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Abstract. In this paper, we introduce the covering $L-$semi uniform spaces (briefly CLS-Uniform spaces). Interior operator in the context of CLS-uniform spaces have been introduced and it is extended to be topological interior operator. Semi-uniformly continuous function have also been studied in the context of CLS-uniform spaces. Lastly we have shown that every CLS-uniform spaces is $L-$semi-pseudo-metrizable if it has countable base.

2020 AMS Classification: 54A20, 54A40, 54E15

Keywords: L-topology, covering L-semi-uniform spaces, Semi-uniformly continuous, L-semi-pseudo-metric.

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1. Introduction

Uniform spaces through the entourage approach in the fuzzy set have been studied by several authors[1, 2, 3, 4] in the category ***I*-TOP** and ***L*-TOP**. Uniform spaces through covering approach was introduced by Soetens and Wuyts [5], Chandrika and Meenakshi [6], and Chandrika [7] in the category $I-$**TOP** and García et al [8] in the category ***L*-TOP**. Hazarika and Mitra have developed generalised uniform structures such as semi-uniformity, locally uniformity and quasi-uniformity through the entourage approach in [9, 10, 11, 12, 13, 14, 15, 16] in the category ***I*-TOP** and ***L*-TOP** many interesting result were obtained such as completeness, compactness, uniform continuity and metrization. It is been always observed that the uniform spaces generates completely regular topological spaces which is strong topological spaces and then the finding for the weaker topological spaces to study uniform properties it is necessary to generalised uniform structures. Recently, we have introduced localization uniform spaces through the covering approach in [17] in the category ***L*-TOP** and many spectacular results were obtained and it generates regular L-topological spaces. For this purpose we further generalised to find the answer of the question for weaker topological to study the uniform structures. However non of the authors were consider semi-uniform structure through the covering approach in the fuzzy setting.

In this paper, we introduce the notion of covering L-semi uniform spaces (in briefly CLS-uniform spaces) in the sense of Garcı́a et al [8]. Several results on L-topoloical spaces, semi uniformly continuous, metrization have been obtained. In next article, we will study various notion in the context of CLS-uniform spaces such as completeness and compactness.

2. Preliminaries

Throughout this paper $(L, \leq, \bigwedge, \bigvee)$ denotes a fuzzy lattice with order reversing involution $'$; 0_L and 1_L are respectively inf and sup in L. X is an arbitrary (ordinary) set and L^X denotes the collection of all mappings $A : X \to L$. Any member of L^X is an L-fuzzy set. The L-fuzzy sets $x_\alpha : X \to L$ defined by $x_\alpha(y) = 0_L$ if $x \neq y$ and $x_\alpha(y) = \alpha$ if $x = y$ are the L-fuzzy points. The mappings $A : X \to L$ and $B : X \to L$ defined by $A(x) = 1_L,\ \forall x \in X$ and $B(x) = 0_L,\ \forall x \in X$ are denoted by $\underline{1}$ and $\underline{0}$ respectively. For any $A,\ B \in L^X$, the union and intersection of A and B are defined as $A \cup B(x) = A(x) \vee B(x)$ and $A \bigcap B(x) = A(x) \wedge B(x)$ respectively. Further, we say that $A \subseteq B$ if and only if $A(x) \leq B(x)$ and $x_\alpha \in A$ if and only if $\alpha < A(x)$, where x_α is an L-fuzzy point; complement A' of A is defined as $A'(x) = A(x)'$. An L-topology $\mathbb{F}$ on L^X is a subset of L^X closed under finite intersection and arbitrary union. In this case, the pair $(L^X, \mathbb{F})$ is known as L-topological space. The elements of $\mathbb{F}$ are called open sets and their complements are the closed sets. For any $A \in L^X$, the interior and closure of A in L-topological space $(L^X, \mathbb{F})$, are respectively denoted by A^o and $\overline{A}$. For basic definitions and results of product of L-topological spaces we refer to [18, 19]. Covering L-valued uniformity referred in this paper is in the sense of García et al. [8].

Definition 2.1 ([19])**.** Let $L-$ be a lattice $\alpha \in L$. α is said to be *join-irreducible*, if $\alpha < 1_L$ and for all $a, b \in L, \alpha = a \vee b \Rightarrow \alpha = a$ or $\alpha = b$. A join-irreducible element of L is called an *molecule* in L.
The set of all molecules in L is denoted by $M(L)$.

Definition 2.2 ([19])**.** Let L be a complete lattice. Define a relation $\preceq$ in L as follows $\forall a, b \in L,\ a \preceq b$ iff $\forall S \subseteq L, b \leq \vee S \Rightarrow \exists s \in S$ such that $a \leq s$, denote $\beta_L(a) = \{b \in L : b \preceq a\}$, $\beta^*_L(a) = M(\beta_L(a))$; or denote them respectively by $\beta(a)$ and $\beta^*(a)$, in short. $\forall a \in L.D \subseteq \beta(a)$ is called a *minimal set* of a, if $\vee D = a$.

Definition 2.3. [20] Let $int : L^X \to L^X$ be a mapping on L^X. Then int is called an *interior operator* on L^X, if it fulfills the following conditions:

(io1) $int(\underline{1}) = \underline{1}$.
(io2) $int(A) \subseteq A, \forall A \in L^X$.
(io3) $int(A \bigcap B) = int(A) \bigcap int(B)$.

L^X together with an interior operator "*int*" shall be called an *interior space*.

An interior operator "*int*" is said to be a *topological interior*, if in addition it satisfies the following:

(io4) $int(int(A)) = int(A), \forall A \in L^X$.
For any $A \in L^X$ with $(int(A'))'$ is closure of A with respect to interior operator int denoted by $cl(A)$.

Definition 2.4 ([8])**.** A collection $\mathscr{A}$ of L^X is called an *L-cover* of L^X, if $\bigcup \mathscr{A} = \underline{1}$. For any $\mathscr{A}, \mathscr{B} \subseteq L^X$ then $\mathscr{A}$ refines $\mathscr{B}$ if and only if for each $A \in \mathscr{A}$, there exits $B \in \mathscr{B}$ such that $A \subseteq B$. we write $\mathscr{A} \preccurlyeq \mathscr{B}$. The set of all $L-$covers of L^X, defined as $L - Cov(X)$, is a preordered set with respect to the relation '$\preccurlyeq$'.

Proposition 2.5 ([8])**.** *For every $L-$covers $\mathscr{A}$ and $\mathscr{B}$ of L^X, we have $\mathscr{A} \bigcap \mathscr{B} = \{A \bigcap B : \ A \in \mathscr{A}, \ B \in \mathscr{B}\}$ is also L-cover of L^X.*

Definition 2.6 ([8])**.** For each $A \in L^X$ and $\mathscr{A} \subseteq L^X$, the star of A with respect to $\mathscr{A}$ is defined as $st(A, \mathscr{A}) := \bigcup \{B \in \mathscr{A} : B \bigcap A \neq \underline{0}\}$. The collection $st(\mathscr{A}) := \{st(A, \mathscr{A}) : A \in \mathscr{A}\}$, is an L-cover of L^X, whenever $\mathscr{A}$ is so.

Theorem 2.7 ([8])**.** *Let $\mathscr{A}, \mathscr{B} \subseteq L^X$ and $A, \ B \in L^X$.*

1. *If $\mathscr{A}$ is an L-cover of L^X, then $A \subseteq st(A, \mathscr{A})$ and consequently, $\mathscr{A} \preccurlyeq st(\mathscr{A})$.*
2. *If $A \subseteq B$, then $st(A, \mathscr{A}) \subseteq st(B, \mathscr{A})$.*
3. *If $\mathscr{A} \preccurlyeq \mathscr{B}$, then $st(A, \mathscr{A}) \subseteq st(A, \mathscr{B})$.*
4. *$st(\bigcup \mathscr{B}, \mathscr{A}) = \bigcup_{B \in \mathscr{B}} st(B, \mathscr{A})$.*
5. *If $\mathscr{A}$ is an $L-$cover, then $st(st(A, \mathscr{A}), \mathscr{A}) \subseteq st(A, st(\mathscr{A}))$.*
6. *Let $f^{\rightarrow} : L^X \rightarrow L^Y$ be an $L-$fuzzy mapping and $\mathscr{B} \subseteq L^Y$. Also, let $f^{-1}(\mathscr{B}) = \{f^{\leftarrow}(B) : B \in \mathscr{B}\}$ and $C \in L^Y$. Then $st(f^{\leftarrow}(C), f^{-1}(\mathscr{B})) \subseteq f^{\leftarrow}(st(C, \mathscr{B}))$.*

Remark 2.8. Let $\mathscr{A}$ and $\mathscr{B}$ be two L-covers of L^X such that $st(\mathscr{A}) \preccurlyeq \mathscr{B}$, then $st(A, st(\mathscr{A}) \subseteq st(A, \mathscr{B}) \ \ \forall \ A \in L^X$.

Definition 2.9 ([17])**.** A pair $(L^X, \mathfrak{U})$, consisting of L^X and a non-empty family $\mathfrak{U}$ of L-covers of L^X, is said to be a *covering L-locally uniform space*, whenever the following conditions are satisfied:
(lc1) $\mathscr{A} \preccurlyeq \mathscr{B}, \mathscr{A} \in \mathfrak{U} \Rightarrow \mathscr{B} \in \mathfrak{U}$.
(lc2) For every $\mathscr{A}, \mathscr{B} \in \mathfrak{U}, \mathscr{A} \bigcap \mathscr{B} \in \mathfrak{U}$.
(lc3) For each $x_\alpha \in L^X$ and $\mathscr{A} \in \mathfrak{U}$, there exits $\mathscr{B} \in \mathfrak{U}$ such that

$$st(x_\alpha, st(\mathscr{B})) \subseteq st(x_\alpha, \mathscr{A}).$$

Theorem 2.10 ([19])**.** *A mapping $P : L^X \times L^X \rightarrow [0, +\infty]$ is an L-pseudo quasi-metric(L-pseudo metric, respectively) on L^X if P satisfies the following conditions (SEMI1)-(SEM3)((SEM1)-(SEM4) respectively:*
(SEM1) $B \subseteq A \Rightarrow P(A, B) = \underline{0}, \ B \neq \underline{0} \Rightarrow P(\underline{0}, B) = +\infty$.
(SEM2) $P(A, B) \leq P(A, C) + P(C, B)$.
(SEM3) $A, B \neq \underline{0} \Rightarrow P(A, B) = \cup_{x_\alpha \in \beta^*(B)} \cap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha)$.
(SEM4) "$P(A, C) < r \Rightarrow C \subseteq B$" $\Leftrightarrow$ "$P(B', D) < r \Rightarrow D \subseteq A'$".

Definition 2.11 ([14])**.** A mapping $P : L^X \times L^X \rightarrow [0, +\infty]$ is called an *L-semi-pseudo-metric* on L^X, if P satisfies the axioms (SEM1),(SEM3), (SEM4) and the following:
(SEM5) $A \subseteq B \Rightarrow P(B, C) \leq P(A, C)$.
The pair (L^X, P) is called an *L-semi-pseudo-metric space*.

Definition 2.12. Let P be a L-semi-pseudo-metric on L^X. The for any $x_\alpha \in L^X$ and $\epsilon > 0$, $B_\epsilon(x_\alpha) = \bigcup\{y_\beta : d(x_\alpha, y\beta) < \epsilon$ is a fuzzy set, which is called an $\epsilon-$*open batt* of x_α.

3. Covering L-Semi-uniform structure

Definition 3.1. A non-empty family $\mathfrak{S}$ of $L-$covers of L^X is said to be *covering semi-uniform spaces* (in short, CLS-uniform space), if it satisfies the following conditions:
(sc1) $\mathscr{A} \preccurlyeq \mathscr{B}, \mathscr{A} \in \mathfrak{S} \Rightarrow \mathscr{B} \in \mathfrak{S}$.
(sc2) For every $\mathscr{A}, \mathscr{B} \in \mathfrak{S}, \mathscr{A} \bigcap \mathscr{B} \in \mathfrak{S}$.
A CLS-uniform space will be denoted by $(L^X, \mathfrak{S})$.

Definition 3.2. A non-empty sub-family $\mathfrak{B}$ of $\mathfrak{S}$ is called a *base* for CLS-uniform space on L^X, if for any $\mathscr{S} \in \mathfrak{S}$, there is $\mathscr{B} \in \mathfrak{S}$ such that $\mathscr{B} \preccurlyeq \mathscr{S}$.

Theorem 3.3. *Every covering L-locally uniform space is a covering CLS-uniform space.*

Proof. It follows from the Definition 2.9. □

Converse of above theorem is not true, for this we cite the following example.

Example 3.4. Let $X = \{a, b, c\}$ with $L = [0, 1]$. Consider $\mathscr{A} = \{\{a\}, \{b\}, \{a, b\}\{c\}\}$ and $\mathscr{B} = \{\{a\}, \{a, b\}, \{b, c\}, \{c\}\}$ are L-covers, then $\mathfrak{B} = \{\mathscr{A}, \mathscr{B}\}$ is a base for CLS-uniform space. But for a and $\mathscr{A}$, there is no $\mathscr{B}$ such that $st(a, st(\mathscr{B})) \subseteq st(a, \mathscr{A})$.

From theorem 3.3, and example 3.4, we can conclude that every CLS-unform spaces is generalisation of covering L-locally uniform spaces form our previous paper [17].

Lemma 3.5. *Let* $(L^X, \mathfrak{S})$ *be an CLS-uniform space. The mapping* $int : L^X \to L^X$ *defined by*

$$int(A) = \bigcup\{x_\alpha : st(x_\alpha, \mathscr{C}) \subseteq A,\ for\ some\ \mathscr{C} \in \mathfrak{S}\}$$

is an interior operator on L^X

Proof. (io1) Clearly, $int(\underline{1}) = \underline{1}$.
(io2) $int(A) \subseteq A$.
(io3) By (sc2), we have $int(A \bigcap B) = int(A) \bigcap int(B)$. □

Every CLS-uniform spaces generates an interior spaces. Generated interior space of CLS-Uniform space may not be topological interior for this we cite the following example.

Example 3.6. Let $X = \{a, b, c\}$ and $L = [0, 1]$ Then the collection $\mathfrak{S} = \{\mathscr{A} = \{\{a, b\}, \{b, c\}\}, \mathscr{B} = \{a, b, c\}\}$ is CLS-Uniform space. Here $A = \{\{a, b\}\}$. Thus $int(A) = \{a\}$ and $int(int(A)) = \underline{0}$, and clearly, $int(A) \neq int(int(A))$.

Theorem 3.7. *Let* $(L^X, \mathfrak{S})$ *be an CLS-uniform space.Then the required condition for generated interior space to be topological* $\forall \mathscr{A} \in \mathfrak{S}$ *and* $\forall x_\alpha$ *there exits* $\mathscr{B} \in \mathfrak{S}$ *such that* $\forall y_\beta \in st(x_\alpha, \mathscr{B})$ *there corresponds* $\mathscr{C} \in \mathfrak{S}$ *with* $st(y_\beta, \mathscr{C}) \subseteq st(x_\alpha, \mathscr{A})$.

Proof. Let $x_\alpha \in int(A)$. Then there exist some $\mathscr{A} \in \mathfrak{S}$ such that $st(x_\alpha, \mathscr{A}) \subseteq A$. Let $\mathscr{B} \in \mathfrak{S}$ such that for any $y_\beta \in st(x_\alpha, \mathscr{B})$, there is $\mathscr{C} \in \mathfrak{S}$ such that $st(y_\beta, \mathscr{C}) \subseteq st(x_\alpha, \mathscr{A})$. Since $x_\alpha \in st(x_\alpha, \mathscr{B})$, we may choose $\mathscr{C} \in \mathfrak{S}$ such that $st(x_\alpha, \mathscr{C}) \subseteq st(x_\alpha, \mathscr{A})$. This implies $x_\alpha \in int(int(A))$ and since the other inclusion follows by (io2) in lemma 3.5, we have $int(A) = int(int(A))$. □

Theorem 3.8. *An CLS- uniform spaces with the condition in Theorem 3.7 generates an L-topological spaces.*

Proof. It follows from Lemma 3.5 and Theorem 3.7. □

The L-topology induced by an CLS-uniform spaces is denoted by $\mathbb{F}(\mathfrak{S})$. We can conclude that the topological CLS-uniform spaces lies between CLS-uniform space and covering L-locally uniform spaces.

Theorem 3.9. *Let $(L^X, \mathfrak{S}_1)$ and $(L^X, \mathfrak{S}_2)$ be two CLS-uniform spaces. If $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, then $\mathbb{F}(\mathfrak{S}_1) \subseteq \mathbb{F}(\mathfrak{S}_2)$.*

Proof. Straightforward. □

Theorem 3.10. *Let $(L^X, \mathfrak{S})$ be an CLS-uniform spaces. Then $\{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{S}\}$ is a base for nbds(=neighborhoods) of x_α in interior spaces*

Proof. Suppose $G \in L^X$ is open and $x_\alpha \in G$. Since $int(G) = G$, there exits $\mathscr{A} \in \mathfrak{S}$ such that $st(x_\alpha, \mathscr{A}) \subseteq G$. Then $\{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{S}\}$ is a base for nbds of x_α. □

4. Uniform Continuous

Definition 4.1. Let $(L^X, \mathfrak{S}_1)$ and $(L^Y, \mathfrak{S}_2)$ be two CLS-uniform spaces. Then $f^{\rightarrow} : L^X \to L^Y$ is said to be *semi-uniformly continuous*, if $f^{-1}(\mathscr{B}) \in \mathfrak{S}_1$ for each $\mathscr{B} \in \mathfrak{S}_2$, where $f^{-1}(\mathscr{B}) = \{f^{\leftarrow}(B) : B \in \mathfrak{B}\}$.

Theorem 4.2. *Let $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \to (L^Y, \mathfrak{S}_2)$ be semi-uniformly continuous is continuous. Then $f^{\rightarrow} : (L^X, \mathbb{F}(\mathfrak{S}_1)) \to (L^Y, \mathbb{F}(\mathfrak{S}_2))$ is continuous function.*

Proof. Let $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \to (L^Y, \mathfrak{S}_2)$ be a semi-uniformly continuous function. Let $A \in L^Y$. Then $int(A) = \bigcup\{x_\alpha : st(x_\alpha, \mathscr{A}) \subseteq A\}$ for some $\mathscr{A} \in \mathfrak{S}_2$. Since $f^{\leftarrow}$ is arbitrary join preserving, then by Theorem 2.1.17 (i) in [19], we have

$$f^{\leftarrow}(int(A)) = \bigcup\{f^{\leftarrow}(x_\alpha) : st(x_\alpha, \mathscr{A}) \subseteq A \ for \ some \ \mathscr{A} \in \mathfrak{S}_2\}. \tag{4.1}$$

Also since $f^{\leftarrow}$ is order preserving, we have

$$st(x_\alpha, \mathscr{A}) \subseteq A \Rightarrow f^{\leftarrow}(st(x_\alpha, \mathscr{A})) \subseteq f^{\leftarrow}(A). \tag{4.2}$$

By the Theorem 2.7 (6) and (4.2), we have

$$st(f^{\leftarrow}(x_\alpha), f^{-1}(\mathscr{A})) \subseteq f^{\leftarrow}(st(x_\alpha, \mathscr{A})) \subseteq f^{\leftarrow}(A). \tag{4.3}$$

Again from (4.1), we have

$$f^{\leftarrow}(int(A)) = \bigcup\{f^{\leftarrow}(x_\alpha) : (st(f^{\leftarrow}(x_\alpha), f^{-1}(\mathscr{A})) \subseteq f^{\leftarrow}(A) \ for \ some \ \mathscr{A} \in \mathfrak{S}_2\}. \tag{4.4}$$

Since $f^{\rightarrow}$ is semi-uniformly continuous, $f^{-1}(\mathscr{A}) \in \mathfrak{S}_1$. Then by (4.4), $f^{\leftarrow}(int(A)) \subseteq int(f^{\leftarrow}(int(A)))$ implies $f^{\leftarrow}(int(A)) \in \mathbb{F}(\mathfrak{S}_1)$. Thus the theorem holds. □

Theorem 4.3. *The composition of semi-uniformly continuous function is semi-uniformly continuous.*

Proof. Let $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ and $g^{\rightarrow} : (L^Y, \mathfrak{S}_2) \rightarrow (L^Z, \mathfrak{S}_3)$ be two semi-uniformly continuous functions. Let $\mathscr{C} \in \mathfrak{S}_3$. Then by the Theorem 2.1.23 (ii) in [19], we have $(g \circ f)^{-1}(\mathscr{C}) = f^{-1}(g^{-1}(\mathscr{C}))$. Since $g^{\rightarrow}$ is semi-uniformly continuous $g^{-1}(\mathscr{C}) \in \mathfrak{S}_2$. Also f^{-1} is uniformly continuous implies $f^{-1}(g^{-1}(\mathscr{C})) \in \mathfrak{S}_1$. Thus $(g \circ f)^{\rightarrow}$ is semi-uniformly continuous. □

5. Metrization

The problem of metrization has occupied an important place in the study of uniform spaces. Having developed the theory of CLS-uniform spaces, we proceed to discuss the problem of metrization(semi-pseudo-metrization) in the same context.

Theorem 5.1. *Every L-semi-pseudo metric generates a CLS-unform space.*

Proof. let (L^X, P) be an L-semi-pseudo-metric space and for any $s > 0$, let $\mathscr{U}_s$ be an L-cover of L^X such that $\mathscr{U}_s = \{B_\epsilon(x_\alpha) : x_\alpha \in L^X\}$. Then clearly, $\mathscr{U}_{\frac{1}{2}s} \preccurlyeq \mathscr{U}_s$ and $\mathscr{U}_s \bigcap \mathscr{U}_t \preccurlyeq \mathscr{U}_{\max[s,t]}$. Thus $\psi(P) = \{\mathscr{U}_s : s > 0\}$ is a base for CLS-uniformity. □

Definition 5.2. We say that a CLS-uniform pace $(L^X \mathfrak{S})$ is *L-semi-pseudo-metrizable*, if there is an L-semi-pseudo-metric that generates $\mathfrak{S}$.

We now proceed the following theorem which is the main result of the section.

Definition 5.3. A CLS-uniform space is said to be *L-semi-pseudo-metrizable*, if it is induced by a L-semi-pseudo-metric.

Lemma 5.4. *Let $(L^X, \mathfrak{S})$ be a CLS-uniform space. For $\mathscr{C} \in \mathfrak{S}$ define a mapping $\psi(\mathscr{C}) : L^X \rightarrow L^X$ such that $[\psi(\mathscr{C})](A) = st(A, \mathscr{C})$. Then $[\psi(\mathscr{C})](\bigcup_i A_i) = st(\bigcap_i A_i, \mathscr{C}) = \bigcup_i [\psi(\mathscr{C})](A_i)$.*

Theorem 5.5. *A CLS-uniform space is $L-$semi-pseudo-metrizable, if it has a countable base.*

Proof. Let $\{\mathscr{C}_n : n \in N\}$ be a base for CLS-uniform space $(L^X, \mathfrak{S})$. Without lost of generality, we can assume $\mathscr{C}_{n+1} \preccurlyeq \mathscr{C}_n$ for each $n \in \mathbb{N}$. For any $r > 0$, let $\psi_r : L^X \rightarrow L^X$ be a mapping defined by

$$\forall A \in L^X, \text{ if } \tfrac{1}{2^n} < r \leq \tfrac{1}{2^{n-1}}, \text{ then } [\psi_r(\mathscr{C}_n)](A) = st(A, \mathscr{C}_n)$$

and if $[\psi_r(\mathscr{C}_n)](A) = \underline{1}$ or $\underline{0}$ according $A \neq \underline{0}$ or $A = \underline{0}$.

For every $r > 0$, let $\mathscr{F}_r : L^X \rightarrow L^X$ be a mapping defined by

$$\mathscr{F}_r = \{\psi_{r_k} : \sum_{i=0}^{k} r_k = r, \forall i \leq k, r_i > 0, k < \omega\}.$$

Then clearly, $\{\mathscr{F}_r : r > 0\}$ is a base for $\mathfrak{S}$ and also define

$$\mathscr{F}_r(A) = \bigcup_{x_\alpha \in A} st(x_\alpha, \mathscr{F}_r) \text{ for all } A \in L^X.$$

Let $P : L^X \times L^X \to [0, \infty]$ be a mapping defined by:

$$P(A, B) = \bigwedge\{r : B \subseteq \mathscr{F}_r(A)\},$$

where we assume that $\bigwedge \Phi = +\infty$.
We claim that P is the required L-semi-pseudo-metric that generates $\mathfrak{S}$.

(SEM1) By Theorem 2.10, P fulfils (SEM1).

