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Quenching for discretization of a nonlinear diffusion equation with singular boundary flux

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Abstract. In this paper, we study the discrete approximation for the following nonlinear diffusion equation with nonlinear source and singular boundary flux

$$\begin{cases} \dfrac{\partial A(u)}{\partial t} = u_{xx} + (1-u)^{-\alpha}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = -B(u(1,t)), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases}$$

with $\alpha > 0$.
We find some conditions under which the solution of a discrete form of above problem quenches in a finite time and estimate its discrete quenching time. We also establish the convergence of the discrete quenching time to the theoretical one when the mesh size tends to zero. Finally, we give some numerical experiments for a best illustration of our analysis.

2020 AMS Classification: 35B50, 35B51, 35K55, 65M06

Keywords: Nonlinear diffusion equation, Numerical quenching, Singular boundary, Discretization, Convergence, Finite difference method.

Corresponding Author: N'Guessan Koffi (nkrasoft@yahoo.fr)

1. Introduction

In this paper, we consider the nonlinear diffusion equation with nonlinear source and singular boundary flux

$$\frac{\partial A(u)}{\partial t} = u_{xx} + (1-u)^{-\alpha}, \quad 0 < x < 1, \quad t > 0, \tag{1.1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -B(u(1,t)), \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le 1, \tag{1.3}$$

where $A(s)$ is an appropriately smooth function which satisfies

$$A(0) = 0,\ A(1) = 1,\ A'(s) > 0,\ A''(s) \leq 0\ \forall s > 0,$$

$B(s)$ satisfies

$$B(s) > 0,\ B'(s) < 0,\ B''(s) \geq 0, \text{ for } s > 0,\ \lim_{s\to 0^+} B(s) = +\infty$$

and $u_0 : [0,1] \longrightarrow (0,1)$ is nonincreasing and satisfies some compatibility conditions and α is a nonnegative constant.

Definition 1.1. We say that the solution u of (1.1)–(1.3) quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty < 1$ for $t \in [0, T_q)$, but

$$\lim_{t\to T_q} \|u(.,t)\|_\infty = 1,$$

where $\|u(.,t)\|_\infty = \max_{0\leq x\leq 1} |u(x,t)|$. The time T_q is called quenching time of the solution u.

When $A(u) = u^m$, the problem (1.1)–(1.3) is known as the classical porous medium equation which shows a number of physical phenomenon in the nature such as the flow of an isentropic gas through a porous medium [1] and heat transfer or diffusion [2] .
The problem (1.1)–(1.3) may be rewritten in the following model

$$u_t = \gamma(u)u_{xx} + \gamma(u)(1-u)^{-\alpha}, \quad 0 < x < 1, \quad t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = -B(u(1,t)), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1, \tag{1.6}$$

where $\gamma(u) = \dfrac{1}{A'(u)}$.

In recent years, the theoretical study of quenching phenomenon for semilinear parabolic equations has been carried out by many researchers (See [2, 3, 4, 5] and references therein). Local in time existence and uniqueness of the solution have been proved (See [3, 4]). Concerning problem (1.1)–(1.3), the author in [5] shows that the solution u of (1.1)–(1.3) quenches in finite time T_q and $x = 0$ is the unique quenching point. He also shows that the time derivative u_t blow-up at the quenching point and he gives a lower bound of the quenching time.

In this paper, we deal with a numerical study using a discrete form obtained by the finite difference method. For previous study on numerical approximations of parabolic system we refer to [6, 7, 8].

In the next section, we present a discrete scheme of (1.4)–(1.6) and give some properties of the discrete solution. In the third section, we prove that the solution of the discrete form of (1.4)–(1.6) quenches in a finite time and we give a estimation of the discrete quenching time. In the fourth section, we study the convergence of the discrete quenching time. In last section, we give some numerical results.

2. Properties of the discrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the discrete scheme. Let $I \geq 3$ be a nonnegative integer and let $h = \frac{1}{I}$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution u of (1.4)–(1.6) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, ..., U_I^{(n)})^T$ and the initial condition u_0 by the initial condition $\varphi_h = (\varphi_0, \varphi_1, ..., \varphi_I)^T$ of the following discrete equations

$$\delta_t U_i^{(n)} = \gamma(U_i^{(n)})\delta^2 U_i^{(n)} + \gamma(U_i^{(n)})(1 - U_i^{(n)})^{-\alpha}, \quad 0 \leq i \leq I-1, \tag{2.1}$$

$$\delta_t U_I^{(n)} = \gamma(U_I^{(n)})\delta^2 U_I^{(n)} - \frac{2\gamma(U_I^{(n)})B(U_I^{(n)})}{h} + \gamma(U_I^{(n)})(1 - U_I^{(n)})^{-\alpha}, \tag{2.2}$$

$$U_i^{(0)} = \varphi_i, \quad 0 \leq i \leq I, \tag{2.3}$$

where

$$n \geq 0, \quad \alpha > 0,$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 0 \leq i \leq I,$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}, \quad \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2},$$

$$\varphi_i > 0, \quad 0 \leq i \leq I,$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}, \quad \delta^+ \varphi_i < 0, \quad 0 \leq i \leq I-1.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches to the quenching time T_q, we need to adapt the size of the time step. We choose

$$\Delta t_n = \min\left\{\frac{h^2}{2}, \tau(1 - \|U_h^{(n)}\|_\infty)^{\alpha+1}\right\} \text{ with } \tau \in (0,1) \text{ and } \|U_h^{(n)}\|_\infty = \max_{0 \leq i \leq I} |U_i^{(n)}|.$$

Lemma 2.1. *Let $b_h^{(n)}$, $V_h^{(n)}$ and $f_h^{(n)}$ be three sequences with $n \geq 0$, $f_h^{(n)} > 0$ and $b_h^{(n)} \leq 0$ such that for $0 \leq i \leq I$*

$$\delta_t V_i^{(n)} - f_i^{(n)} \delta^2 V_i^{(n)} + b_i^{(n)} V_i^{(n)} \geq 0, \tag{2.4}$$

$$V_i^{(0)} \geq 0. \tag{2.5}$$

Then we have

$$V_i^{(n)} \geq 0,\ 0 \leq i \leq I,\ n \geq 0 \text{ when } \Delta t_n \leq \frac{h^2}{2\|f_h^{(n)}\|_\infty}.$$

Proof. A straightforward computation shows that

$$V_0^{(n+1)} \geq \left(1 - 2\frac{\Delta t_n \|f_h^{(n)}\|_\infty}{h^2}\right) V_0^{(n)} + \frac{2\Delta t_n \|f_h^{(n)}\|_\infty}{h^2} V_1^{(n)} - \Delta t_n b_0^{(n)} V_0^{(n)},$$

$$V_i^{(n+1)} \geq \left(1 - 2\frac{\Delta t_n \|f_h^{(n)}\|_\infty}{h^2}\right) V_i^{(n)} + \frac{\Delta t_n \|f_h^{(n)}\|_\infty}{h^2} \left(V_{i+1}^{(n)} + V_{i-1}^{(n)}\right) - \Delta t_n b_i^{(n)} V_i^{(n)},$$

$1 \leq i \leq I-1$,

$$V_I^{(n+1)} \geq \left(1 - 2\frac{\Delta t_n \|f_h^{(n)}\|_\infty}{h^2}\right) V_I^{(n)} + \frac{2\Delta t_n \|f_h^{(n)}\|_\infty}{h^2} V_{I-1}^{(n)} - \Delta t_n b_I^{(n)} V_I^{(n)}.$$

If $V_h^{(n)} \geq 0$, then using an argument of recursion, we easily see that

$$V_h^{(n+1)} \geq 0,$$

because $-b_h^{(n)} \geq 0$ and $\Delta t_n \leq \dfrac{h^2}{2\|f_h^{(n)}\|_\infty}$. This end the proof. □

Lemma 2.2. *Let $g_h^{(n)}$, $V_h^{(n)}$ and $W_h^{(n)}$ be three sequences, with $n \geq 0$ and $g_h^{(n)} \leq 0$, such that for $0 \leq i \leq I$*

$$\delta_t V_i^{(n)} - \gamma(V_i^{(n)})\delta^2 V_i^{(n)} + g_i^{(n)} V_i^{(n)} \leq \delta_t W_i^{(n)} - \gamma(W_i^{(n)})\delta^2 W_i^{(n)} + g_i^{(n)} W_i^{(n)},$$

$$V_i^{(0)} \leq W_i^{(0)}.$$

Then we have

$$V_i^{(n)} \leq W_i^{(n)},\ 0 \leq i \leq I,\ n \geq 0.$$

Proof. Define the vector $Z_h^{(n)} = W_h^{(n)} - V_h^{(n)}$. For $0 \leq i \leq I$, a straightforward calculation gives

$$\delta_t Z_i^{(n)} - \gamma(V_i^{(n)})\delta^2 Z_i^{(n)} + \left(g_i^{(n)} - \gamma'(\theta_i^{(n)})\delta^2 W_i^{(n)}\right) Z_i^{(n)} \geq 0.$$

