Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 22, No. 2, (October 2021) pp. 205–222
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2021.22.2.205

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Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 22, No. 2, October 2021

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Received 31 May 2021; Revised 11 July 2021; Accepted 8 August 2021

Abstract. The paper at first briefly delineates some properties of multisets. The notion of multisemigroups and left(right) multi-ideals in multiset framework are introduced and several properties are investigated. Relationships between multisemigroups and left(right) multi-ideals are discussed. Characterizations of left(right) multi-deals, and multi-ideals generated by multisets are also considered.

2020 AMS Classification: 20N25, 03E72

Keywords: Multisemigroup, Left (right) multi-ideal, Multi-ideal generated by a multiset.

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1. Introduction

The development of multiset theory has provided an avenue to generalize several basic notions of group theory and algebra in general. Multisets are extension of classical sets, which accommodate repeated elements unlike the classical sets that wholly exclude repetition of elements. For more details, we refer the readers to [1, 2, 3].

Semigroups play an important role in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. Generalization of semigroups owing to classical structures has been studied by many authors. Among others are the notion of left almost semigroups (LA-semigroups) introduced by Kazim and Naseeruddin [4]. The structure is also known as AG-groupoid and modular groupoid and has a variety of applications in topology, matrices, flock theory, finite mathematics and geometry. In 2013, Akram *et al.* [5] discussed some properties of (m, n)-ideals in a locally associative LA-semigroup. Gulistan *et al.* [6] introduced H_v-LA-semigroups and showed that every LA-semihypergroup is an H_v-LA-semigroup. Kudryavtseva and Mazorchuk [7] was motivated by the appearance of multivalued structures and proposed multisemigroups as an extension of semigroups. Forsberg [8] presented multisemigroup with multistructures. Very recently, Chinram *et al.* [9] presented necessary and sufficient conditions for elements in semigroups of partial transformations to be left or right magnifiers.

The algebraic extensions of a semigroup in non-classical structures have also been studied. Fuzzy semigroups were introduced by Kuroki [10], which is a generalization of classical semigroups. Extensive studies in this direction have been carried by several researchers (See [11] for more details). Shabir and Ali [12] defined soft semigroup and its substructures. In 2015, Khan *et al.* [13] defined the concept of generalized cubic subsemigroups (ideals) of a semigroup and investigated some of its related properties.

Motivated by semigroup theory, the present paper introduces multisemigroups depicting multiset perspective as different from the concept of multisemigroups discussed by [8] and [7] and investigates some properties analogous to semigroup concept.

2. Preliminaries

A non empty set X together with a binary associative operation "." is called a *semigroup*. A semigroups is said to be *commutative*, if $xy = yx$ for all $x, y \in X$. An element $1 \in X$ is called an *identity*, if for all $x \in X$, we have $1x = x1 = x$. A semigroup containing such an identity element is called a *monoid*. A monoid in which, for each $x \in X$ there exists a unique $x^{-1} \in X$ such that $xx^{-1} = x^{-1}x = 1$ is called a *group*.

Unlike a group, a semigroup does not necessarily contain an identity element. We denote the monoid obtained from the semigroup X by adjoining an identity element 1 by X^1. It is routine to verify that $X^1 = X \cup \{1\}$ is a monoid. Identity elements are necessarily unique.

A semigroup X is said to be $left$ $(right)$ $zero$, if $y \in X$ satisfies $yx = y\,(xy = y)$ for all $x \in X$. If X does not have a zero element, we may adjoin one and obtain a new semigroup $X^0 = X \cup \{0\}$ which satisfies $x0 = 0x = 0^2 = 0$ for all $x \in X$.

An *idempotent semigroup* is a system of elements closed under an associative multiplication such that, for every element x of the semigroup X, $x^2 = x$. One-sided identity and zero elements are idempotent.

A subset $S \subseteq X$ is called a *subsemigroup* of X, denoted $S \leq X$, if S forms a semigroup under the operation inherited from X. A non-empty subsemigroup S satisfying $xy \in S$ for all $x \in X$ and $y \in S$ is called a *left ideal*. A *right ideal* is a subsemigroup S satisfying $yx \in S$ for all $x \in X$ and $y \in S$. A subsemigroup which is both a left and right ideal is called a *two-sided ideal* or simply an *ideal* for short.

A semigroup X is said to be $left$ $(right)$ *cancellative*, provided that $xz = yz \Longrightarrow x = y\,(zx = zy \Longrightarrow x = y)$ for all $x, y, z \in X$.

If X is both left and right cancellative, then it is said to be *cancellative*.

3. Multiset properties

Definition 3.1 ([14])**.** A *multiset* $\mathcal{A}$ is a countable set X together with a function $C_{\mathcal{A}} : X \longrightarrow \mathbb{N}_{\geq 0} = \mathbb{N} \cup \{0\}$ that defines the count or multiplicity of the elements of X in $\mathcal{A}$. An expedient notation of $\mathcal{A}$ drawn from $X = \{x_1, .., x_n\}$ is $[x_1, .., x_n]_{C_{\mathcal{A}}(x_1),...,C_{\mathcal{A}}(x_n)}$ such that $C_{\mathcal{A}}(x_i)$ is the number of times x_i occurs in $\mathcal{A}$, $(i = 1, ..., n)$.

The customary set operations can be carried over to multisets. Let $\mathcal{A}$ and $\mathcal{B}$ be multisets over a semigroup X. Then

(i) $\mathcal{A} \sqsubseteq \mathcal{B} \Longleftrightarrow C_{\mathcal{A}}(x) \leq C_{\mathcal{B}}(x)\ \forall\ x \in X$.
(ii) $\mathcal{A} = \mathcal{B} \Longleftrightarrow C_{\mathcal{A}}(x) = C_{\mathcal{B}}(x)\ \forall\ x \in X$.
(iii) $\mathcal{A} \bigcup \mathcal{B} \Longleftrightarrow C_{\mathcal{A} \bigcup \mathcal{B}}(x) = C_{\mathcal{A}}(x) \bigvee C_{\mathcal{B}}(x)\ \forall\ x \in X$.
(iv) $\mathcal{A} \bigcap \mathcal{B} \Longleftrightarrow C_{\mathcal{A} \bigcap \mathcal{B}}(x) = C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{B}}(x)\ \forall\ x \in X$.

Definition 3.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two multisets over a semigroup X such that the count functions are $C_{\mathcal{A}} : X \longrightarrow \mathbb{N}_{\geq 0}$ and $C_{\mathcal{B}} : X \longrightarrow \mathbb{N}_{\geq 0}$ respectively. Then the *product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \circ \mathcal{B}$, is defined by: for all $x \in X$,

$$C_{\mathcal{A} \circ \mathcal{B}}(x) = \begin{cases} \bigvee_{x=yz}\{C_{\mathcal{A}}(y) \bigwedge C_{\mathcal{B}}(z)\}, & if\ \exists\ y, z \in X\ such\ that\ x = yz, \\ 0, & otherwise. \end{cases}$$

Following Definition 3.2 terminology, the m *times multiplication* of the multiset $\mathcal{A}$ can be defined as $\mathcal{A}^m = \mathcal{A} \circ \mathcal{A} \circ ... \circ \mathcal{A}$ and its count function is

$$C_{\mathcal{A}^m}(x) = \begin{cases} \bigvee\{\bigwedge_{i \in \{1,..,m\}} C_{\mathcal{A}}(x_i)\}, & if\ \exists\ x_i \in X\ such\ that\ \prod_{i=1}^m x_i = x, \\ 0, & otherwise. \end{cases}$$

We denote the set of all multisets over a semigroup X by $M(X)$.

Proposition 3.3. *Let $\mathcal{A}, \mathcal{B}_i \in M(X)$ and $i = 1, .., k$. Then*

(1) $\mathcal{A} \bigcup \left(\bigcap_{i=1}^k \mathcal{B}_i\right) = \bigcap_{i=1}^k (\mathcal{A} \bigcup \mathcal{B}_i)$,
(2) $\mathcal{A} \bigcap \left(\bigcup_{i=1}^k \mathcal{B}_i\right) = \bigcup_{i=1}^k (\mathcal{A} \bigcap \mathcal{B}_i)$,
(3) $\mathcal{A} \circ \left(\bigcup_{i=1}^k \mathcal{B}_i\right) = \bigcup_{i=1}^k (\mathcal{A} \circ \mathcal{B}_i)$,
(4) $\mathcal{A} \circ \left(\bigcap_{i=1}^k \mathcal{B}_i\right) \sqsubseteq \bigcap_{i=1}^k (\mathcal{A} \circ \mathcal{B}_i)$.

Proof. (1)-(2) immediate.

(3) Let $x \in X$. If $x \neq yz$, then $C_{\mathcal{A} \circ \left(\bigcup_{i=1}^k \mathcal{B}_i\right)}(x) = 0 = C_{\bigcup_{i=1}^k (\mathcal{A} \circ \mathcal{B}_i)}(x)$. Thus we have

$$\mathcal{A} \circ \left(\bigcup_{i=1}^k \mathcal{B}_i\right) = \bigcup_{i=1}^k (\mathcal{A} \circ \mathcal{B}_i).$$

If $x = yz$ for some $x, y \in X$, then

$$\begin{aligned}
C_{\mathcal{A}\circ\left(\bigcup_{i=1}^{k}\mathcal{B}_i\right)}(x) &= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\bigcup_{i=1}^{k}\mathcal{B}_i}(z)\} \\
&= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge\left(\bigvee_{i=1}^{k}C_{\mathcal{B}_i}(z)\right)\} \\
&= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge\left(C_{\mathcal{B}_1}(z)\bigvee\ldots\bigvee C_{\mathcal{B}_n}(z)\right)\} \\
&= \left(\bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\mathcal{B}_1}(z)\}\right)\bigvee\ldots\bigvee\left(\bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\mathcal{B}_n}(z)\}\right) \\
&= \bigvee_{i=1}^{k}C_{\mathcal{A}\circ\mathcal{B}_i}(x) = C_{\bigcup_{i=1}^{k}(\mathcal{A}\circ\mathcal{B}_i)}(x).
\end{aligned}$$

Thus $\mathcal{A}\circ\left(\bigcup_{i=1}^{k}\mathcal{B}_i\right) = \bigcup_{i=1}^{k}(\mathcal{A}\circ\mathcal{B}_i)$.

(4) Let $x \in X$. If $x \neq yz$ for any $y, z \in X$, then the result is obvious. Otherwise, there exist $y, z \in X$ such that $x = yz$. Thus

$$\begin{aligned}
C_{\mathcal{A}\circ\left(\bigcap_{i=1}^{k}\mathcal{B}\right)}(x) &= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\bigcap_{i=1}^{k}\mathcal{B}}(z)\} \\
&= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge\left(\bigwedge_{i=1}^{k}C_{\mathcal{B}_i}(z)\right)\} \\
&= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge\left(C_{\mathcal{B}_1}(z)\bigwedge\ldots\bigwedge C_{\mathcal{B}_n}(z)\right)\} \\
&\leq \left(\bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\mathcal{B}_1}(z)\}\right)\bigwedge\ldots\bigwedge\left(\bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\mathcal{B}_n}(z)\}\right) \\
&= \bigwedge_{i=1}^{k}C_{\mathcal{A}\circ\mathcal{B}_i} = C_{\bigcap_{i=1}^{k}(\mathcal{A}\circ\mathcal{B}_i)}(x).
\end{aligned}$$

So $\mathcal{A}\circ\left(\bigcap_{i=1}^{k}\mathcal{B}_i\right) \sqsubseteq \bigcap_{i=1}^{k}(\mathcal{A}\circ\mathcal{B}_i)$. □

Proposition 3.4. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M(X)$. If $\mathcal{A} \sqsubseteq \mathcal{B}$, then we get*

$$\mathcal{A}\circ\mathcal{C} \sqsubseteq \mathcal{B}\circ\mathcal{C} \text{ and } \mathcal{C}\circ\mathcal{A} \sqsubseteq \mathcal{C}\circ\mathcal{B}.$$

Proof. Let $x \in X$. If $x \neq yz$ for any $y, z \in X$, then $C_{\mathcal{A}\circ\mathcal{C}}(x) = 0 \leq C_{\mathcal{B}\circ\mathcal{C}}(x)$. Otherwise,

$$\begin{aligned}
C_{\mathcal{A}\circ\mathcal{B}}(x) &= \bigvee_{x=yz}\{C_{\mathcal{A}}(y)\bigwedge C_{\mathcal{C}}(z)\} \\
&\leq \bigvee_{x=yz}\{C_{\mathcal{B}}(y)\bigwedge C_{\mathcal{C}}(z)\} \quad [\textit{Since } C_{\mathcal{A}}(y) \leq C_{\mathcal{B}}(y)] \\
&= C_{\mathcal{B}\circ\mathcal{C}}(x).
\end{aligned}$$

□

Thus $\mathcal{A} \circ \mathcal{C} \sqsubseteq \mathcal{B} \circ \mathcal{C}$.

Similarly, we may prove $C_{\mathcal{C}\circ\mathcal{A}} \sqsubseteq C_{\mathcal{C}\circ\mathcal{B}}$.

Remark 3.5. If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M(X)$ such that $xy = y$ for every $x, y \in X$, then $\mathcal{A} \circ \mathcal{C} = \mathcal{B} \circ \mathcal{C} = \mathcal{C}$. Analogously, $\mathcal{C} \circ \mathcal{A} = \mathcal{C} \circ \mathcal{B} = \mathcal{C}$ is such that $xy = x$ for every $x, y \in X$.

Proposition 3.6. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M(X)$. Then $\mathcal{A} \circ \mathcal{C} = \mathcal{B} \circ \mathcal{C} \nRightarrow \mathcal{A} = \mathcal{B}$ and $\mathcal{C} \circ \mathcal{A} = \mathcal{C} \circ \mathcal{B} \nRightarrow \mathcal{A} = \mathcal{B}$.*

Proof. The collection of multisets over a semigroup X is not cancellative because there may exists $y \in X$ such that $C_{\mathcal{A}}(y) < C_{\mathcal{B}}(y)$. □

Proposition 3.7. *Let $\mathcal{A}, \mathcal{B} \in M(X)$. If $\mathcal{A} \sqsubseteq \mathcal{B}$, then $\mathcal{A}^m \sqsubseteq \mathcal{B}^m$.*

Proof. Let $x \in X$. Suppose $x \neq \prod_{i=1}^m x_i$. Since $\mathcal{A} \sqsubseteq \mathcal{B}$ implies $C_{\mathcal{A}}(x) \leq C_{\mathcal{B}}(x)$ for all $x \in X$, $C_{\mathcal{A}^m}(x) = 0 \leq C_{\mathcal{B}^m}(x)$. Suppose $x = \prod_{i=1}^m x_i$ for some $x_i \in X$. Then we have

$$\bigvee\{\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{A}}(x_i) \mid x = \prod_{i=1}^m x_i\} \leq \bigvee\{\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{B}}(x_i) \mid x = \prod_{i=1}^m x_i\}.$$

Thus $\mathcal{A}^m \sqsubseteq \mathcal{B}^m$. □

Definition 3.8. Let $\mathcal{A}$ and $\mathcal{B}$ be multisets over semigroups X_1 and X_2 respectively. The *Cartesian product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times \mathcal{B}$, is a function

$$C_{\mathcal{A}\times\mathcal{B}} : X_1 \times X_2 \longrightarrow \mathbb{N}_{\geq 0}$$

defined by

$$C_{\mathcal{A}\times\mathcal{B}}(x, y) = \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{B}}(y) \mid x \in X_1,\ y \in X_2\}.$$

Proposition 3.9. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in M(X)$. Then*

(1) $\mathcal{A} \times (\mathcal{B} \bigcup \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) \bigcup (\mathcal{A} \times \mathcal{C})$,
(2) $\mathcal{A} \times (\mathcal{B} \bigcap \mathcal{C}) = (\mathcal{A} \times \mathcal{B}) \bigcap (\mathcal{A} \times \mathcal{C})$.

Proof. Straightforward. □

Proposition 3.10. *Let $\mathcal{A} \in M(X_1)$ and $\mathcal{B} \in M(X_2)$. Then $(\mathcal{A} \times \mathcal{B})^m = \mathcal{A}^m \times \mathcal{B}^m$.*

Proof. Let $(x, y) \in X_1 \times X_2$. Suppose $(x, y) \neq \prod_{i=1}^m (x_i, y_i)$. Then we get

$$C_{(\mathcal{A}\times\mathcal{B})^m}(x, y) = 0 = C_{\mathcal{A}^m}(x, y) \times C_{\mathcal{B}^m}(x, y).$$

Thus $(\mathcal{A} \times \mathcal{B})^m = \mathcal{A}^m \times \mathcal{B}^m$. Suppose $(x, y) = \prod_{i=1}^m (x_i, y_i)$ for some $(x_i, y_i) \in X_1 \times X_2$. Then

$$\begin{aligned}
&C_{(\mathcal{A}\times\mathcal{B})^m}(x, y) \\
&= \bigvee\{\textstyle\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{A}\times\mathcal{B}}(x_i, y_i) \mid \prod_{i=1}^m (x_i, y_i) = (x, y),\ x_i \in X_1,\ y_i \in X_2\} \\
&= \bigvee\{\textstyle\bigwedge_{i\in\{1,\ldots,m\}} (C_{\mathcal{A}}(x_i) \bigwedge C_{\mathcal{B}}(y_i)) \mid \prod_{i=1}^m (x_i, y_i) = (x, y),\ x_i \in X_1,\ y_i \in X_2\} \\
&= \bigvee\{\Big(\textstyle\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{A}}(x_i)\Big) \bigwedge \Big(\textstyle\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{B}}(y_i)\Big) \mid \prod_{i=1}^m x_i = x, \\
&\qquad\qquad \textstyle\prod_{i=1}^m y_i = y,\ x_i \in X_1, y_i \in X_2\} \\
&= \bigvee\{\textstyle\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{A}}(x_i) \mid x_i \in X_1,\ \prod_{i=1}^m x_i = x\}
\end{aligned}$$

$\bigwedge \bigvee\{\bigwedge_{i\in\{1,\ldots,m\}} C_{\mathcal{B}}(y_i) \mid y_i \in X_2,\ \prod_{i=1}^{m} y_i = y\}$
$= C_{\mathcal{A}^m}(x) \bigwedge C_{\mathcal{B}^m}(y)$
$= C_{\mathcal{A}^m\times\mathcal{B}^m}(x, y).$
Thus $(\mathcal{A}\times\mathcal{B})^m = \mathcal{A}^m \times \mathcal{B}^m$. □

Definition 3.11. Let $\mathcal{A}$ be a multiset over a set X and $n \in \mathbb{Z}^+$. Then the sets

$$\mathcal{A}_n = \{x \in X \mid C_{\mathcal{A}}(x) \geq n\} \text{ and } \mathcal{A}_n^{>} = \{x \in X \mid C_{\mathcal{A}}(x) > n\}$$

are called *n-level sets* and *strong n-level sets* of $\mathcal{A}$.

Clearly for $n \geq C_{\mathcal{A}}(x)$, $\mathcal{A}_n^{>}$ is always empty.

Proposition 3.12. *Let $A, B \in M(X)$ and $n \in \mathbb{Z}^+$. Then $(\mathcal{A}\circ\mathcal{B})_n^{>} = \mathcal{A}_n^{>} \cdot \mathcal{B}_n^{>}$ for every $n < C_{\mathcal{A}}(x)$.*

Proof. Let $x \in (\mathcal{A}\circ\mathcal{B})_n^{>}$

$\iff C_{\mathcal{A}\circ\mathcal{B}}(x) > n$

$\iff \bigvee_{x=ab} (C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{B}}(b)) > n$

$\iff C_{\mathcal{A}}(a_o) \bigwedge C_{\mathcal{B}}(b_o) > n$ for some $a_o, b_o \in X$ such that $x = a_o b_o$

$\iff C_{\mathcal{A}}(a_o) > n$ and $C_{\mathcal{B}}(b_o) > n$

$\iff a_o \in \mathcal{A}_n^{>}$ and $b_o \in \mathcal{B}_n^{>}$

$\iff x = a_o b_o \in \mathcal{A}_n^{>}.\ \mathcal{B}_n^{>}.$ □

4. Multisemigroup and ideals

Definition 4.1. Let a map $\cdot : X \times X \longrightarrow X$ be a composition law such that $(X, \cdot)$ forms a semigroup. A multiset $\mathcal{A}$ constructed from X is called a *multisemigroup*, if

$$C_{\mathcal{A}}(ab) = C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{A}}(b)\ \forall\ a, b \in X.$$

Particularly, if $C_{\mathcal{A}}(x) = 1,\ \forall\, x \in X$, such a multisemigroup is called a *semigroup*. Undeniably, every semigroup is a multisemigroup in an obvious manner, however, not every multisemigroup is a semigroup. The set of all multisemigroups over a semigroup X is denoted by $MS(X)$.

Example 4.2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a semigroup with the multiplication (See the below table). Let $t_i \in \mathbb{N}_{\geq 0},\ 0 \leq i \leq 2$ be such that $t_0 > t_1 > t_2$. Define a multiset

$$C_{\mathcal{A}} : X \longrightarrow \mathbb{N}_{\geq 0} \text{ as follows :}$$
$$C_{\mathcal{A}}(0) = t_0,\ C_{\mathcal{A}}(1) = C_{\mathcal{A}}(5) = t_1\ , C_{\mathcal{A}}(2) = C_{\mathcal{A}}(3) = C_{\mathcal{A}}(4) = t_2.$$

Then clearly, $\mathcal{A}$ is a multisemigroup of X. However, if $C_{\mathcal{A}}(5) < t_2$, then $\mathcal{A}$ is not a multisemigroup of X.

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Theorem 4.3. *Let $\mathcal{A} \in M(X)$. Then $\mathcal{A} \in MS(X)$ if and only if $\mathcal{A} \circ \mathcal{A} \sqsubseteq \mathcal{A}$.*

Proof. Let $\mathcal{A} \in MS(X)$ and $a \in X$. If $a \neq xy$ for any $x, y \in X$, then $C_{\mathcal{A}\circ\mathcal{A}}(a) = 0 \leq C_{\mathcal{A}}(a)$. If such exists, let $a = xy$ for some $x, y \in X$. Then

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{A}}(a) &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{A}}(y)\} \\ &\leq \bigvee_{a=xy} \{C_{\mathcal{A}}(xy)\} \\ &= \bigvee \{C_{\mathcal{A}}(a)\} = C_{\mathcal{A}}(a). \end{aligned}$$

Thus $\mathcal{A} \circ \mathcal{A} \sqsubseteq \mathcal{A}$.

Conversely, let $\mathcal{A} \circ \mathcal{A} \sqsubseteq \mathcal{A}$ and $x, y \in X$. Then $xy \in X$. Let $a = xy$. Then

$$\begin{aligned} C_{\mathcal{A}}(xy) = C_{\mathcal{A}}(a) &\geq C_{\mathcal{A}\circ\mathcal{A}}(a) \\ &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{A}}(y)\} \\ &\geq C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{A}}(y). \end{aligned}$$

Thus $\mathcal{A} \in MS(X)$. □

Theorem 4.4. *Let X be a semigroup with identity e. If $\mathcal{A} \in MS(X)$ such that $C_{\mathcal{A}}(e) \geq C_{\mathcal{A}}(a)$ for all $a \in X$, then $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$.*

Proof. For any $a \in X$, we have

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{A}}(a) &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{A}}(y)\} \\ &= \bigvee_{a=ae} \{C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{A}}(e)\} \\ &\geq C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{A}}(e) \\ &= C_{\mathcal{A}}(a). \end{aligned}$$

□

This shows that $\mathcal{A} \sqsubseteq \mathcal{A} \circ \mathcal{A}$. Since $\mathcal{A} \circ \mathcal{A} \sqsubseteq \mathcal{A}$ by Theorem 4.1, we have that $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$.

Remark 4.5. It follows from Definition 3.2 and Theorem 4.3 that $\mathcal{A}^m = \mathcal{A}$ $\forall\ m \in \mathbb{Z}^+$.

Theorem 4.6. *Let $\mathcal{A}, \mathcal{B} \in MS(X)$. Then $\mathcal{A} \bigcap \mathcal{B} \in MS(X)$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in MS(X)$ and $x, y \in X$. Then

$$\begin{aligned} C_{\mathcal{A}\bigcap\mathcal{B}}(xy) &= C_{\mathcal{A}}(xy) \bigwedge C_{\mathcal{B}}(xy) \\ &\geq \Big[C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{A}}(y)\Big] \bigwedge \Big[C_{\mathcal{B}}(x) \bigwedge C_{\mathcal{B}}(y)\Big] \\ &= \Big[C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{B}}(x)\Big] \bigwedge \Big[C_{\mathcal{A}}(y) \bigwedge C_{\mathcal{B}}(y)\Big] \\ &= C_{\mathcal{A}\bigcap\mathcal{B}}(x) \wedge C_{\mathcal{A}\bigcap\mathcal{B}}(y). \end{aligned}$$

Thus $\mathcal{A} \bigcap \mathcal{B} \in MS(X)$. □

Remark 4.7. Let $\{\mathcal{A}_i : i \in I\}$ be a non-empty family of multisemigroups over a semigroup X. Then $\bigcap_{i\in I} \mathcal{A}_i$ is a multisemigroup over X.

The union of any two multisemigroups may not be a multisemigroup as is shown in the following.

Example 4.8. By Example 4.1, let $\mathcal{A} = [1, 2, 4]_{3,2,2}$ and $\mathcal{B} = [1, 3]_{2,1}$. Then $\mathcal{A} \bigcup \mathcal{B} = [1, 2, 3, 4]_{3,2,1,2}$. Clearly, $C_{\mathcal{A}\bigcup\mathcal{B}}(4.3) = C_{\mathcal{A}\bigcup\mathcal{B}}(5) = 0 \ngeq 1 = C_{\mathcal{A}\bigcup\mathcal{B}}(4) \wedge C_{\mathcal{A}\bigcup\mathcal{B}}(3)$.

Proposition 4.9. *Let X be a left zero semigroup. If $\mathcal{A} \in MS(X)$ such that $C_{\mathcal{A}}(x) > C_{\mathcal{A}}(y)$ and also interchangeably satisfies the inequality, then $\mathcal{A}$ is a constant function.*

Proof. Let $x, y \in X$. Then $xy = x$ and $yx = y$. Thus

$$\begin{aligned} C_{\mathcal{A}}(x) &= C_{\mathcal{A}}(xy) \\ &\geq C_{\mathcal{A}}(y) \qquad (C_{\mathcal{A}}(x) > C_{\mathcal{A}}(y)) \\ &= C_{\mathcal{A}}(yx) \\ &\geq C_{\mathcal{A}}(x). \qquad (C_{\mathcal{A}}(y) > C_{\mathcal{A}}(x)) \end{aligned}$$

So $C_{\mathcal{A}}(x) = C_{\mathcal{A}}(y)$ for all $x, y \in X$. Hence the proof is complete.

Similarly, we can prove for right zero semigroup. □

Remark 4.10. If $\mathcal{A} \in MS(X)$ with a fixed element $a \in X$ and setting $xy = a$ for all $x, y \in X$, then $C_{\mathcal{A}}(xy) = C_{\mathcal{A}}(yx)$ for all $x, y \in X$.

Definition 4.11. Let X be a semigroup with identity e and $\mathcal{A} \in MS(X)$. Then the subsemigroup $\mathcal{A}_e$ is a constant function defined as follows:

$$\mathcal{A}_e = \{x \in X \mid C_{\mathcal{A}}(x) = C_{\mathcal{A}}(e)\}.$$

Proposition 4.12. *Let X be a semigroup with identity e and $\mathcal{A}, \mathcal{B} \in MS(X)$. Then $\mathcal{A}_e \cap \mathcal{B}_e \subseteq (\mathcal{A} \cap \mathcal{B})_e$.*

Proof. Let $x \in \mathcal{A}_e \cap \mathcal{B}_e$. Then $x \in \mathcal{A}_e$ and $x \in \mathcal{B}_e$.
$\Longrightarrow C_{\mathcal{A}}(x) = C_{\mathcal{A}}(e)\ \forall\ x \in X$ and $C_{\mathcal{B}}(x) = C_{\mathcal{B}}(e)\ \forall\ x \in X$
$\Longrightarrow C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{B}}(x) = C_{\mathcal{A}}(e) \bigwedge C_{\mathcal{B}}(e)\ \forall\ x \in X$
$\Longrightarrow C_{\mathcal{A}\cap\mathcal{B}}(x) = C_{\mathcal{A}\cap\mathcal{B}}(e)\ \forall\ x \in X$
$\Longrightarrow x \in (\mathcal{A} \cap \mathcal{B})_e$.
Thus $\mathcal{A}_e \cap \mathcal{B}_e \subseteq (\mathcal{A} \cap \mathcal{B})_e$. □

Theorem 4.13. *Let $\mathcal{A} \in M(X)$. Then $\mathcal{A} \in MS(X)$ if and only if every $\mathcal{A}_n \neq \varnothing$ is a subsemigroup of X.*

Proof. Let $\mathcal{A} \in MS(X)$ and $x, y \in \mathcal{A}_n$ $\forall$ $n \in \mathbb{Z}^+$. Then $C_\mathcal{A}(x) \geq n$ and $C_\mathcal{A}(y) \geq n$. It follows from Definition 4.1 that $C_\mathcal{A}(xy) \geq C_\mathcal{A}(x) \bigwedge C_\mathcal{A}(y) \geq n$. Thus $xy \in \mathcal{A}_n$. So $\mathcal{A}_n$ is a subsemigroup of X.

Conversely, let $n \in \mathbb{Z}^+$ be such that $\mathcal{A}_n \neq \varnothing$ and $\mathcal{A}_n$ is a subsemigroup of X. Assume that $C_\mathcal{A}(xy) \ngeq C_\mathcal{A}(x) \bigwedge C_\mathcal{A}(y)$. Then there exist $n_0 \in \mathbb{Z}^+$ such that $C_\mathcal{A}(xy) < n_0 \leq C_\mathcal{A}(x) \bigwedge C_\mathcal{A}(y)$ for some $n_0 \in \mathbb{Z}^+$ implies $x, y \in \mathcal{A}_{n_0}$ but $xy \notin \mathcal{A}_{n_0}$. This is a contradiction. Thus the proof is complete. □

Proposition 4.14. *Let $\mathcal{A} \in MS(X_1)$ and $\mathcal{B} \in MS(X_2)$. Then $\mathcal{A} \times \mathcal{B} \in MS(X_1 \times X_2)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$. Then

$$\begin{aligned} C_{\mathcal{A}\times\mathcal{B}}((x_1, y_1), (x_2, y_2)) &= C_{\mathcal{A}\times\mathcal{B}}(x_1x_2, y_1y_2) \\ &= C_\mathcal{A}(x_1x_2) \bigwedge C_\mathcal{B}(y_1y_2) \\ &\geq \Big(C_\mathcal{A}(x_1) \bigwedge C_\mathcal{A}(x_2)\Big) \bigwedge \Big(C_\mathcal{B}(y_1) \bigwedge C_\mathcal{B}(y_2)\Big) \\ &= \Big(C_\mathcal{A}(x_1) \bigwedge C_\mathcal{B}(y_1)\Big) \bigwedge \Big(C_\mathcal{A}(x_2) \bigwedge C_\mathcal{B}(y_2)\Big) \\ &= C_{\mathcal{A}\times\mathcal{B}}(x_1, y_1) \bigwedge C_{\mathcal{A}\times\mathcal{B}}(x_2, y_2). \end{aligned}$$

Thus $\mathcal{A} \times \mathcal{B} \in MS(X_1 \times X_2)$. □

Remark 4.15. If $\mathcal{A}_1, ..., \mathcal{A}_k$ are multisemigroups over $X_1, ..., X_k$ respectively, then $\mathcal{A}_1 \times ... \times \mathcal{A}_k$ is a multisemigroups over $X_1 \times ... \times X_k$.

Theorem 4.16. *Let $\mathcal{A} \in MS(X_1)$ and $\mathcal{B} \in MS(X_2)$. If $\mathcal{A} = \mathcal{B}$, then $\mathcal{A}\times\mathcal{B} = \mathcal{B}\times\mathcal{A}$.*

Proof. Suppose $\mathcal{A} = \mathcal{B}$ and $x, y \in X_1$. Then

$$\begin{aligned} C_{\mathcal{A}\times\mathcal{B}}(x, y) &= C_\mathcal{A}(x) \bigwedge C_\mathcal{B}(y) \\ &= C_\mathcal{B}(x) \bigwedge C_\mathcal{A}(y) \\ &= C_{\mathcal{B}\times\mathcal{A}}(x, y). \end{aligned}$$

Thus $\mathcal{A} \times \mathcal{B} = \mathcal{B} \times \mathcal{A}$.

□

However, the converse problem above does not hold. For example, let

$$C_{\mathcal{A}\times\mathcal{B}}(x, y) = \{C_\mathcal{A}(x) \bigwedge C_\mathcal{B}(y) \mid x \in X_1,\ y \in X_2\}$$

and

$$C_{\mathcal{B}\times\mathcal{A}}(y, x) = \{C_\mathcal{B}(y) \bigwedge C_\mathcal{A}(x) \mid x \in X_1,\ y \in X_2\}.$$

Then

$$C_{\mathcal{A}\times\mathcal{B}}(x, y) = C_{\mathcal{B}\times\mathcal{A}}(y, x).$$

Thus we obtain $\mathcal{A} \times \mathcal{B} = \mathcal{B} \times \mathcal{A}$ but $\mathcal{A} \neq \mathcal{B}$, if $x \neq y$.

Definition 4.17. Let X be a semigroup and $\mathcal{A}$ be a multisemigroup over X. A submultisemigroup $\mathcal{J}$ of $\mathcal{A}$ is called a *left multi-ideal* of $\mathcal{A}$, if $C_{\mathcal{J}}(ab) \geq C_{\mathcal{J}}(b)\ \forall\, a, b \in X$. Analogously, $\mathcal{J}$ is called a *right multi-ideal* of $\mathcal{A}$, if $C_{\mathcal{J}}(ab) \geq C_{\mathcal{J}}(a)$.

Equivalently, a submultisemigroup $\mathcal{J}$ of $\mathcal{A}$ is called a $left$ ($right$) *multi-ideal*, if $\mathcal{A} \circ \mathcal{J} \sqsubseteq \mathcal{J}$ ($\mathcal{J} \circ \mathcal{A} \sqsubseteq \mathcal{J}$).

A submultisemigroup $\mathcal{J}$ of $\mathcal{A}$ is called a *multi-ideal*, if it is a left and a right multi-ideal of $\mathcal{A}$.

Remark 4.18.
1. The union of any collection of left(right) multi-ideals of $\mathcal{A}$ is a left(right) multi-ideal of $\mathcal{A}$.
2. The product of two left(right) multi-ideals of $\mathcal{A}$ is a left(right) multi-ideal of $\mathcal{A}$.
3. The intersection of any collection of left(right) multi-ideals of $\mathcal{A}$ is also a left(right) multi-ideal of $\mathcal{A}$.
4. If $\mathcal{J}$ and $\mathcal{K}$ are two left(right) multi-ideals of $\mathcal{A}$, then $\mathcal{J} \cap \mathcal{K}$ is a left(right) multi-ideal of $\mathcal{A}$.
5. If X is a right zero semigroup, then $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$. However, $\mathcal{J}$ is a right multi-ideal of $\mathcal{A}$ if X is a left zero semigroup.
6. The intersection of a left multi-ideal and a right multi-ideal of $\mathcal{A}$ need not be a multi-ideal of $\mathcal{A}$.

Example 4.19. Let $X = \{1, 2, 3, 4, 5\}$ be a semigroup with the following multiplication table below:

.	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	3	3	5
4	1	1	3	4	5
5	1	1	3	3	5

Let $\mathcal{A} = [1, 2, 3, 4, 5]_{5,3,2,1,4}$ and $\mathcal{J} = [1, 3, 4, 5]_{4,2,1,3}$ imply that

$$C_{\mathcal{J}}(1) = 4,\ C_{\mathcal{J}}(2) = 0,\ C_{\mathcal{J}}(3) = 2,\ C_{\mathcal{J}}(4) = 1,\ C_{\mathcal{J}}(5) = 3.$$

Then it is easy to verify that $\mathcal{J}$ is a multi ideal of $\mathcal{A}$.

Proposition 4.20. *Let X be a semigroup. Then every left(right) multi-ideal is a multisemigroup.*

Proof. Let $\mathcal{J}$ be a left(right) multi-ideal of $\mathcal{A}$. Since $\mathcal{J} \sqsubset \mathcal{A}$, we get

$$\mathcal{J} \circ \mathcal{J} \sqsubseteq \mathcal{A} \circ \mathcal{J}\, (\mathcal{J} \circ \mathcal{A}).$$

Then $\mathcal{J} \circ \mathcal{J} \sqsubseteq \mathcal{A} \circ \mathcal{J}\, (\mathcal{J} \circ \mathcal{A}) \sqsubseteq \mathcal{J}$. Thus $\mathcal{J} \circ \mathcal{J} \sqsubseteq \mathcal{J}$. □

The converse of the preceding Proposition may not be true in general. For example, let $X = \{\varepsilon, \alpha, \beta, \gamma\}$ be a semigroup with the following multiplication table below:

Let $\mathcal{J} = [\varepsilon, \alpha, \beta, \gamma]_{3,2,3,1}$ and let $\mathcal{A} = [\varepsilon, \alpha, \beta, \gamma]_{4,2,3,2}$. Then it is easy to verify that $\mathcal{J}$ is a multisemigroup over X but it is not a left multi-ideal of $\mathcal{A}$, since $C_{\mathcal{J}}(\gamma\beta) = C_{\mathcal{J}}(\alpha) = 2 \ngeq 3 = C_{\mathcal{J}}(\beta)$.

$\cdot$	ε	α	β	γ
ε	ε	ε	ε	ε
α	ε	ε	ε	ε
β	ε	ε	ε	α
γ	ε	ε	α	β

Proposition 4.21. *Let X be a semigroup. Then $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$ if and only if $\mathcal{A} \circ \mathcal{J} \sqsubseteq \mathcal{J}$.*

Proof. Let $\mathcal{J}$ be a left multi-ideal of $\mathcal{A}$ and $a \in X$. If $a \neq xy$ for any $x, y \in X$, then it is obvious that $\mathcal{A} \circ \mathcal{J} \sqsubseteq \mathcal{J}$. Suppose $a = xy$ for some $x, y \in X$. Then

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{J}}(a) &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}}(y)\} \\ &= \bigvee_{a=xy} \{C_{\mathcal{J}}(y)\} \\ &\leq \bigvee_{a=xy} \{C_{\mathcal{J}}(xy)\} \\ &= \bigvee \{C_{\mathcal{J}}(a)\} = C_{\mathcal{J}}(a). \end{aligned}$$

Thus $\mathcal{A} \circ \mathcal{J} \sqsubseteq \mathcal{J}$.

Conversely, let $\mathcal{A} \circ \mathcal{J} \sqsubseteq \mathcal{J}$ and $x, y \in X$. Then $xy \in X$. Let $a = xy$. Then

$$\begin{aligned} C_{\mathcal{J}}(xy) = C_{\mathcal{J}}(a) &\geq C_{\mathcal{A}\circ\mathcal{J}}(a) \\ &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}}(y)\} \\ &\geq C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}}(y) \\ &= C_{\mathcal{J}}(y). \end{aligned}$$

Thus $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$.

Similarly, we can prove for right multi-ideal of $\mathcal{A}$. □

Remark 4.22. Let X be a commutative semigroup. If $\mathcal{J}$ is a submultisemigroup of $\mathcal{A}$, then $\mathcal{A} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{A} \sqsubseteq \mathcal{J}$.

Proposition 4.23. *Let X be a semigroup. If $\mathcal{J}$ and $\mathcal{K}$ are two left(right) multi-ideals of $\mathcal{A}$, then $\mathcal{J} \times \mathcal{K}$ is a left (right) multi-ideal of $\mathcal{A} \times \mathcal{A}$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\begin{aligned} C_{\mathcal{J}\times\mathcal{K}}((x_1, x_2), (y_1, y_2)) &= C_{\mathcal{J}\times\mathcal{K}}(x_1y_1, x_2y_2) \\ &= C_{\mathcal{J}\times\mathcal{K}}(x_1y_1) \bigwedge C_{\mathcal{J}\times\mathcal{K}}(x_2y_2) \\ &\geq C_{\mathcal{J}}(y_1) \bigwedge C_{\mathcal{K}}(y_2) \\ &= C_{\mathcal{J}\times\mathcal{K}}(y_1, y_2). \end{aligned}$$

Similarly,

$$\begin{aligned} C_{\mathcal{J}\times\mathcal{K}}((x_1,x_2),(y_1,y_2)) &= C_{\mathcal{J}\times\mathcal{K}}(x_1y_1,x_2y_2) \\ &= C_{\mathcal{J}\times\mathcal{K}}(x_1y_1)\bigwedge C_{\mathcal{J}\times\mathcal{K}}(x_2y_2) \\ &\geq C_{\mathcal{J}}(x_1)\bigwedge C_{\mathcal{K}}(x_2) \\ &= C_{\mathcal{J}\times\mathcal{K}}(x_1,x_2). \end{aligned}$$

□

Proposition 4.24. *Let X be a semigroup. Then $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$ if and only if $\mathcal{J}\times\mathcal{J}$ is a left multi-ideal of $\mathcal{A}\times\mathcal{A}$.*

Proof. Let $\mathcal{J}$ be a submultisemigroup of $\mathcal{A}$. If $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$, then by Proposition 4.23, $\mathcal{J}\times\mathcal{J}$ is a left multi-ideal of $\mathcal{A}\times\mathcal{A}$.

Conversely, suppose $\mathcal{J}\times\mathcal{J}$ is a left multi-ideal of $\mathcal{A}\times\mathcal{A}$ and $x_1,x_2,y_1,y_2\in X$. Then

$$\begin{aligned} C_{\mathcal{J}}(x_1y_1)\bigwedge C_{\mathcal{J}}(x_2y_2) &= C_{\mathcal{J}\times\mathcal{J}}(x_1y_1,x_2y_2) \\ &= C_{\mathcal{J}\times\mathcal{J}}((x_1,x_2),(y_1,y_2)) \\ &\geq C_{\mathcal{J}}(y_1,y_2) \\ &= C_{\mathcal{J}}(y_1)\bigwedge C_{\mathcal{J}}(y_2). \end{aligned}$$

Now, setting $x_1=x$, $x_2=a$, $y_1=y$ and $y_2=a$ such that $aa=a$ in the above inequality and noticing that $C_{\mathcal{J}}(a)\geq C_{\mathcal{J}}(x)\ \forall\ x\in X$, we have $C_{\mathcal{J}}(xy)\geq C_{\mathcal{J}}(y)$. Thus $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$. □

Similarly, we can prove for right multi-ideal of $\mathcal{A}$.

Proposition 4.25. *Let X be a left zero semigroup. If $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$, then $C_{\mathcal{J}}(x)=C_{\mathcal{J}}(y)$ for all $x,y\in X$.*

Proof. Let $x,y\in X$. Then $xy=x$ and $yx=y$. Thus

$$\begin{aligned} C_{\mathcal{J}}(x) &= C_{\mathcal{J}}(xy) \\ &\geq C_{\mathcal{J}}(y) \\ &= C_{\mathcal{J}}(yx) \\ &\geq C_{\mathcal{J}}(x). \end{aligned}$$

So $C_{\mathcal{J}}(x)=C_{\mathcal{J}}(y)$ for all $x,y\in X$.

□

Similarly, we can prove for right multi-ideal of $\mathcal{A}$ over a right zero semigroup.

Proposition 4.26. *Let X be a semigroup and $E(X)$ be the set of all idempotent elements of X such that $ab=a$ and $ba=b$. If $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$, then $C_{\mathcal{J}}(a)=C_{\mathcal{J}}(b)$ for all $a,b\in E(X)$.*

Proof. Suppose $a, b \in E(X)$. Then $ab = a$ and $ba = b$. Thus

$$\begin{aligned} C_{\mathcal{J}}(a) &= C_{\mathcal{J}}(ab) \\ &\geq C_{\mathcal{J}}(b) \\ &= C_{\mathcal{J}}(ba) \\ &\geq C_{\mathcal{J}}(a). \end{aligned}$$

So $C_{\mathcal{J}}(a) = C_{\mathcal{J}}(b)$. □

Similarly, we can prove for right multi-ideal of $\mathcal{A}$.

Proposition 4.27. *Let X be a semigroup. If $\mathcal{J}$ is a right multi-ideal and $\mathcal{K}$ is a left multi-ideal of $\mathcal{A}$, then $\mathcal{J} \circ \mathcal{K} \sqsubseteq \mathcal{J} \cap \mathcal{K}$.*

Proof. Let $\mathcal{J}$ and $\mathcal{K}$ be a right multi-ideal and a left multi-ideal of $\mathcal{A}$, and $a \in X$. If $a \neq xy$ for any $x, y \in X$, then $C_{\mathcal{J} \circ \mathcal{K}}(a) = 0 \leq C_{\mathcal{J} \cap \mathcal{K}}(a)$. Suppose $a = xy$ for some $x, y \in X$. Then

$$\begin{aligned} C_{\mathcal{J} \circ \mathcal{K}}(a) &= \bigvee_{a=xy} \{C_{\mathcal{J}}(x) \bigwedge C_{\mathcal{K}}(y)\} \\ &\leq \bigvee_{a=xy} \{C_{\mathcal{J}}(xy) \bigwedge C_{\mathcal{K}}(xy)\} \\ &= C_{\mathcal{J}}(a) \bigwedge C_{\mathcal{K}}(a) \\ &= C_{\mathcal{J} \cap \mathcal{K}}(a). \end{aligned}$$

Thus $\mathcal{J} \circ \mathcal{K} \sqsubseteq \mathcal{J} \cap \mathcal{K}$. □

Proposition 4.28. *Let X be a semigroup. If $\mathcal{J}$ is a multi-ideal of $\mathcal{A}$ and $\mathcal{B}$ is a submultisemigroup of $\mathcal{A}$, then $\mathcal{B} \cap (\mathcal{A} \circ \mathcal{J})$ $(\mathcal{B} \cap (\mathcal{J} \circ \mathcal{A}))$ is a multi-ideal of multi-semigroup $\mathcal{B}$.*

Proof. Suppose $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$ and $\mathcal{B} \sqsubseteq \mathcal{A}$. Then we have

$$\mathcal{B} \circ (\mathcal{B} \cap (\mathcal{A} \circ \mathcal{J})) = (\mathcal{B} \circ \mathcal{B}) \cap (\mathcal{B} \circ (\mathcal{A} \circ \mathcal{J})) \sqsubseteq \mathcal{B} \cap (\mathcal{A} \circ \mathcal{J}).$$

Thus $\mathcal{B} \cap (\mathcal{A} \circ \mathcal{J})$ is a left multi-ideal of $\mathcal{B}$. Also, we get

$$(\mathcal{B} \cap (\mathcal{J} \circ \mathcal{A})) \circ \mathcal{B} = (\mathcal{B} \circ \mathcal{B}) \cap ((\mathcal{J} \circ \mathcal{A}) \circ \mathcal{B}) \sqsubseteq \mathcal{B} \cap (\mathcal{J} \circ \mathcal{A}).$$

So $\mathcal{B} \cap (\mathcal{J} \circ \mathcal{A})$ is a right multi-ideal of $\mathcal{B}$. □

Proposition 4.29. *Let X be a semigroup. If $\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$, then $C_{\mathcal{J}}(a^n) \leq C_{\mathcal{J}}(a^{1+n})$, $\forall\ n \in \mathbb{Z}^+$.*

Proof. For any $n \in \mathbb{Z}^+$, we have

$$\begin{aligned} C_{\mathcal{J}}(a^{1+n}) &\geq C_{\mathcal{A} \circ \mathcal{J}}(a^{1+n}) \\ &= \bigvee_{a^{1+n}=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}}(y)\} \\ &\geq C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{J}}(a^n) \\ &= C_{\mathcal{J}}(a^n). \end{aligned}$$

Then the result holds. □

Similarly, we can prove for right multi-ideal of $\mathcal{A}$.

Proposition 4.30. *Let X be a semigroup. If $\mathcal{J}$ is a left(right) multi-ideal of $\mathcal{A}$, then every non-empty $\mathcal{J}_n$ of $\mathcal{J}$ is a left(right) ideal of X.*

Proof. Suppose $\mathcal{J}_n \neq \varnothing$. Let $a, b \in \mathcal{J}_n\ \forall\ n \in \mathbb{Z}^+$. Then $C_{\mathcal{J}}(a) \geq n$ and $C_{\mathcal{J}}(b) \geq n$ imply $C_{\mathcal{J}}(ab) \geq C_{\mathcal{J}}(a) \bigwedge C_{\mathcal{J}}(b) \geq n$. Thus $ab \in \mathcal{J}_n$. So $\mathcal{J}_n$ is a subsemigroup of X. Now, let $x \in X$ and $a \in \mathcal{J}_n\ \forall\ n \in \mathbb{Z}^+$. Then $C_{\mathcal{J}}(xa) \geq C_{\mathcal{J}}(a) \geq n$ $(C_{\mathcal{J}}(ax) \geq C_{\mathcal{J}}(a) \geq n)$. Thus $\mathcal{J}_n$ is a left (right) ideal of X. □

Proposition 4.31. *Let X be a semigroup. If $\mathcal{J}$ is a left(right) multi-ideal of $\mathcal{A}$, then every non-empty $\mathcal{J}_n^>$ of $\mathcal{J}$ is a left(right) ideal of X.*

Proof. Assume that $\mathcal{J}$ is left multi-ideal of $\mathcal{A}$. Let $\mathcal{J}_n^> \neq \varnothing$ be strong n-level sets of $\mathcal{J}$. We show that $\mathcal{J}_n^>$ is a left ideal of X. Suppose, if possible, $\mathcal{J}_n^>$ is not a left ideal of X. Then $X \cdot \mathcal{J}_n^> \nsubseteq \mathcal{J}_n^>$. This implies that there exists $z \in X \cdot \mathcal{J}_n^>$ but $z \notin \mathcal{J}_n^>$. Thus let $z = xy_0$ for some $y_0 \in \mathcal{J}_n^>$ and $x \in X$. Since $y_0 \in \mathcal{J}_n^>$, $C_{\mathcal{J}}(y_0) > n$. So $C_{\mathcal{J}}(z) = C_{\mathcal{J}}(xy_0) \geq C_{\mathcal{J}}(y_0) > n$. Hence, $z \in \mathcal{J}_n^>$, which is a contradiction. This shows that $\mathcal{J}_n^>$ is a left ideal of X. □

Remark 4.32. The non-empty $\mathcal{J}_n$, $\mathcal{J}_n^>$ of $\mathcal{J}$ may not necessarily be an ideal of X.

Proposition 4.33. *Let X be a semigroup. Suppose $\mathcal{J}$ is a left(right) multi-ideal of $\mathcal{A}$. Then two n-level left(right) ideals $\mathcal{J}_{n_1}, \mathcal{J}_{n_2}$ of $\mathcal{J}$ with $n_1 < n_2$ are equal if and only if there is no $x \in X$ such that $n_1 \leq C_{\mathcal{J}}(x) < n_2$.*

Proof. Assume that $\mathcal{J}_{n_1} = \mathcal{J}_{n_2}$ for $n_1 < n_2$ and if there exists $x \in X$ such that $n_1 \leq C_{\mathcal{J}}(x) < n_2$, then $\mathcal{J}_{n_2} \subset \mathcal{J}_{n_1}$. This is a contradiction.

Conversely, suppose there is no $x \in X$ such that $n_1 \leq C_{\mathcal{J}}(x) < n_2$. We have that $n_1 < n_2$ implies $\mathcal{J}_{n_2} \subseteq \mathcal{J}_{n_1}$. If $x \in \mathcal{J}_{n_1}$, then $C_{\mathcal{J}}(x) \geq n_1$. Since $C_{\mathcal{J}}(x) \nless n_2$, we have $C_{\mathcal{J}}(x) \geq n_2$ or $x \in \mathcal{J}_{n_2}$. Thus $\mathcal{J}_{n_1} = \mathcal{J}_{n_2}$. So the result holds. □

Proposition 4.34. *Let $\mathcal{J}$ be a left(right) multi-ideal of $\mathcal{A}$. If $n_1, n_2 \in Im(J)$ such that $\mathcal{J}_{n_1} = \mathcal{J}_{n_2}$, then $n_1 = n_2$.*

Proof. Assume that $n_1 \neq n_2$, say $n_1 < n_2$. Then there exists $x \in X$ such that $C_{\mathcal{J}}(x) = n_1 < n_2$. Thus $x \in \mathcal{J}_{n_1}$ and $x \notin \mathcal{J}_{n_2}$. So $\mathcal{J}_{n_1} \neq \mathcal{J}_{n_2}$, a contradiction. Hence the result holds. □

Definition 4.35. Let X be a semigroup and $\mathcal{A} \in MS(X)$. The smallest left(right) multi-ideal of $\mathcal{A}$ containing $\mathcal{J}$ is called the left(right) multi-ideal of $\mathcal{A}$ generated by $\mathcal{J}$.

By Remark 4.18 (3), it follows that the intersection of all multi-ideals of $\mathcal{A}$ containing $\mathcal{J}$ is a multi-ideal generated by $\mathcal{J}$.

Proposition 4.36. *Let X be a semigroup. Then $\mathcal{J} \cup \mathcal{A} \circ \mathcal{J}$ is the left multi-ideal of $\mathcal{A}$ generated by $\mathcal{J}$.*

Proof. Let $\{\mathcal{H}_i\}_{i\in I}$ be the collection of all left multi-ideals of $\mathcal{A}$ containing $\mathcal{J}$. Since $C_{\mathcal{H}_i}(y) \leq C_{\mathcal{H}_i}(xy)$, we get

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{H}_i}(a) &= \bigvee_{a=xy} \Big(C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{H}_i}(y)\Big) \\ &\leq \bigvee_{a=xy} C_{\mathcal{H}_i}(xy) \\ &= C_{\mathcal{H}_i}(a). \end{aligned}$$

Since $\mathcal{A}\circ\mathcal{H}_i \sqsubseteq \mathcal{H}_i$ for each $i \in I$, $\mathcal{A}\circ\mathcal{J} \sqsubseteq \bigcap_{i\in I}\mathcal{H}_i$. As a result, $\mathcal{J}\cup\mathcal{A}\circ\mathcal{J} \sqsubseteq \bigcap_{i\in I}\mathcal{H}_i$. Since $\mathcal{A}$ is a multisemigroup, $\mathcal{A}\circ\mathcal{A} \sqsubseteq \mathcal{A}$. Then we have

$$\mathcal{A}\circ(\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}) = \mathcal{A}\circ\mathcal{J}\cup\mathcal{A}\circ(\mathcal{A}\circ\mathcal{J}) = \mathcal{A}\circ\mathcal{J}\cup(\mathcal{A}\circ\mathcal{A})\circ\mathcal{J} \sqsubseteq \mathcal{A}\circ\mathcal{J}\cup\mathcal{A}\circ\mathcal{J} \sqsubseteq \mathcal{J}\cup\mathcal{A}\circ\mathcal{J}.$$

Moreover, we get

$$\begin{aligned} C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}}(ab) \geq C_{\mathcal{A}\circ(\mathcal{J}\cup\mathcal{A}\circ\mathcal{J})}(ab) &= \bigvee_{ab=xy} \Big(C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}}(y)\Big) \\ &\geq C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}}(b) \\ &= C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}}(b). \end{aligned}$$

Thus $\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}$ is a left multi-ideal of $\mathcal{A}$ containing $\mathcal{J}$, that is,

$$\bigcap_{i\in I}\mathcal{H}_i \sqsubseteq \mathcal{J}\cup\mathcal{A}\circ\mathcal{J}.$$

So $\bigcap_{i\in I}\mathcal{H}_i = \mathcal{J}\cup\mathcal{A}\circ\mathcal{J}$. □

Similarly, we can prove for right multi-ideal of $\mathcal{A}$ generated by $\mathcal{J}$.