(SEM3) By Theorem 1.3.24 (ii) in [19], we have $\beta^*[\mathscr{F}_r] = \bigcup_{A\in\mathscr{F}_r} \beta^*(A)$.
Now for any arbitrary $A,\ B \neq \underline{0}$, by assumption, we get

$$\begin{aligned} P(A, B) < r &\Longrightarrow B \subseteq \mathscr{F}_r(A) \\ &\Longrightarrow \forall x_\alpha \in \beta^*(B), x_\alpha \in \beta^*(\mathscr{F}_r(A)) \\ &\Longrightarrow \forall x_\beta \in \beta^*(B), \exists y_\beta \in \beta^*(A), x_\alpha \in \beta^*(\mathscr{F}_r(y_\beta)) \\ &\Longrightarrow x_\alpha \in \beta^*(B), \exists y_\beta \in \beta^*(A), P(y_\beta, x_\alpha) < r \\ &\Longrightarrow \textstyle\bigcup_{x_\alpha\in\beta^*(B)} \bigcap_{y_\beta\in\beta^*(A)} P(y_\beta, x_\alpha) < r. \end{aligned}$$

Again suppose that $\bigcup_{x_\alpha\in\beta^*(B)} \bigcap_{y_\beta\in\beta^*(A)} P(y_\beta, x_\alpha) < r$. Then

$$\forall x_\alpha \in \beta^*(B), \exists y_\beta(x_\alpha) \in \beta^*(A), P(y_\beta(x_\alpha), x_\alpha) < r,$$

where $y_\beta(x_\alpha)$ is an L-fuzzy point corresponding to the L-fuzzy point x_α.
Thus we have

$$\begin{aligned} & x_\alpha \in \beta^*(B), \exists y_\beta(x_\alpha) \in \beta^*(A), x_\alpha \subseteq \mathscr{F}_r(y_\beta(x_\alpha)) \\ \Longrightarrow\ & B = \textstyle\bigcup \beta^*(B) \subseteq \bigcup_{x_\alpha\in\beta^*(B)} \mathscr{F}_r(y_\beta(x_\alpha)) \\ \Longrightarrow\ & \mathscr{F}_r(\textstyle\bigcup_{x_\alpha\in\beta^*(B)} y_\beta(x_\alpha)) \subseteq \mathscr{F}_r(\bigcup \beta^*(A)) = \mathscr{F}_r(A). \end{aligned}$$

Which implies $P(A, B) < r$. So we get

$$A, B \neq \underline{0} \Rightarrow P(A, B) = \bigcap_{x_\alpha\in\beta^*(B)} \bigcap_{y_\beta\in\beta^*(A)} P(y_\beta, x_\alpha).$$

(SEM4) Suppose
$$\begin{aligned} \mathscr{F}_r(A) \subseteq B &\Longleftrightarrow \textstyle\bigcup\{C : P(A, C) < r\} \subseteq B \\ &\Longleftrightarrow P(A, C) \Rightarrow C \subseteq B \\ &\Longleftrightarrow P(B', D) < r \Rightarrow D \subseteq A' \\ &\Longleftrightarrow \textstyle\bigcup\{D : P(B', D) < r\} \subseteq A' \\ &\Longleftrightarrow \mathscr{F}_r(B') \subseteq A'. \end{aligned}$$

Which implies "$P(A, C) < r \Rightarrow C \subseteq B$" $\Longleftrightarrow$ "$P(B', D) < r \Rightarrow D \subseteq A'$".

For any $x_\alpha \in L^X$, they have same neigbourhood at x_α viz,

$$\{\psi_r[\mathscr{C}_n](x_\alpha)r > 0\} = \{\mathscr{F}_r(x_\alpha) : r > 0\}.$$

Which implies they induced the same interior operator.

Conversely, suppose $\mathfrak{S}$ is a CLS-uniform space generated by L-semi-pseudo-metric P. For any $s > 0$, let $\mathscr{U}_s$ be an L-cover of L^X such that $\mathscr{U}_s = \{B_\epsilon(x_\alpha) : x_\alpha \in L^X\}$. Then clearly, $\mathscr{U}_{\frac{1}{2}s} \preccurlyeq \mathscr{U}_s$ and $\mathscr{U}_s \bigcap \mathscr{U}_t \preccurlyeq \mathscr{U}_{\max[s,t]}$. Thus $\psi(P) = \{\mathscr{U}_s : s > 0\}$ is a base for CLS-uniformity. □

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