x,x,1,1) \in \tau_{e_L}$.

By Theorem 2.10, $(\tau_{e_L},\vee,\wedge,\otimes,\Rightarrow,0_L,1_L)$ is a c-quantale with r-fuzzy preorder $e_{\tau_{e_L}}$.

Define $P_{\tau_{e_L}} : \tau_{e_L}\times\tau_{e_L}\times\tau_{e_L} \to L$ as follows:

$$P_{\tau_{e_L}}(A,B,C) = \bigwedge_{x\in L}((A\otimes B)(x) \to C(x))$$

where $(A\otimes B)(x) = \bigvee_{y,z}(A(y)\odot B(z)\odot e_L(y\odot z,x)$. By Theorem 2.12 (3), $(\tau_{e_L}, e_{\tau_{e_L}}, P)$ is a cq-frame. By Theorem 2.10, $(\tau_{e_{\tau_{e_L}}},\vee,\otimes,\Rightarrow,0,1,e_{\tau_{e_{\tau_{e_L}}}})$ is a c-quantale.

By Theorem 2.12 (3), $(\tau_{e_{\tau_{e_L}}}, e_{\tau_{e_{\tau_{e_L}}}}, \hat{P})$ is a cq-frame, where

$$\hat{P}(\alpha,\beta,\gamma) = \bigwedge_{A\in\tau_{e_L}}(\alpha\otimes\beta(A)) \to \gamma(A))).$$

Since $(B\otimes C)(z) = \bigvee_{x,y} B(x)\odot C(y)\odot P(x,y,z)$ and $(B\otimes C) \in \tau_{e_L}$ from Theorem 2.10, we have

$$\begin{aligned}\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_L}}(B\otimes C,A) &\leq B(x)\odot C(y)\odot \Big(B(x)\odot C(y)\odot P(x,y,0) \to A(0)\Big)\\ &\leq P(x,y,0) \to A(0).\end{aligned}$$

$$\begin{aligned}&\hat{P}(\hat{x},\hat{y},\hat{0}) = \bigwedge_{A\in\tau_{e_L}}(\hat{x}\otimes\hat{y})(A) \to \hat{0}(A))\\ &= \bigwedge_{A\in\tau_{e_L}}(\bigvee_{B,C}(\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_L}}(B\otimes C,A)) \to \hat{0}(A)))\\ &\geq \bigwedge_{A\in\tau_{e_L}}((P(x,y,0) \to A(0)) \to A(0))\\ &\geq P(x,y,0) = x\odot y \to 0 = x.\end{aligned}$$

Put $B = e_x = e_L(x,-) = (x,1,1,1)$ and $C = e_y = e_L(y,-) = (0,x,1,1)$ for $x,y \in L$. Then $e_x, e_y \in \tau_{e_L}$ because $e_x(y)\odot e_L(y,z) \leq e_x(z)$. Since $e_x(x')\odot e_y(y') \leq$

$e_{x*y}(x' * y')$, we have

$$\begin{array}{ll}\hat{P}(\hat{x},\hat{y},\hat{0}) & = \bigwedge_{A\in\tau_{e_L}}(\hat{x}\otimes\hat{y})(A)\to\hat{0}(A)) \\ & = \bigwedge_{A\in\tau_{e_L}}(\bigvee_{B,C}(\hat{x}(B)\odot\hat{y}(C)\odot e_{\tau_{e_L}}(B\otimes C,A))\to\hat{0}(A))) \\ & \leq \bigwedge_{A\in\tau_{e_L}}(\hat{x}(e_L)\odot\hat{y}(e_y)\odot e_{\tau_{e_L}}(e_x\otimes e_y,A))\to\hat{z}(A))) \\ & \leq \hat{x}(e_x)\odot\hat{y}(e_y)\odot e_{\tau_{e_L}}(e_x\otimes e_y,e_x\otimes e_y)\to\hat{z}(e_x\otimes e_y) \\ & = (e_x\otimes e_y)(0) = \bigvee_{x',y'}e_L(x'\odot y',0)\odot e_x(x')\odot e_y(y') \\ & \leq \bigvee_{x',y'}e_L(x'\odot y',0)\odot e_L(x\odot y,x'\odot y') \\ & = e_L(x\odot y,0) = P(x,y,0).\end{array}$$

Thus $\hat{P}(\hat{x},\hat{y},\hat{0}) = P(x,y,0)$. So $(L,\wedge,\vee,\odot,\to,0,1,e_L)$ is a c-quantale.

4. Conclusion

In this paper, we prove two main theorems. Let $(X,\wedge,\vee,*,\nearrow,0,1)$ be a c-quantale with r-fuzzy preorder e_X and $\tau_{e_{\tau_{e_X}}} = \{\alpha\in L^{\tau_{e_X}} \mid \alpha(A)\odot e_{\tau_{e_X}}(A,B)\leq \alpha(B)\}$. Then a map $h:(X,\wedge,\vee,*,\nearrow,0,1,e_X)\to(\tau_{e_{\tau_{e_X}}},\vee,\otimes,\Rightarrow,0,1,e_{\tau_{e_{\tau_{e_X}}}})$ defined as $h(x)(A)=\hat{x}(A)=A(x)$ is a embedding map.

Let (X,e_X,P) be a cq-frame. A map $k:X\to\tau_{e_{\tau_{e_X}}}$ as $k(x)(A)=\hat{x}(A)=A(x)$. Then $k:(X,e_X,P)\to(\tau_{e_{\tau_{e_X}}},e_{\tau_{e_{\tau_{e_X}}}},\hat{P})$ defined as $k(x)(A)=\hat{x}(A)=A(x)$ is a frame embedding map.

From the above results, left continuous t-norms and complete residuated lattices (complete BL-algebra, complete MV-algebra, complete Boolean algebra) can be embedded in above space.

In the future, by using the concepts of embedding maps, we must find the the problems for information systems and decision rules on commutative quantales can be solved on the Alexandrov fuzzy topologies.

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Ju-mok Oh (jumokoh@gwnu.ac.kr)
Department of Mathematics, Faculty of Natural Science, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea

Yong Chan Kim (yck@gwnu.ac.kr)
Department of Mathematics, Faculty of Natural Science, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea