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Abstract. This paper concerns the study of the numerical approximation for the following initial-boundary value problem .

$$\begin{cases} u_t = \left(|u_x|^{p-2}u_x\right)_x + (1-u)^{-h}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases}$$

where $p \ge 2$, $h > 0$, $q > 0$. $u_0 : [0,1] \to (0,1)$ and satisfies compatiblity conditions. We find some conditions under which the solution of a discrete form of above problem quenches in a finite time and estimate its discrete quenching time. We also establish the convergence of the discrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

2020 AMS Classification: 35B50, 35K55, 35K20, 65M06

Keywords: Non-Newtonian filtration equations, Discretization, Discrete quenching time, Convergence.

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1. Introduction

In this paper, we consider the following boundary value problem

$$u_t = (|u_x|^{p-2}u_x)_x + (1-u)^{-h}, \quad 0 < x < 1, t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{1.3}$$

where, $p \geq 2, h > 0,\ q > 0$. $u_0 : [0,1] \to (0,1)$ and satisfies some compatibility conditions such that $u_0^{'}(0) = 0$, $u_0^{'}(1) = -u_0^{-q}(1)$, $u_0^{'}(x) \leq 0$
and $(|u_0^{'}(x)|^{p-2}u_0^{'}(x))^{'} + (1-u_0(x))^{-h} \geq 0,\ 0 \leq x \leq 1$. The quenching behavior describes the phenomenon that there exists a finite time T_q such that the solution of the problem (1.1)–(1.3) satisfied the following definition

Definition 1.1. We say that the classical solution u of the problem (1.1)–(1.3) quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$ but

$$\lim_{t \to T_q} \|u(.,t)\|_\infty = 1,$$

where $\|u(.,t)\|_\infty = \max_{0 \leq x \leq 1} |u(x,t)|$. The time T_q is called the quenching time of the solution u.

The problem (1.1)–(1.3) may be rewritten in the following form

$$u_t = (p-1)|u_x|^{p-2}u_{xx} + (1-u)^{-h}, \quad 0 < x < 1, t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \leq x \leq 1, \tag{1.6}$$

where, $p \geq 2, h > 0,\ q > 0$. $u_0 : [0,1] \to (0,1)$ and satisfies some compatibility conditions such that $u_0^{'}(0) = 0$, $u_0^{'}(1) = -u_0^{-q}(1)$, $u_0^{'}(x) \leq 0$
and $(p-1)|u_0^{'}(x)|^{p-2}u_0^{''}(x) + (1-u_0(x))^{-h} \geq 0,\ 0 \leq x \leq 1$. Equation (1.4) is known as the classical non-Newtonian filtration equation that incorporates the effects of nonlinear reaction source and nonlinear boundary outflux. The variable u representes the speed of the fluid flow. Kawarada first studied the quenching phenomenon for semilinear heart equation $u_t = u_{xx} + (1-u)^{-1}$. He obtained the results that, when the solution reaches level $u = 1$, the reaction term and the time derivative blow up. Since then, the theoretical study of quenching phenomena for semilinear parabolic equations have been the subject of investigations of many researchers(See for examples [1, 2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein).

In the problem (1.4)–(1.6), the authors prove under certain conditions that quenching occurs in finite time and they show that the only quenching point is $x = 0$. They have also established the bounds for quenching rate and the lower bound for the quenching time (See [7, 8]).

In this paper, we are interesting in the numerical study of the phenomenon of quenching using a discrete form of the problem (1.4)–(1.6). This method of study has been used by many researchers (See [11, 12, 13, 14, 15, 16, 17, 18]). We give some conditions under which the solution of the discrete form of the problem (1.4)–(1.6) quenches in finite time and estimate its discrete quinching time. We also prove that the discrete quenching time converges to the real one when the mesh size goes to zero.

This paper is organised as follows. In the next section, we give some properties concerning our discrete sheme. In section 3, under some conditions, we prove that the solution of a discrete form of the problem (1.4)–(1.6) quenches in a finite time and estimate its discrete quenching time. In section 4, we show that the quenching time converges to the theoretical one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the discrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the semidiscrete scheme. Let $I \geq 3$ be a positive integer and let $s = 1/I$. Define the grid $x_i = is$, $0 \leq i \leq I$. Approximate the solution u of problem (1.4)–(1.6) by the solution $U_s^{(n)} = (U_0^{(n)}, U_1^{(n)}, \dots, U_I^{(n)})^T$ and the initial condition u_0 by the initial condition $\varphi_s = (\varphi_0, \varphi_1, \dots, \varphi_I)^T$ the following discrete equations

$$\delta_t U_i^{(n)} = (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 U_i^{(n)} + (1-U_i^{(n)})^{-h},\ 0 \leq i \leq I-1, \tag{2.1}$$

$$\delta_t U_I^{(n)} = (p-1)|(U^{-q})_I^{(n)}|^{p-2}\delta_*^2 U_I^{(n)} + (1-U_I^{(n)})^{-h}, \tag{2.2}$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I, \tag{2.3}$$

where $n \geq 0$, $p \geq 2$, $q > 0$, $h > 0$,

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 0 \leq i \leq I,$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{s^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{s^2}, \quad \delta_*^2 U_I^{(n)} = \delta^2 U_I^{(n)} - \frac{2}{s}(U^{-q})_I^{(n)}, \quad \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{s^2},$$

$$\delta^0 U_0^{(n)} = 0,\ \delta^0 U_i^{(n)} = \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2s}, \quad 1 \leq i \leq I-1,$$

$$0 < \varphi_s < 1, \quad \varphi_{i+1} < \varphi_i, \quad 0 \leq i \leq I-1.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the quenching time T_q , we need to adapt the size of the time step. We choose

$$\Delta t_n = \min\left(\frac{s^2}{2(p-1)\max\{a(j-1,1)\}}, \tau(1-\|U_s^{(n)}\|_\infty)^{h+1}\right),$$

with $0 < \tau < 1$ and $a(j-1,1) = \left(\dfrac{|U_{j+1}^{(n)} - U_{j-1}^{(n)}|}{2s}\right)^{p-2}$ for $2 \leq j \leq I$.

Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution when this one is decreasing.

Lemma 2.1. *Let $\alpha_s^{(n)}, a_s^{(n)}, \gamma_s^{(n)}$ and let $V_s^{(n)}$ be the three sequences with $n \geq 0$; $\alpha_s^{(n)} \geq 0; a_s^{(n)} \leq 0; \gamma_s^{(n)} \leq 0$ such that*

$$\delta_t V_i^{(n)} - \alpha_i^{(n)}\delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \geq 0,\ 0 \leq i \leq I \tag{2.4}$$

$$V_i^{(0)} \geq 0, \quad 0 \leq i \leq I. \tag{2.5}$$

Then we have

$$V_i^{(n)} \geq 0,\ 0 \leq i \leq I,\ n \geq 0,\ \textit{when}\ \Delta t_n \leq \frac{s^2}{2\|\alpha_s^{(n)}\|_\infty},\ 1 \leq i \leq I. \tag{2.6}$$

Proof. A straightforward computation shows that for $1 \leq i \leq I-1$,

$$\begin{aligned}
&V_i^{(n+1)} \\
&\geq \left(1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\right) V_i^{(n)} + \frac{\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\left(V_{i+1}^{(n)} + V_{i-1}^{(n)}\right) - \Delta t_n a_i^{(n)} V_i^{(n)}, \\
&V_0^{(n+1)} \geq \left(1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\right) V_0^{(n)} + \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2} V_1^{(n)} - \Delta t_n a_0^{(n)} V_0^{(n)}, \\
&V_I^{(n+1)} \\
&\geq \left(1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\right) V_I^{(n)} + \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2} V_{I-1}^{(n)} - \frac{2a_I^{(n)}\gamma_I^{(n)}}{s} V_I^{(n)} - \Delta t_n a_I^{(n)} V_I^{(n)}.
\end{aligned}$$

If $V_s^{(n)} \geq 0$, then using an argument of recursion, we easily see that $V_s^{(n+1)} \geq 0$, because $1 - \dfrac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2} \geq 0$, $a_s^{(n)} \leq 0$, $\gamma_s^{(n)} \leq 0$, $\alpha_s^{(n)} \geq 0$. This ends the proof. □

A direct consequence of the above result is the following comparison Lemma.

Lemma 2.2. *Let $a_s^{(n)}, \alpha_s^{(n)}, V_s^{(n)}$ and $W_s^{(n)}$ be four sequences, with $n \geq 0$, $a_s^{(n)} \leq 0, \alpha_s^{(n)} \geq 0$, such that*

$$\begin{aligned}
&\delta_t V_i^{(n)} - \alpha_i^{(n)} \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} < \\
&\delta_t W_i^{(n)} - \alpha_i^{(n)} \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)}, 0 \leq i \leq I,
\end{aligned} \tag{2.7}$$

$$V_i^{(0)} < W_i^{(0)}, \quad 0 \leq i \leq I. \tag{2.8}$$

Then we have

$$V_i^{(n)} < W_i^{(n)},\ 0 \leq i \leq I,\ n \geq 0,\ \textit{when}\ \Delta t_n \leq \frac{s^2}{2\|\alpha_s^{(n)}\|_\infty},\ 1 \leq i \leq I.$$

Proof. Define the sequence $Z_s^{(n)} = W_s^{(n)} - V_s^{(n)}$. A straightforward calculation gives

$$\delta_t W_i^{(n)} - \delta_t V_i^{(n)} - \alpha_i^{(n)} \delta^2 (W_i^{(n)} - V_i^{(n)}) + a_i^{(n)} (W_i^{(n)} - V_i^{(n)}) > 0,\ 0 \leq i \leq I,$$

which is equivalent to

$$\delta_t Z_i^{(n)} - \alpha_i^{(n)} \delta^2 Z_i^{(n)} + a_i^{(n)} Z_i^{(n)} > 0,\ 0 \leq i \leq I.$$

Knowing that $Z_s^{(0)} > 0$, from Lemma (2.1), we have $Z_s^{(n)} > 0$, which implies that $V_i^{(n)} < W_i^{(n)}$, $0 \leq i \leq I$, and the proof is complete. □

Lemma 2.3. *Let $a_s^{(n)}, \alpha_s^{(n)}, V_s^{(n)}$ and $W_s^{(n)}$ be four sequences, with $n \geq 0$, $a_s^{(n)} \leq 0, \alpha_s^{(n)} \geq 0$, such that*

$$\delta_t V_i^{(n)} - \alpha_i^{(n)} \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \leq \tag{2.9}$$

$$\delta_t W_i^{(n)} - \alpha_i^{(n)} \delta^2 W_i^{(n)} + a_i^{(n)} W_i^{(n)}, 0 \leq i \leq I,$$

(2.10) $$V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I.$$

Then we have

$$V_i^{(n)} \leq W_i^{(n)},\ 0 \leq i \leq I,\ n \geq 0,\ when\ \Delta t_n \leq \frac{s^2}{2\|\alpha_s^{(n)}\|_\infty},\ 1 \leq i \leq I.$$

The lemma below reveals the positivity of the discrete solution.

Lemma 2.4. *Let $U_s^{(n)}, n \geq 0$, be the solution of the discrete problem* (2.1)–(2.3). *Then we have*

(2.11) $$U_i^{(n)} > 0,\ 0 \leq i \leq I,\ when\ \Delta t_n = \frac{s^2}{2\|\alpha_s^{(n)}\|_\infty}, 1 \leq i \leq I.$$

Proof. A routine calculation reveals that for $1 \leq i \leq I-1$,

$$U_i^{(n+1)}$$
$$\geq \left(1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\right) U_i^{(n)} + \frac{\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\left(U_{i+1}^{(n)} + U_{i-1}^{(n)}\right) - \Delta t_n a_i^{(n)} U_i^{(n)},$$
$$U_0^{(n+1)} \geq \left(1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\right) U_0^{(n)} + \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2} U_1^{(n)} - \Delta t_n a_0^{(n)} U_0^{(n)},$$
$$U_I^{(n+1)}$$
$$\geq \left(1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2}\right) U_I^{(n)} + \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2} U_{I-1}^{(n)} - \frac{2a_I^{(n)}\gamma_I^{(n)}}{s} U_I^{(n)} - \Delta t_n a_I^{(n)} U_I^{(n)}.$$

If $U_s^{(n)} > 0$, then using an argument of recursion, we easily see that $U_s^{(n+1)} > 0$, because $1 - \frac{2\Delta t_n \|\alpha_s^{(n)}\|_\infty}{s^2} \geq 0$, $a_s^{(n)} < 0$, $\gamma_s^{(n)} < 0$, $\alpha_s^{(n)} > 0$. This ends the proof. □

Lemma 2.5. *Let $U_s^{(n)}, n \geq 0$, be the solution of the discrete problem* (2.1)–(2.3). *Then we have*

(2.12) $$U_{i+1}^{(n)} < U_i^{(n)},\ 0 \leq i \leq I-1.$$

Proof. Difine the vector $Z_s^{(n)}$ such that $Z_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)},\ 0 \leq i \leq I-1$. We have $Z_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)},\ 1 \leq i \leq I-2,\ \ Z_0^{(n)} = U_1^{(n)} - U_0^{(n)},\ Z_{I-1}^{(n)} = U_I^{(n)} - U_{I-1}^{(n)}$.