Where $\theta_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $W_i^{(n)}$, $0 \leq i \leq I$. Knowing that $Z_h^{(0)} \geq 0$, from Lemma (2.1), we have $Z_h^{(n)} \geq 0$. □

Lemma 2.3. *Let $g_h^{(n)}$, $V_h^{(n)}$ and $W_h^{(n)}$ be three sequences, with $n \geq 0$ and $g_h^{(n)} \leq 0$, such that for $0 \leq i \leq I$,*

$$\delta_t V_i^{(n)} - \gamma(V_i^{(n)})\delta^2 V_i^{(n)} + g_i^{(n)} V_i^{(n)} < \delta_t W_i^{(n)} - \gamma(W_i^{(n)})\delta^2 W_i^{(n)} + g_i^{(n)} W_i^{(n)},$$

$$V_i^{(0)} < W_i^{(0)}.$$

Then we have

$$V_i^{(n)} < W_i^{(n)},\ 0 \leq i \leq I,\ n \geq 0.$$

Proof. Define the vector $Z_h^{(n)} = W_h^{(n)} - V_h^{(n)}$. For $0 \le i \le I$, a straightforward calculation gives

$$\delta_t Z_i^{(n)} - \gamma(V_i^{(n)})\delta^2 Z_i^{(n)} + \left(g_i^{(n)} - \gamma'(\theta_i^{(n)})\delta^2 W_i^{(n)}\right) Z_i^{(n)} > 0.$$

Where $\theta_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $W_i^{(n)}$, $0 \le i \le I$. Knowing that $Z_h^{(0)} > 0$, from Lemma (2.1), we have $Z_h^{(n)} > 0$. □

Lemma 2.4. *Let $U_h^{(n)}$, $n \ge 0$ be a sequence such that $\|U_h^{(n)}\|_\infty < 1$. Then we have*

$$\delta_t(1 - U_i^{(n)})^{-\alpha} \ge \alpha(1 - U_i^{(n)})^{-\alpha-1}\delta_t U_i^{(n)}, \quad 0 \le i \le I.$$

Proof. Using Taylor's expansion, we get

$$\delta_t(1 - U_i^{(n)})^{-\alpha}$$
$$= \alpha(1 - U_i^{(n)})^{-\alpha-1}\delta_t U_i^{(n)} + \frac{\alpha(\alpha+1)}{2}\Delta t_n(1 - \theta_i^{(n)})^{-\alpha-2}(\delta_t U_i^{(n)})^2, 0 \le i \le I,$$

where $\theta_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_i^{(n+1)}$, $0 \le i \le I$. We use the fact that $\|U_h^{(n)}\|_\infty < 1$, $n \ge 0$ to complete the proof. □

Lemma 2.5. *Let $U_h^{(n)}$, $n \ge 0$, be the solution of the discrete problem* (2.1)–(2.3) *Then*

$$\delta_t U_i^{(n)} \ge 0, \quad 0 \le i \le I.$$

Proof. Consider the vector $Z_h^{(n)}$ such that $Z_i^{(n)} = \delta_t U_i^{(n)}$, $0 \le i \le I$. Then by using Lemma (2.4), a straightforward calculation gives

$$\delta_t Z_i^{(n)} - \gamma(U_i^{(n)})\delta^2 Z_i^{(n)} - \alpha\gamma(U_i^{(n)})(1 - U_i^{(n)})^{-\alpha-1} Z_i^{(n)} - \gamma'(U_i^{(n)}) Z_i^{(n)}\delta^2 U_i^{(n)}$$
$$-\gamma'(U_i^{(n)})(1 - U_i^{(n)})^{-\alpha} Z_i^{(n)} \ge 0,\ 0 \le i \le I-1,$$

$$\delta_t Z_I^{(n)} - \gamma(U_I^{(n)})\delta^2 Z_I^{(n)} - \alpha\gamma(U_I^{(n)})(1 - U_I^{(n)})^{-\alpha-1} Z_I^{(n)} - \gamma'(U_I^{(n)})[\delta^2 U_I^{(n)}$$
$$+(1 - U_I^{(n)})^{-\alpha}] Z_I^{(n)} + \frac{2}{h}[\gamma'(U_I^{(n)})B(U_I^{(n)}) + \gamma(U_I^{(n)})B'(U_I^{(n)})] Z_I^{(n)} \ge 0.$$

Since $Z_h^{(0)} \ge 0$, from Lemma (2.1), we have $Z_h^{(n)} \ge 0$. Thus we get $\delta_t U_i^{(n)} \ge 0$, $0 \le i \le I$. □

Lemma 2.6. *Let $U_h^{(n)}$, $n \ge 0$ be the solution of the discrete problem* (2.1)–(2.3). *Then*

$$U_{i+1}^{(n)} < U_i^{(n)}, \quad 0 \le i \le I-1. \tag{2.6}$$

Proof. Define the vector $Z_i^{(n)}$ such that $Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)}$, $0 \le i \le I-1$. Then we have

$$Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)}, \quad 0 \le i \le I-2,$$

$$Z_{I-1}^{(n)} = U_{I-1}^{(n)} - U_I^{(n)}.$$

By a straightforward computations, we have

$$\delta_t Z_i^{(n)} = \gamma(U_{i+1}^{(n)})\delta^2 Z_i^{(n)} + \gamma'(\omega_i^{(n)}) Z_i^{(n)}\delta^2 U_i^{(n)} + \gamma'(\omega_i^{(n)})(1 - U_i^{(n)})^{-\alpha} Z_i^{(n)}$$
$$+\alpha\gamma(U_{i+1}^{(n)})(1 - \zeta_i^{(n)})^{-\alpha-1} Z_i^{(n)},\ 0 \le i \le I-2,$$

$$\delta_t Z_{I-1}^{(n)} = \gamma(U_I^{(n)})\delta^2 Z_{I-1}^{(n)} + \gamma'(\omega_{I-1}^{(n)}) Z_{I-1}^{(n)} \delta^2 U_{I-1}^{(n)} + \gamma'(\omega_{I-1}^{(n)})(1-U_{I-1}^{(n)})^{-\alpha} Z_{I-1}^{(n)}$$
$$+\alpha\gamma(U_I^{(n)})(1-\zeta_{I-1}^{(n)})^{-\alpha-1} Z_{I-1}^{(n)} + \frac{2\gamma(U_I^{(n)})B(U_I^{(n)})}{h}.$$

Where $\zeta_i^{(n)}$ and $\omega_i^{(n)}$ are an intermediate values between $U_i^{(n)}$ and $U_{i+1}^{(n)}$, $0 \le i \le I-1$. Knowing that $Z_h^{(0)} > 0$, from Lemma (2.1), we have $Z_h^{(n)} > 0$, which implies that $U_{i+1}^{(n)} < U_i^{(n)}$, $0 \le i \le I-1$. □

3. Quenching in the discrete problem

In this section, under some assumptions, we show that the solution $U_h^{(n)}$ of the discrete problem (2.1)–(2.3) quenches in a finite time and estimate its numerical quenching time. Now let us set $V_h^{(n)} = 1 - U_h^{(n)}$. The problem (2.1)–(2.3) is equivalent to

$$\delta_t V_i^{(n)} = \gamma(1-V_i^{(n)})\delta^2 V_i^{(n)} - \gamma(1-V_i^{(n)})(V_i^{(n)})^{-\alpha}, \quad 0 \le i \le I-1, \tag{3.1}$$

$$\delta_t V_I^{(n)} = \gamma(1-V_I^{(n)})\delta^2 V_I^{(n)} + \frac{2}{h}\gamma(1-V_I^{(n)})B(1-V_I^{(n)}) - \gamma(1-V_I^{(n)})(V_I^{(n)})^{-\alpha}, \tag{3.2}$$

$$V_i^{(0)} = \phi_i = 1 - \varphi_i, \quad 0 \le i \le I, \tag{3.3}$$

where

$$n \ge 0, \quad \alpha > 0.$$

Lemma 3.1. *Let $V_h^{(n)}$, $n \ge 0$ be a sequence such that $\|V_h^{(n)}\|_{\inf} > 0$. Then we have*

$$\delta_t (V_i^{(n)})^{-\alpha} \ge -\alpha (V_i^{(n)})^{-\alpha-1} \delta_t V_i^{(n)}, \quad 0 \le i \le I.$$

Proof. Using Taylor's expansion, we get

$$\delta_t (V_i^{(n)})^{-\alpha} = -\alpha (V_i^{(n)})^{-\alpha-1} \delta_t V_i^{(n)} + \frac{\alpha(\alpha+1)}{2} \Delta t_n (\theta_i^{(n)})^{-\alpha-2} (\delta_t V_i^{(n)})^2, 0 \le i \le I,$$

where $\theta_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $V_i^{(n+1)}$, $0 \le i \le I$. We use the fact that $\|V_h^{(n)}\|_{\inf} > 0$, $n \ge 0$ to complete the proof. □

Lemma 3.2. *Let $V_h^{(n)}$, $n \ge 0$ be a sequence such that $\|V_h^{(n)}\|_{\inf} > 0$. Then we have*

$$\delta^2 (V_i^{(n)})^{-\alpha} \ge -\alpha (V_i^{(n)})^{-\alpha-1} \delta^2 V_i^{(n)}, \quad 0 \le i \le I.$$

Proof. Applying Taylor's expansion, we obtain

$$\delta^2 (V_i^{(n)})^{-\alpha} = -\alpha (V_i^{(n)})^{-\alpha-1} \delta^2 V_i^{(n)} + (V_{i-1}^{(n)} - V_i^{(n)})^2 \frac{\alpha(\alpha+1)}{2h^2} (\theta_i^{(n)})^{-\alpha-2}$$
$$+(V_{i+1}^{(n)} - V_i^{(n)})^2 \frac{\alpha(\alpha+1)}{2h^2} (\varepsilon_i^{(n)})^{-\alpha-2}, \ 1 \le i \le I-1,$$

$$\delta^2 (V_0^{(n)})^{-\alpha} = -\alpha (V_0^{(n)})^{-\alpha-1} \delta^2 V_0^{(n)} + (V_1^{(n)} - V_0^{(n)})^2 \frac{\alpha(\alpha+1)}{h^2} (\theta_0^{(n)})^{-\alpha-2},$$

$$\delta^2 (V_I^{(n)})^{-\alpha} = -\alpha (V_I^{(n)})^{-\alpha-1} \delta^2 V_I^{(n)} + (V_{I-1}^{(n)} - V_I^{(n)})^2 \frac{\alpha(\alpha+1)}{h^2} (\theta_I^{(n)})^{-\alpha-2},$$

where $\theta_0^{(n)}$ is an intermediate value between $V_0^{(n)}$ and $V_1^{(n)}$, $\theta_i^{(n)}$ is an intermediate value between $V_{i-1}^{(n)}$ and $V_i^{(n)}$, $1 \leq i \leq I-1$, $\theta_I^{(n)}$ is an intermediate value between $V_{I-1}^{(n)}$ and $V_I^{(n)}$, $\varepsilon_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $V_{i+1}^{(n)}$, $1 \leq i \leq I-1$. Using the fact that $\|V_h^{(n)}\|_{inf} > 0$, we complete the proof. □

Theorem 3.3. *Let $U_h^{(n)}$ be the solution of (2.1)–(2.3) Suppose that there exists a constant $\lambda \in (0,1]$ such that the initial data at (3.3) satisfies*

$$\gamma(1-\phi_i)\delta^2\phi_i - \gamma(1-\phi_i)\phi_i^{-\alpha} \leq -\lambda\phi_i^{-\alpha}, \quad 0 \leq i \leq I-1, \tag{3.4}$$

$$\gamma(1-\phi_I)\delta^2\phi_I + \frac{2}{h}\gamma(1-\phi_I)B(1-\phi_I) - \gamma(1-\phi_I)\phi_I^{-\alpha} \leq -\lambda\phi_I^{-\alpha}. \tag{3.5}$$

Then $U_h^{(n)}$ quenches in a finite time $T_h^{\Delta t} = \sum_{n=0}^{+\infty} \Delta t_n$, which satisfies the estimate

$$T_h^{\Delta t} \leq \frac{\tau(1-\|\varphi_h\|_\infty)^{\alpha+1}}{1-(1-\tau')^{\alpha+1}},$$

where $\Delta t_n = \min\left\{\frac{h^2}{2}, \tau(V_{hmin}^{(n)})^{\alpha+1}\right\}$ with $\tau \in (0,1)$, $V_{hmin}^{(n)} = (1-\|U_h^{(n)}\|_\infty)$ and $\tau' = \lambda\min\{\frac{h^2(\phi_{hmin})^{-\alpha-1}}{2}, \tau\}$.