Proposition 4.37. *Let X be a semigroup. Then $\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}$ is the multi-ideal of $\mathcal{A}$ generated by $\mathcal{J}$.*

Proof. Let $\{\mathcal{H}_i\}_{i\in I}$ be the collection of all multi-ideals of $\mathcal{A}$ containing $\mathcal{J}$. We can show $\mathcal{A}\circ\mathcal{J} \sqsubseteq \bigcap_{i\in I}\mathcal{H}_i$ and $\mathcal{J}\circ\mathcal{A} \sqsubseteq \bigcap_{i\in I}\mathcal{H}_i$ by the same way as shown in Proposition

4.36. Since $\mathcal{A} \circ \mathcal{H}_i \circ \mathcal{A} = (\mathcal{A} \circ \mathcal{H}_i) \circ \mathcal{A}$, we have

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{H}_i\circ\mathcal{A}}(a) &= \bigvee_{a=xy} \left(C_{\mathcal{A}\circ\mathcal{H}_i}(x) \bigwedge C_{\mathcal{A}}(y)\right) \\ &= \bigvee_{a=xy} \left(\left(\bigvee_{x=pq} \left(C_{\mathcal{A}}(p) \bigwedge C_{\mathcal{H}_i}(q)\right)\right) \bigwedge C_{\mathcal{A}}(y)\right) \\ &= \bigvee_{a=xy} \left(\bigvee_{x=pq} C_{\mathcal{H}_i}(q)\right) \\ &\leq \bigvee_{a=xy} \left(\bigvee_{x=pq} C_{\mathcal{H}_i}(pq)\right) \\ &= \bigvee_{a=xy} (C_{\mathcal{H}_i}(x)) \\ &\leq \bigvee_{a=xy} (C_{\mathcal{H}_i}(xy)) = C_{\mathcal{H}_i}(a). \end{aligned}$$

Since $\mathcal{A} \circ \mathcal{J} \circ \mathcal{A} \sqsubseteq \bigcap_{i\in I} \mathcal{H}_i$ for each $i \in I$, we have

$$\mathcal{J} \cup \mathcal{A} \circ \mathcal{J} \cup \mathcal{J} \circ \mathcal{A} \cup \mathcal{A} \circ \mathcal{J} \circ \mathcal{A} \sqsubseteq \bigcap_{i\in I} \mathcal{H}_i.$$

Since $\mathcal{A}$ is a multisemigroup, we get

$$\begin{aligned} &\mathcal{A} \circ (\mathcal{J} \cup \mathcal{A} \circ \mathcal{J} \cup \mathcal{J} \circ \mathcal{A} \cup \mathcal{A} \circ \mathcal{J} \circ \mathcal{A}) \\ &= \mathcal{A} \circ \mathcal{J} \cup \mathcal{A} \circ (\mathcal{A} \circ \mathcal{J}) \cup \mathcal{A} \circ (\mathcal{J} \circ \mathcal{A}) \cup \mathcal{A} \circ (\mathcal{A} \circ \mathcal{J} \circ \mathcal{A}) \\ &= \mathcal{A} \circ \mathcal{J} \cup (\mathcal{A} \circ \mathcal{A}_{\mathcal{R}}) \circ \mathcal{J} \cup \mathcal{A} \circ \mathcal{J} \circ \mathcal{A} \cup (\mathcal{A} \circ \mathcal{A}) \circ \mathcal{J} \circ \mathcal{A} \\ &\sqsubseteq \mathcal{J} \cup \mathcal{A} \circ \mathcal{J} \cup \mathcal{J} \circ \mathcal{A} \cup \mathcal{A} \circ \mathcal{J} \circ \mathcal{A}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(ab) &\geq C_{\mathcal{A}\circ(\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A})}(ab) \\ &= \bigvee_{ab=xy} \left(C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(y)\right) \\ &\geq C_{\mathcal{A}}(a) \bigwedge C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(b) \\ &= C_{\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(b). \end{aligned}$$

Then $\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}$ is a left multi-ideal of $\mathcal{A}$. Similarly, we can show that $\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}$ is a right multi-ideal of $\mathcal{A}$. Thus $\mathcal{J}\cup\mathcal{A}\circ\mathcal{J}\cup\mathcal{J}\circ\mathcal{A}\cup\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}$ is a multi-ideal of $\mathcal{A}$ containing $\mathcal{J}$, that is, $\bigcap_{i\in I} \mathcal{H}_i \sqsubseteq \mathcal{J} \cup \mathcal{A} \circ \mathcal{J} \cup \mathcal{J} \circ \mathcal{A} \cup \mathcal{A} \circ \mathcal{J} \circ \mathcal{A}$. So $\bigcap_{i\in I} \mathcal{H}_i = \mathcal{J} \cup \mathcal{A} \circ \mathcal{J} \cup \mathcal{J} \circ \mathcal{A} \cup \mathcal{A} \circ \mathcal{J} \circ \mathcal{A}$. □

Example 4.38. Let $X = \{a, b, c, d, e\}$ be a semigroup with the following multiplication (See the below table). Let $\mathcal{A} = [a, b, c, d, e]_{7,6,5,4,5}$ and $\mathcal{J} = [a, b, c, d, e]_{3,2,1,4,5}$. Then

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{J}}(a) &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}}(y)\} \\ &= \bigvee \{C_{\mathcal{J}}(a), C_{\mathcal{J}}(c)\} = 3. \end{aligned}$$

.	a	b	c	d	e
a	a	d	a	d	e
b	e	b	c	e	e
c	c	b	c	b	e
d	e	d	a	e	e
e	e	e	e	e	e

Similarly, we can easily check that that
$C_{\mathcal{A}\circ\mathcal{J}}(b) = C_{\mathcal{A}\circ\mathcal{J}}(d) = 4,\ C_{\mathcal{A}\circ\mathcal{J}}(c) = 3,\ C_{\mathcal{J}\circ\mathcal{A}}(a) = C_{\mathcal{J}\circ\mathcal{A}}(d) = 4,$
$C_{\mathcal{J}\circ\mathcal{A}}(b) = C_{\mathcal{J}\circ\mathcal{A}}(c) = 2,\ C_{\mathcal{A}\circ\mathcal{J}}(e) = C_{\mathcal{J}\circ\mathcal{A}}(e) = 5.$
Also, we get

$$\begin{aligned} C_{\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(a) &= \bigvee_{a=xy} \{C_{\mathcal{A}}(x) \bigwedge C_{\mathcal{J}\circ\mathcal{A}}(y)\} \\ &= \bigvee\{C_{\mathcal{J}\circ\mathcal{A}}(a), C_{\mathcal{J}\circ\mathcal{A}}(c).\} = 4 \end{aligned}$$

We can analogously show that

$$C_{\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(b) = C_{\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(c) = C_{\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(d) = 4,\ C_{\mathcal{A}\circ\mathcal{J}\circ\mathcal{A}}(e) = 5.$$

Let $\mathcal{G} = \mathcal{J} \cup \mathcal{A}\circ\mathcal{J} \cup \mathcal{J}\circ\mathcal{A} \cup \mathcal{A}\circ\mathcal{J}\circ\mathcal{A}$. Then we have

$$C_{\mathcal{G}}(a) = C_{\mathcal{G}}(b) = C_{\mathcal{G}}(c) = C_{\mathcal{G}}(d) = 4,\ C_{\mathcal{G}}(e) = 5.$$

It is easily checked that $\mathcal{G}$ is a multi-ideal of $\mathcal{A}$. Let $\mathcal{K}$ be a multi-ideal of $\mathcal{A}$ containing $\mathcal{J}$. Then $C_{\mathcal{K}}(a) = C_{\mathcal{K}}(dc) \geq C_{\mathcal{K}}(d) \geq C_{\mathcal{J}}(d) = 4 = C_{\mathcal{G}}(a)$. Similarly, we can show that $C_{\mathcal{G}}(b) \leq C_{\mathcal{K}}(b)$, $C_{\mathcal{G}}(c) \leq C_{\mathcal{K}}(c)$, $C_{\mathcal{G}}(d) \leq C_{\mathcal{K}}(d)$, and $C_{\mathcal{G}}(e) \leq C_{\mathcal{K}}(e)$. Thus $\mathcal{G} = \mathcal{J} \cup \mathcal{A}\circ\mathcal{J} \cup \mathcal{J}\circ\mathcal{A} \cup \mathcal{A}\circ\mathcal{J}\circ\mathcal{A}$ such that $C_{\mathcal{G}}(a) = C_{\mathcal{G}}(b) = C_{\mathcal{G}}(c) = C_{\mathcal{G}}(d) = 4$, and $C_{\mathcal{G}}(e) = 5$ is the multi-ideal generated by $\mathcal{J}$.

5. Conclusion

Using multiset theory, we introduced the concept of multisemigroups and left(right) multi-ideals, and several properties were investigated. In addition, we discussed the relationships between multisemigroups and left(right) multi-ideals, and showed by an example that every multisemigroup is not a left(right) multi-ideal. Finally, we described multi-ideals generated by multisets.

6. Acknowledgement

The authors are grateful to the anonymous reviewers for their constructive suggestions on this paper.

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