A straightforward computations reveals that

$$\delta_t Z_i^{(n)} - (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 Z_i^{(n)} + (p-1)(p-2)|\delta^0 U_i^{(n)}|^{p-3}\delta^2 U_{i+1}^{(n)} \delta^0 Z_i^{(n)}$$
$$-h(1-\theta_i^{(n)})^{-h-1} Z_i^{(n)} = 0,\ 0 \leq i \leq I-2,$$

$$\delta_t Z_{I-1}^{(n)} - (p-1)|U_I^{(n)}|^{-q(p-2)}\delta_*^2 Z_{I-1}^{(n)} + q(p-1)(p-2)(U_I^{(n)})^{-q(p-2)-1}\delta^0 (U_I^{(n)})^{-q(p-2)}$$
$$-h(1-\xi_I^{(n)})^{-h-1} Z_{I-1}^{(n)} = 0,$$

where $\theta_i^{(n)} \in (U_{i+1}^{(n)}, U_i^{(n)})$ and $\xi_I^{(n)} \in (U_I^{(n)}, U_{I-1}^{(n)})$.
Knowing that $Z_s^{(n)} < 0$, from Lemma (2.1), we have $Z_s^{(n)} < 0$, which implies that $U_{i+1}^{(n)} < U_i^{(n)}, 0 \leq i \leq I-1$, and we obtain the desired result. □

Lemma 2.6. *Let $U_s^{(n)}, n \geq 0$, be the solution of the problem (2.1)–(2.3) and the initial data at (2.3) verifies some compatibility conditions. Then $\delta_t U_i^{(n)} \geq 0$ for $0 \leq i \leq I$.*

Proof. Consider the vector $Z_s^{(n)}$ such that $Z_i^{(n)} = \delta_t U_i^{(n)}$ for $0 \leq i \leq I$
A straightforward calculation gives

$$\begin{aligned}\delta_t Z_i^{(n)} &= (p-1)(p-2)\delta^0 U_i^{(n)} |\delta^0 U_i^{(n)}|^{p-2} \delta^0 Z_i^{(n)} \delta^2 U_i^{(n)} + (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 Z_i^{(n)} \\ &\qquad + h(1-U_i^{(n)})^{-h-1} Z_i^{(n)}, 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}\delta_t Z_I^{(n)} &= -q(p-1)(p-2)(U^{-q(p-2)-1})_I^{(n)} Z_I^{(n)} \delta^2 U_I^{(n)} + (p-1)(U^{-q(p-2)})_I^{(n)} \delta^2 Z_I^{(n)} \\ &\qquad + \frac{2q(p-1)^2}{s}(U^{-q(p-1)-1})_I^{(n)} Z_I^{(n)} + h(1-U_I^{(n)})^{-h-1} Z_I^{(n)}.\end{aligned}$$

We finally have

$$\begin{aligned}&\delta_t Z_i^{(n)} - (p-1)(p-2)\delta^0 U_i^{(n)} |\delta^0 U_i^{(n)}|^{p-2} \delta^0 Z_i^{(n)} \delta^2 U_i^{(n)} - (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 Z_i^{(n)} \\ &\qquad - h(1-U_i^{(n)})^{-h-1} Z_i^{(n)} = 0, 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}&\delta_t Z_I^{(n)} + q(p-1)(p-2)(U^{-q(p-2)-1})_I^{(n)} Z_I^{(n)} \delta^2 U_I^{(n)} - (p-1)(U^{-q(p-2)})_I^{(n)} \delta^2 Z_I^{(n)} \\ &\qquad - \frac{2q(p-1)^2}{s}(U^{-q(p-1)-1})_I^{(n)} Z_I^{(n)} - h(1-U_I^{(n)})^{-h-1} Z_I^{(n)} = 0.\end{aligned}$$

Knowing that $Z_i^{(0)} = (p-1)|\delta^0\varphi_i|^{p-2}\delta^2\varphi_i + (1-\varphi_i)^{-h} \geq 0,\ 0 \leq i \leq I,$ from Lemma (2.1), we have $Z_s^{(n)} \geq 0,$ which implies that $\delta_t U_i^{(n)} \geq 0,\ 0 \leq i \leq I$, we have the wished result. □

3. Quenching in the discrete problem

In this section, under some assumptions, we show that the solution $U_s^{(n)}$ of the problem (2.1)–(2.3) quenches in a finite time and estimate its discrete quenching time. To facilitate our discussion, we need to define the notion of numerical quenching.

Definition 3.1. We say that the solution $U_s^{(n)}$ of the problem (2.1)–(2.3) *quenches in a finite time*, if $\|U_s^{(n)}\|_\infty < 1$ for $n \geq 0$ but $\lim_{n\to+\infty} \|U_s^{(n)}\|_\infty = 1$ and

$$T_s^{\Delta t} = \lim_{n\to+\infty} \sum_{i=0}^{n-1} \Delta t_i < +\infty.$$

We call $T_s^{\Delta t}$ the *numerical quenching time* of $U_s^{(n)}$.

Now let us set $W_s^{(n)} = 1 - U_s^{(n)}, n \geq 0$. The problem (1.4)–(1.6) may be rewritten in the following form

$$\delta_t W_i^{(n)} = (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2 W_i^{(n)} - (W_i^{(n)})^{-h},\ 0 \leq i \leq I-1,\ n \geq 0, \tag{3.1}$$

$$\delta_t W_I^{(n)} = (p-1)|(1-W_I^{(n)})|^{-q(p-1)}\delta^2 W_I^{(n)} + \frac{2(p-1)}{s}(1-W_I^{(n)})^{-q(p-1)} - (W_I^{(n)})^{-h},\ n \geq 0, \tag{3.2}$$

$$W_i^{(0)} = \nu_i = 1 - \varphi_i > 0,\ 0 \leq i \leq I, \tag{3.3}$$

where $n \geq 0$, $p \geq 2$, $q > 0$, $h > 0$.

Lemma 3.2. *Let $W_s^{(n)}, n \geq 0$, be a sequence such that $W_s^{(n)} > 0$. Then we have for $0 \leq i \leq I$,*

$$\delta^2 (W_i^{(n)})^{-h} \geq -h(W_i^{(n)})^{-h-1}\delta^2 W_i^{(n)},\ n \geq 0.$$

Proof. Applying Taylor's expansion, we obtain

$$\delta^2 (W_i^{(n)})^{-h} = -h(W_i^{(n)})^{-h-1}\delta^2 W_i^{(n)} + (W_{i-1}^{(n)} - W_i^{(n)})^2 \frac{2h(h+1)}{2s^2}(\theta_i^{(n)})^{-h-2} + (W_{i+1}^{(n)} - W_i^{(n)})^2 \frac{2h(h+1)}{2s^2}(\xi_i^{(n)})^{-h-2},\ 1 \leq i \leq I-1,\ n \geq 0,$$

$$\delta^2 (W_0^{(n)})^{-h} = -h(W_0^{(n)})^{-h-1}\delta^2 W_0^{(n)} + (W_1^{(n)} - W_0^{(n)})^2 \frac{2h(h+1)}{2s^2}(\theta_0^{(n)})^{-h-2},\ n \geq 0,$$

$$\delta^2 (W_I^{(n)})^{-h} = -h(W_I^{(n)})^{-h-1}\delta^2 W_I^{(n)} + (W_{I-1}^{(n)} - W_I^{(n)})^2 \frac{2h(h+1)}{2s^2}(\theta_I^{(n)})^{-h-2},\ n \geq 0,$$

where $\theta_0^{(n)}$ is an intermediate value between $W_1^{(n)}$ and $W_0^{(n)}$, $\theta_i^{(n)}$ is an intermediate value between $W_{i-1}^{(n)}$ and $W_i^{(n)}$, for $1 \leq i \leq I-1$, $\theta_I^{(n)}$ is an intermediate value between $W_{I-1}^{(n)}$ and $W_I^{(n)}$, $\xi_i^{(n)}$ is an intermediate value between $W_i^{(n)}$ and $W_{i+1}^{(n)}$, $1 \leq i \leq I-1$. The result follows taking into account the fact that $W_s^{(n)} > 0$ □

Theorem 3.3. *Let $U_s^{(n)}$ be the solution of the problem (2.1)–(2.3) and assume that there exists a constant $A \in (0,1]$ such that the initial data at (3.3) verifies the hypothesis*

$$(p-1)|\delta^0 \nu_i|^{p-2}\delta^2 \nu_i - (\nu_i)^{-h} \leq -A(\nu_i)^{-h}, \quad 0 \leq i \leq I-1, \tag{3.4}$$

$$(p-1)|(1-\nu_I)|^{-q(p-2)}\delta^2 \nu_I + \frac{2(p-1)}{s}(1-\nu_I)^{-q(p-1)} - (\nu_I)^{-h} \leq -A(\nu_I)^{-h}. \tag{3.5}$$

Then there exists a finite time $T_s^{\Delta t}$ *such that* $U_s^{(n)}$ *quenches in this time and we have the following estimate*

$$T_s^{\Delta t} \leq \frac{(1-\|\varphi_s\|_\infty)^{h+1}}{1-(1-\tau')^{h+1}},$$

where $\Delta t_n = \min\left(\frac{s^2}{2(p-1)\max\limits_{2\leq j\leq I}\{a(j-1,1)\}}, \tau\|W_s^{(n)}\|_{inf}^{h+1}\right)$, *with* $0<\tau<1$, $\|W_s^{(n)}\|_{inf} = 1-\|U_s^{(n)}\|_\infty$, $\|\nu_s\|_{inf} = 1-\|\varphi_s\|_\infty$ *and*

$$\tau^{'} = A\min\left(\frac{s^2\|\nu_s\|_{\text{inf}}^{-h-1}}{2(p-1)\max\limits_{2\leq j\leq I}\{a(j-1,1)\}}, \tau\right).$$

Proof. Consider the vector $J_s^{(n)}, n \geq 0$ such that

$$J_i^{(n)} = \delta_t W_i^{(n)} + A(W_i^{(n)})^{-h}, \quad 0 \leq i \leq I.$$

A straightforward computation gives

$$\begin{aligned}
&\delta_t J_i^{(n)} - (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2 J_i^{(n)}\\
&= \delta_t^2 W_i^{(n)} - hA(W_i^{(n)})^{-h-1}\delta_t W_i^{(n)} - (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2(\delta_t W_i^{(n)})\\
&\qquad +A(p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2(W_i^{(n)})^{-h}.
\end{aligned}$$

From Lemma (3.2), we can show that $\delta^2(W_i^{(n)})^{-h} \geq -h(W_i^{(n)})^{-h-1}\delta^2 W_i^{(n)}$, which implies that

$$\begin{aligned}
&\delta_t J_i^{(n)} - (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2 J_i^{(n)}\\
&\leq \delta_t^2 W_i^{(n)} - hA(W_i^{(n)})^{-h-1}\delta_t W_i^{(n)} - (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2(\delta_t W_i^{(n)})\\
&\qquad +hA(p-1)|\delta^0 W_i^{(n)}|^{p-2}(W_i^{(n)})^{-h-1}\delta^2 W_i^{(n)}.
\end{aligned}$$

Using(3.1) and (3.2), we arrive at

$$\delta_t J_i^{(n)} - (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2 J_i^{(n)} \leq h(W_i^{(n)})^{-h-1}J_i^{(n)}, \quad 0 \leq i \leq I-1,$$

$$\begin{aligned}
&\delta_t J_I^{(n)} - (p-1)|(1-W_I^{(n)})|^{-q(p-2)}\delta^2 J_I^{(n)} - h(W_I^{(n)})^{-h-1}J_I^{(n)}\\
&\leq -\frac{2q(p-1)^2}{s}(W_I^{(n)})^{-q(p-1)-1}g(W_I^{(n)}),
\end{aligned}$$

where $g(W_I^{(n)}) = -\delta_t W_I^{(n)} + \dfrac{Ah}{q(p-1)}(1-W_I^{(n)})(W_I^{(n)})^{-h-1} \geq 0$. It is not hard to see that

$$\delta_t J_i^{(n)} - (p-1)|\delta^0 W_i^{(n)}|^{p-2}\delta^2 J_i^{(n)} - h(W_i^{(n)})^{-h-1}J_i^{(n)} \leq 0, \quad 0 \leq i \leq I-1,$$

$$\delta_t J_I^{(n)} - (p-1)|1-W_I^{(n)}|^{-q(p-2)}\delta^2 J_I^{(n)} - h(W_I^{(n)})^{-h-1}J_I^{(n)} \leq 0,$$

From (3.4) and (3.5), we see that $J_s^{(0)} \leq 0$. We deduce from Lemma (2.1) that $J_s^{(n)} \leq 0$, for $n \geq 0$, which implies that

(3.6) $$\delta_t W_i^{(n)} \leq -A(W_i^{(n)})^{-h},\ 0 \leq i \leq I,$$

These estimate may be rewritten in the following form

$$W_i^{(n+1)} \leq W_i^{(n)} - A\Delta t_n (W_i^{(n)})^{-h}.$$

Therefore

$$W_i^{(n+1)} \leq W_i^{(n)}(1 - A\Delta t_n (W_i^{(n)})^{-h-1}), \tag{3.7}$$

which implies that

$$\|W_s^{(n+1)}\|_{inf} \leq \|W_s^{(n)}\|_{inf}(1 - A\Delta t_n \|W_s^{(n)}\|_{inf}^{-h-1}), n \geq 0.$$

From Lemma (2.6), $\|W_s^{(n+1)}\|_{inf} \leq \|W_s^{(n)}\|_{inf}$. By induction, we obtain

$$\|W_s^{(n)}\|_{inf} \leq \|W_s^{(0)}\|_{inf} = \|\nu_s\|_{inf},$$

Then we have

$$\|W_s^{(n)}\|_{inf}^{-h-1} \geq \|\nu_s\|_{inf}^{-h-1}$$

and with $A\Delta t_n \|W_s^{(n)}\|_{inf}^{-h-1} \geq \tau^{'}$, we arrive at

$$\|W_s^{(n+1)}\|_{inf} \leq \|W_s^{(n)}\|_{inf}(1 - \tau^{'}).$$

By induction, we get

$$\|W_s^{(n)}\|_{inf} \leq \|W_s^{(0)}\|_{inf}(1 - \tau^{'})^n, \ n \geq 0,$$

which leads us to

$$\|W_s^{(n)}\|_{inf} \leq \|\nu_s\|_{inf}(1 - \tau^{'})^n, \ n \geq 0.$$

Since the term on the right hand side of the above inequality tends to zero as n approaches infinity, we conclude that $\|W_s^{(n)}\|_{inf}$ tends to zero, therefore, $\|U_s^{(n)}\|_\infty$ tends to 1. Now, let us estimate the numerical quenching time. It is not hard to see that

$$\sum_{n=0}^{+\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \tau \|W_s^{(n)}\|_{inf}^{h+1} \leq \tau \|\nu_s\|_{inf}^{h+1} \sum_{n=0}^{+\infty} \Big((1 - \tau^{'})^{h+1}\Big)^n.$$

Using the fact that the series $\sum_{n=0}^{+\infty} \Big((1 - \tau^{'})^{h+1}\Big)^n$ converges towards $\dfrac{\tau \|\nu_s\|_{inf}^{h+1}}{1 - (1 - \tau^{'})^{h+1}}$. We deduce that

$$T_s^{\Delta t} = \sum_{n=0}^{+\infty} \Delta t_n \leq \frac{\tau \|\nu_s\|_{inf}^{h+1}}{1 - (1 - \tau^{'})^{h+1}}.$$

Since $\|\nu_s\|_{inf} = 1 - \|\varphi_s\|_\infty$, we have

$$T_s^{\Delta t} = \sum_{n=0}^{+\infty} \Delta t_n \leq \frac{\tau (1 - \|\varphi_s\|_\infty)^{h+1}}{1 - (1 - \tau^{'})^{h+1}}.$$

We conclude the proof. □

Remark 3.4. Using Taylor's expansion, we get

$$(1-\tau^{'})^{h+1} = 1-(h+1)\tau^{'} + O(\tau^{'}),$$

which implies that

$$\frac{\tau}{1-(1-\tau')^{h+1}} = \frac{\tau}{\tau'}\frac{1}{(h+1)} \leq \frac{C}{(h+1)}.$$

If we take $\tau = \dfrac{s^2}{2(p-1)}$, we have

$$\frac{\tau^{'}}{\tau} = A\min\left(\frac{\|\nu_s\|_{inf}^{-h-1}}{\max\limits_{2\leq j\leq I}\{a(j-1;1)\}}, 1\right),$$

and therefore

$$\frac{\tau}{\tau^{'}} = \frac{1}{A}\min\left(\max_{2\leq j\leq I}\{a(j-1,1)\}\|\nu_s\|_{inf}^{h+1}, 1\right),$$

then

$$\frac{\tau}{1-(1-\tau')^{h+1}} \leq \frac{C}{(h+1)} = \frac{C}{A\,(h+1)}\min\left(\max_{2\leq j\leq I}\{a(j-1,1)\}\|\nu_s\|_{inf}^{h+1}, 1\right).$$

We conclude that $\dfrac{\tau}{1-(1-\tau')^{h+1}}$ is bounded.

Remark 3.5.

$$\|W_s^{(n+1)}\|_{inf} \leq \|W_s^{(n)}\|_{inf}(1-\tau^{'}),$$

we get

$$\|W_s^{(n)}\|_{inf} \leq \|W_s^{(q)}\|_{inf}(1-\tau^{'})^{n-q}, for\ n\geq q,$$

which implies that

$$\sum_{n=q}^{+\infty}\Delta t_n \leq \tau\|\nu_s\|_{inf}^{h+1}\sum_{n=q}^{+\infty}\left((1-\tau^{'})^{h+1}\right)^{n-q},$$

we deduce that

$$T_s^{\Delta t} - t_q \leq \frac{\tau\|W_s^{(n)}\|_{inf}^{h+1}}{1-(1-\tau')^{h+1}}\ with\ \Delta t_q = \sum_{j=0}^{q-1}\Delta t_j,$$

and since $\|W_s^{(n)}\|_{inf} = 1-\|U_s^{(n)}\|_\infty$, we have

$$T_s^{\Delta t} - t_q \leq \frac{\tau(1-\|U_s^{(n)}\|_\infty)^{h+1}}{1-(1-\tau')^{h+1}}\ with\ \Delta t_q = \sum_{j=0}^{q-1}\Delta t_j.$$

In the sequel, we take $\tau = \dfrac{s^2}{2(p-1)}$.