Proof. Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t(V_i^{(n)}) + \lambda(V_i^{(n)})^{-\alpha}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

A straightforward computation yields for $0 \leq i \leq I$ and $n \geq 0$,

$$\begin{aligned}\delta_t J_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 J_i^{(n)} &= \delta_t(\delta_t V_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 V_i^{(n)}) + \lambda\delta_t(V_i^{(n)})^{-\alpha} \\ &\quad -\lambda\gamma(1-V_i^{(n)})\delta^2(V_i^{(n)})^{-\alpha}.\end{aligned}$$

From (3.1)–(3.2), we arrive at

$$\begin{aligned}\delta_t J_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 J_i^{(n)} &= -\delta_t\gamma(1-V_i^{(n)})(V_i^{(n)})^{-\alpha} + \lambda\delta_t(V_i^{(n)})^{-\alpha} \\ &\quad -\lambda\gamma(1-V_i^{(n)})\delta^2(V_i^{(n)})^{-\alpha},\ 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}\delta_t J_I^{(n)} - \gamma(1-V_I^{(n)})\delta^2 J_I^{(n)} &= -\delta_t\gamma(1-V_I^{(n)})(V_I^{(n)})^{-\alpha} + \lambda\delta_t(V_I^{(n)})^{-\alpha} \\ &\quad -\lambda\gamma(1-V_I^{(n)})\delta^2(V_I^{(n)})^{-\alpha} \\ &\quad +\frac{2}{h}\delta_t\left(\gamma(1-V_I^{(n)})B(1-V_I^{(n)})\right),\end{aligned}$$

$$\begin{aligned}\delta_t J_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 J_i^{(n)} &= -(\gamma(1-V_i^{(n)}) - \lambda)\delta_t(V_i^{(n)})^{-\alpha} \\ &\quad -\lambda\gamma(1-V_i^{(n)})\delta^2(V_i^{(n)})^{-\alpha} \\ &\quad -(V_i^{(n)})^{-\alpha}\delta_t\gamma(1-V_i^{(n)}),\ 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}\delta_t J_I^{(n)} - \gamma(1-V_I^{(n)})\delta^2 J_I^{(n)} &= -(\gamma(1-V_I^{(n)}) - \lambda)\delta_t(V_I^{(n)})^{-\alpha} \\ &\quad -\lambda\gamma(1-V_I^{(n)})\delta^2(V_I^{(n)})^{-\alpha} \\ &\quad -(V_I^{(n)})^{-\alpha}\delta_t\gamma(1-V_I^{(n)})\end{aligned}$$

$$+\frac{2}{h}\delta_t\left(\gamma(1-V_I^{(n)})B(1-V_I^{(n)})\right).$$

It follows from Lemma (3.1) and Lemma (3.2) that for $n \geq 0$,

$$\begin{aligned}\delta_t J_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 J_i^{(n)} \leq\ & \alpha\gamma(1-V_i^{(n)})(V_i^{(n)})^{-\alpha-1}\delta_t V_i^{(n)}\\ & -\alpha\lambda(V_i^{(n)})^{-\alpha-1}[\delta_t V_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 V_i^{(n)}]\\ & -(V_i^{(n)})^{-\alpha}\delta_t\gamma(1-V_i^{(n)}),\ 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}\delta_t J_I^{(n)} - \gamma(1-V_I^{(n)})\delta^2 J_I^{(n)} \leq\ & \alpha\gamma(1-V_I^{(n)})(V_I^{(n)})^{-\alpha-1}\delta_t V_I^{(n)}\\ & -\alpha\lambda(V_I^{(n)})^{-\alpha-1}[\delta_t V_I^{(n)}\\ & -\gamma(1-V_I^{(n)})\delta^2 V_I^{(n)}] - (V_I^{(n)})^{-\alpha}\delta_t\gamma(1-V_I^{(n)})\\ & +\frac{2}{h}\delta_t\left(\gamma(1-V_I^{(n)})B(1-V_I^{(n)})\right),\end{aligned}$$