4. Convergence of the discrete quennching time

In this section, under some assumptions, we show that the discrete quenching time converges to the real one when the mesh size goes to zero. We denote by :

$$u_s(t_n) = (u(x_0,t_n), u(x_1,t_n), \dots, u(x_I,t_n))^T \text{ and } \|U_s^{(n)}\|_\infty = \max_{0\le i\le I} |U_i^{(n)}.$$

In order to obtain the convergence of discrete quenching time, we firstly prove the following theorem about the convergence of the discrete scheme.

Theorem 4.1. *Assume that the problem (1.4)–(1.6) has a solution $u \in C^{4,2}([0,1]\times[0,T])$ such that $\sup_{t\in[0,T]} \|u(.,t)\|_{inf} = \lambda < 1$. Suppose that the initial data at (2.3) satisfies*

$$\|\varphi_s - u_s(0)\|_\infty = o(1) \quad as \quad s \to 0, \tag{4.1}$$

Then for s sufficiently small, the problem (2.1)–(2.3) has a unique solution $U_s^{(n)}, 0 \le n \le J$ such that

$$\max_{0\le n\le J} \|U_s^{(n)} - u_s(t_n)\|_\infty = O(\|\varphi_s - u_s(0)\|_\infty + s + \Delta t_n) \quad as \quad s \to 0. \tag{4.2}$$

Where J is such that $\sum_{j=1}^{J-1} \Delta t_j \le T$ and $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. For each s, the discrete problem (2.1)–(2.3) has a unique solution $U_s^{(n)}$. Let $N \le J$, the greatest value of n. There exists a positive real ρ (*with* $\lambda < \rho < 1$) such that

$$\|U_s^{(n)} - u_s(t_n)\|_\infty < \frac{\rho-\lambda}{2},\ for\ n < N. \tag{4.3}$$

We know that $N \ge 1$ because of (4.1). Due to the fact $u \in C^{4,2}([0,1]\times[0,T])$. By the triangular inequality, we obtain

$$\|U_s^{(n)}\|_\infty \le \|u_s(t_n)\|_\infty + \|U_s^{(n)} - u_s(t_n)\|_\infty,\ n < N,$$

which implies that

$$\|U_s^{(n)}\|_\infty \le \lambda + \frac{\rho-\lambda}{2} = \frac{\rho+\lambda}{2} < 1,\ n < N. \tag{4.4}$$

Since $u \in C^{4,2}([0,1]\times[0,T])$. Applying Taylor's expansion, we obtain

$$\begin{aligned}
&\delta_t u(x_i,t_n) - (p-1)|\delta^0 u(x_i,t_n)|^{p-2}\delta^2 u(x_i,t_n) \\
&= (1-u(x_i,t_n))^{-h} + \frac{h^2(p-1)|\delta^0 u(x_i,t_n)|^{p-2}}{12} u_{xxxx}(\widetilde{x}_i,t_n) \\
&\qquad + \frac{\Delta t_n}{2}\delta_{tt} u(x_i,\widetilde{t}_n),\ 0 \le i \le I-1,
\end{aligned}$$

$$\begin{aligned}
&\delta_t u(x_I,t_n) - (p-1)|u^{-q}(x_i,t_n)|^{p-2}\delta^2 u(x_I,t_n) \\
&= (1-u(x_I,t_n))^{-h} + \frac{2(p-1)}{h}|u^{-q}(x_i,t_n)|^{p-2}u^{-q}(x_I,t_n) \\
&\qquad + \frac{h(p-1)}{3}|u^{-q}(x_i,t_n)|^{p-2}u_{xxx}(\widetilde{x}_I,t_n)
\end{aligned}$$

$$-\frac{h^2(p-1)}{12}|u^{-q}(x_i,t_n)|^{p-2}u_{xxxx}(\widetilde{x}_I,t_n)$$
$$+\frac{\Delta t_n}{2}\delta_{tt}u(x_i,\widetilde{t}_n).$$

Let $e_s^{(n)} = U_s^{(n)} - u_s(t_n)$ be the error of discretization. Using Taylor's expansion, we have for $n < N$,

$$\begin{aligned}
&\delta_t e_i^{(n)} - \alpha_i^{(n)}\delta^2 e_i^{(n)} \\
&= h(1-\theta_i^{(n)})^{-h-1}e_i^{(n)} + \frac{h^2}{12}\alpha_i^{(n)}u_{xxxx}(\widetilde{x}_i,t_n) - \frac{\Delta t_n}{2}\delta_{tt}u(x_i,\widetilde{t}_n), \quad 0 \le i \le I-1, \\
&\delta_t e_I^{(n)} - \alpha_I^{(n)}\delta^2 e_I^{(n)} \\
&= h(1-\xi_I^{(n)})^{-h-1}e_I^{(n)} + \frac{2}{h}q\alpha_I^{(n)}(\xi_I^{(n)})^{-q-1}e_I^{(n)} - \tfrac{h}{3}\alpha_I^{(n)}u_{xxx}(\widetilde{x}_I,t) \\
&\qquad + \tfrac{h^2}{12}\alpha_I^{(n)}u_{xxxx}(\widetilde{x}_I,t) - \frac{\Delta t_n}{2}\delta_{tt}u(x_i,\widetilde{t}_n),
\end{aligned}$$

where $\theta_i^{(n)}$ an intermediate value between $u(x_i,t_n)$ and $U_i^{(n)}$ for $i \in \{0,\cdots,I-1\}$ and $\xi_I^{(n)}$ is an intermediate value between $u(x_I,t_n)$ and $U_I^{(n)}$. Since $\alpha(x,t)u_{xxx}(x,t)$, $\alpha(x,t)u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded, there existes a positive constant $K > 0$ such that

$$\delta_t e_i^{(n)} - \alpha_i^{(n)}\delta^2 e_i^{(n)} \le (1-\theta_i^{(n)})|e_i^{(n)}| + Ks^2 + K\Delta t_n,\ 0 \le i \le I-1,\ n < N, \tag{4.5}$$

$$\delta_t e_I^{(n)} - \alpha_I^{(n)}\delta^2 e_I^{(n)} \le \left((1-\xi_I^{(n)}) + \frac{2}{h}q\alpha_I^{(n)}(\xi_I^{(n)})^{-q-1}\right)e_I^{(n)} + Ks + K\Delta t_n,\ n < N. \tag{4.6}$$

Set $L = (1-\frac{\rho+\lambda}{2}) + \frac{2}{h}q\alpha_I^{(n)}(\frac{\rho+\lambda}{2})^{-q-1}$; and introduce the vector $W_s^{(n)}$ defined as follows

$$W_i^{(n)} = e^{(L+1)t_n}(\|\varphi_s - u_s(0)\|_\infty + Ks + K\Delta t_n), \quad 0 \le i \le I, n < N.$$

A direct calculation yields

$$\delta_t W_i^{(n)} - \alpha_i^{(n)}\delta^2 W_i^{(n)} > (1-\theta_i^{(n)})W_i^{(n)} + Ks^2 + K\Delta t_n,\ 0 \le i \le I-1,\ n < N, \tag{4.7}$$

$$\delta_t W_I^{(n)} - \alpha_I^{(n)}\delta^2 W_I^{(n)} > \left((1-\xi_I^{(n)}) + \frac{2}{h}q\alpha_I^{(n)}(\xi_I^{(n)})^{-q-1}\right)W_I^{(n)} + Ks + K\Delta t_n, \tag{4.8}$$

$$W_0^{(n)} > e_0^{(n)}, \quad W_I^{(n)} > e_I^{(n)},\ n < N \tag{4.9}$$

$$W_i^{(0)} > e_i^{(0)}, \quad 0 \le i \le I. \tag{4.10}$$

Applying comparison Lemma (2.2), we arrive at

$$W_i^{(n)} > e_i^{(n)}, \quad 0 \le i \le I.$$

In the same way, we also prove that

$$W_i^{(n)} > -e_i^{(n)}, \quad 0 \le i \le I,$$

which implies that

$$W_i^{(n)} > |e_i^{(n)}|,\ 0 \le i \le I.$$

We deduce that

$$\|U_s^{(n)} - u_s(t_n)\|_\infty \le e^{(L+1)t_n}(\|\varphi_s - u_s(0)\|_\infty + Ks + K\Delta t_n),\ n < N. \tag{4.11}$$

Now, let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (4.11), and taking into account the inequality (4.3), we obtain

$$\frac{\rho - \lambda}{2} \le \|U_s^{(N)} - u_s(t_N)\|_\infty \le \tag{4.12}$$

$$e^{(L+1)T}(\|\varphi_s - u_s(0)\|_\infty + Ks + K\Delta t_n).$$

Since the term on the right hand side of the above inequality goes to zero as s tends to zero, we deduce that $\frac{\rho-\lambda}{2} \le 0$, which is impossible. Consequently $N = J$, and we conclude the proof. □

Theorem 4.2. *Suppose that the solution u of the problem (1.4)–(1.6) quenches in a finite time T_q such that $u \in C^{4,2}([0,1] \times [0,T_q))$ and the initial condition at (2.3) satisfies*

$$\|\varphi_s - u_s(0)\|_\infty = \circ(1) \quad s \to 0.$$

Under the assumptions of the Theorem (3.3), the discrete problem (2.1)–(2.3) has a solution $U_s^{(n)}$ which quenches in a finite time $T_s^{\Delta t}$ and the following relation holds

$$\lim_{s\to 0} T_s^{\Delta t} = T_q.$$

Proof. The Remark (3.4) allows us to say that $\dfrac{\tau}{1-(1-\tau')^{h+1}}$ is bounded. Letting $0 < \varepsilon < \dfrac{T_q}{2}$. Then exists a positive real $\gamma = \rho - \lambda$ (*with* $\lambda < \rho < 1$) such that

$$\frac{\tau(1-z)^{h+1}}{1-(1-\tau')^{h+1}} \le \frac{\varepsilon}{2}, \quad for\ z \in [1-\gamma, 1). \tag{4.13}$$

Since u quenches in a finite time T_q, there exists a time $T_1 \in (T_q - \dfrac{\varepsilon}{2}, T_q)$ and $s_0(\varepsilon) > 0$ such that $1 - \dfrac{\gamma}{2} \le \|u(.,t_n)\|_\infty < 1$ with $t_n \in [T_1, T_q[,\ s \le s_0(\varepsilon)$. Let $T_2 = \dfrac{T_1 + T_q}{2}$ and q be a positive integer such that $t_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T_2]$, for $s \le s_0(\varepsilon)$. We have

$$1 - \frac{\gamma}{2} \le \|u_s(t_n)\|_\infty < 1,\ for\ n \le q,\ s \le s_0(\varepsilon).$$

It follows from Theorem (4.1) that the discrete problem (2.1)–(2.3) has a solution $U_s^{(n)}$ which verifies

$$\|U_s^{(n)} - u_s(t_n)\|_\infty < \frac{\gamma}{2},\ for\ n \le q,\ s \le s_0(\varepsilon).$$

Using the triangle inequality, we get

$$\|U_s^{(q)}\|_\infty \geq \|u_s(t_n)\|_\infty - \|U_s^{(q)} - u_s(t_q)\|_\infty \geq 1 - \frac{\gamma}{2} - \frac{\gamma}{2},\ s \leq s_0(\varepsilon),$$

which implies that

$$\|U_s^{(q)}\|_\infty \geq 1 - \gamma,\ s \leq s_0(\varepsilon).$$

From Theorem (3.3), $U_s^{(n)}$ quenches at the time $T_s^{\Delta t}$. It follows from Remark (3.5) and (4.13) that

$$|T_s^{\Delta t} - t_q| \leq \frac{\tau(1 - \|U_s^{(q)}\|_\infty)^{h+1}}{1 - (1 - \tau')^{h+1}} < \frac{\varepsilon}{2},$$

because, we have $\|U_s^{(q)}\|_\infty \geq 1 - \dfrac{\gamma}{2},\ for\ s \leq s_0(\varepsilon)$. We deduce that for $s \leq s_0(\varepsilon)$,

$$|T_s^{\Delta t} - T_q| \leq |T_s^{\Delta t} - t_q| + |t_q - T_q| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which leads us to the desired result. □

5. Numerical experiments

In this section, we present some numerical approximations to the quenching time of the problem (1.4)–(1.6). We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e}$$
$$= (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 U_i^{(n)} + (1 - U_i^{(n)})^{-h}, \quad 0 \leq i \leq I-1,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n^e}$$
$$= (p-1)|(U^{-q})_I^{(n)}|^{p-2}\delta^2 U_I^{(n)} + (p-1)|(U^{-q})_I^{(n)}|^{p-2}\left(\frac{-(U^{-q})_I^{(n)}}{s}\right) + (1 - U_I^{(n)})^{-h},$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $p \geq 2$, $h > 0$, $q > 0$, $\delta^0 U_i^{(n)} = \dfrac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2s}$, $\delta^2 U_i^{(n)} = \dfrac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{s^2}$, for $1 \leq i \leq I-1$,

$$\delta^2 U_I^{(n)} = \frac{2}{s^2}\left(U_{I-1}^{(n)} - U_I^{(n)}\right),$$

$$\Delta t_n^e = \min\left(\frac{s^2}{2(p-1)\max\{a(j-1,1)\}}, \tau(1 - \|U_s^{(n)}\|_\infty)^{h+1}\right)$$

with $0 < \tau < 1$ and $a(j-1,1) = \left(\dfrac{|U_{j+1}^{(n)} - U_{j-1}^{(n)}|}{2s}\right)^{p-2}$ for $2 \leq j \leq I$.

Now, approximate the solution u of the problem (1.4)–(1.6) by the solution $U_s^{(n)} = \left(U_0^{(n)}, U_1^{(n)}, \cdots, U_I^{(n)}\right)^T$ of the following implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}$$

$$= (p-1)|\delta^0 U_i^{(n)}|^{p-2}\delta^2 U_i^{(n+1)} + (1-U_i^{(n)})^{-h}, \quad 0 \le i \le I-1,$$

$$\frac{U_I^{(n+1)}-U_I^{(n)}}{\Delta t_n}$$

$$= (p-1)|(U^{-q})_I^{(n)}|^{p-2}\delta^2 U_I^{(n+1)} + (p-1)|(U^{-q})_I^{(n)}|^{p-2}(\frac{-(U^{-q})_I^{(n)}}{s}) + (1-U_I^{(n)})^{-h},$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \le i \le I,$$

where $n \ge 0$, $p \ge 2$, $h > 0$, $q > 0$, $\delta^0 U_i^{(n)} = \dfrac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2s}$,

$$\delta^2 U_i^{(n+1)} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{s^2},$$

$$\delta^2 U_I^{(n+1)} = \frac{2}{s^2}\left(U_{I-1}^{(n+1)} - U_I^{(n+1)}\right), \ \Delta t_n = \tau(1 - \|U_s^{(n)}\|_\infty)^{h+1}$$

with $0 < \tau < 1$. In the following tables, in rows, we present the numerical quenching times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. The numerical time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}$. The order(s) of the method is computed from

$$s_0^{'} = \frac{\log((T_{4s} - T_{2s})/(T_{2s} - T_s))}{\log(2)}.$$

For the numerical value, we take: $\varphi_i = 0.5 + \dfrac{1}{6\pi} cos\left(\dfrac{\pi(is)}{2}\right) - \dfrac{1}{3}(is)^{4.5}$, for $i = 0, \cdots, I$.

Table 1: Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method for $q = 0, 1, \quad p = 2, \quad h = 3$

I	T^n	n	$CPU\ time$	$s_0^{'}$
16	0.010005572074890	3502	0.07	-
32	0.009983606857072	13308	0.26	-
64	0.009978123925230	50402	1.71	2.00
128	0.009976753714490	190260	14.21	2.00
256	0.009976411192172	715623	138.75	2.00
512	0.009976325554442	2680794	692.37	2.00
1024	0.009976304108366	9996366	3910.76	2.00

Table 2 : Numerical quenching times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method for $q = 0, 1, \quad p = 2, \quad h = 3$

I	T^n	n	$CPU\ time$	$s_0^{'}$
16	0.010005572074890	3502	0.29	-
32	0.009983606857072	13308	1.65	-
64	0.009978123925230	50402	63.75	2.00
128	0.009976753714490	190260	539.15	2.00
256	0.009976411192172	715623	6848.92	2.00
512	0.009976325554442	2680794	72578.25	2.00
1024	0.009976304108366	9996366	954792.15	2.00

Remark 5.1. The two tables show that the solution of the problem quenches in a finite time. We estimate this time at 10^{-2}.

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 32$ and $(q, p, h) = (0.1, 2, 3)$.
In the figures 1 and 2, we can appreciate the quenching of the discrete solution and in the figures 3 and 4, we observe that the discrete solution quenches at the finite time $T_s^{\Delta t} = 10^{-2}$.

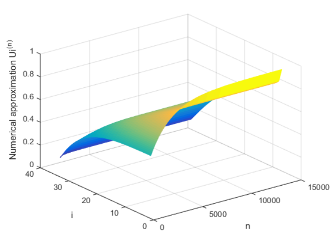

FIGURE 1. Evolution of the discrete solution (explicit scheme).

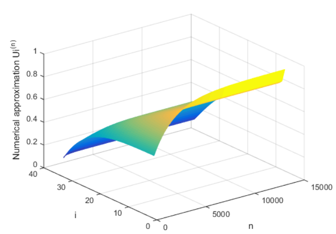

FIGURE 2. Evolution of the discrete solution (implicit scheme).

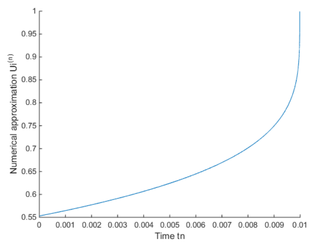

FIGURE 3. Evolution of the norm of the discrete solution according to the time (explicit scheme).

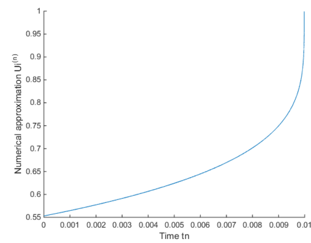

FIGURE 4. Evolution of the norm of the discrete solution according to the time (implicit scheme).

6. Conclusion

In this paper, we have studied the numerical quenching of the solution of the non-Newtonian filtration equation with singular boundary flux (1.1)–(1.3) and we have obtained good approximations of its quenching time.

We have constructed, by the finite difference method, the discrete problem (2.1)–(2.3) associated to the continuous problem (1.1)–(1.3). We have shown that under some conditions, the solution of the discrete problem (2.1)–(2.3) quenches in finite time and we have estimated its discrete quenching time. We have also established the convergence of the discrete time towards the theoretical time when the spatial and temporal discretization steps tend towards zero. Finally, we have given some numerical experiments to illustrate our analysis.

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