$$\begin{aligned}\delta_t J_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 J_i^{(n)} \leq\ & \alpha\gamma(1-V_i^{(n)})(V_i^{(n)})^{-\alpha-1}(\delta_t V_i^{(n)} + \lambda(V_i^{(n)})^{-\alpha})\\ & -(V_i^{(n)})^{-\alpha}\delta_t\gamma(1-V_i^{(n)}),\ 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}\delta_t J_I^{(n)} - \gamma(1-V_I^{(n)})\delta^2 J_I^{(n)} \leq\ & \alpha\gamma(1-V_I^{(n)})(V_I^{(n)})^{-\alpha-1}(\delta_t V_I^{(n)} + \lambda(V_I^{(n)})^{-\alpha})\\ & -(V_I^{(n)})^{-\alpha}\delta_t\gamma(1-V_I^{(n)})\\ & +\frac{2}{h}\delta_t\left(\gamma(1-V_I^{(n)})B(1-V_I^{(n)})\right).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}&\delta_t J_i^{(n)} - \gamma(1-V_i^{(n)})\delta^2 J_i^{(n)} - \alpha\gamma(1-V_i^{(n)})(V_i^{(n)})^{-\alpha-1}J_i^{(n)} + (V_i^{(n)})^{-\alpha}\delta_t\gamma(1-V_i^{(n)})\\ &\leq 0,\ 0 \leq i \leq I-1,\end{aligned}$$

$$\begin{aligned}&\delta_t J_I^{(n)} - \gamma(1-V_I^{(n)})\delta^2 J_I^{(n)} - \alpha\gamma(1-V_I^{(n)})(V_I^{(n)})^{-\alpha-1}J_I^{(n)} + (V_I^{(n)})^{-\alpha}\delta_t\gamma(1-V_I^{(n)})\\ &\quad -\frac{2}{h}\delta_t\left(\gamma(1-V_I^{(n)})B(1-V_I^{(n)})\right) \leq 0.\end{aligned}$$

Using inequalities (3.4) and (3.5), we have $J_h^{(0)} \leq 0$. Applying Lemma (2.1), we get $J_h^{(n)} \leq 0$ for $n \geq 0$, which implies that

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} \leq -\lambda(V_i^{(n)})^{-\alpha}, \quad 0 \leq i \leq I, \quad n \geq 0.$$

Then we get

$$V_i^{(n+1)} \leq V_i^{(n)}\left(1 - \lambda\Delta t_n(V_i^{(n)})^{-\alpha-1}\right), \quad 0 \leq i \leq I, \quad n \geq 0. \tag{3.6}$$

These estimates reveal that the sequence $V_h^{(n)}$ is nonincreasing. By induction, we obtain $V_h^{(n)} \leq V_h^{(0)} = \phi_h$. Thus the following holds

$$\lambda\Delta t_n(V_{hmin}^{(n)})^{-\alpha-1} \geq \lambda\min\{\frac{h^2(\phi_{hmin})^{-\alpha-1}}{2}, \tau\} = \tau'.$$

Let i_0 be such that $V_{hmin}^{(n)} = V_{i_0}^{(n)}$. Replacing i by i_0 in (3.6), we obtain

$$V_{hmin}^{(n+1)} \leq V_{hmin}^{(n)}(1-\tau'), \quad n \geq 0, \tag{3.7}$$

and by iteration, we arrive at

$$V_{hmin}^{(n)} \leq V_{hmin}^{(0)}(1-\tau')^n = \phi_{hmin}(1-\tau')^n, \quad n \geq 0. \tag{3.8}$$

Since the term on the right hand side of the above equality goes to zero as n approaches infinity, we conclude that $V^{(n)}_{hmin}$ tends to zero as n approaches infinity. So $\|U^{(n)}_h\|_\infty$ tends to one as n approaches infinity. Now, let us estimate the numerical quenching time. Due to (3.8) and the restriction $\Delta t_n \leq \tau(V^{(n)}_{hmin})^{\alpha+1}$, it is not hard to see that

$$\sum_{n=0}^{+\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \tau\phi^{\alpha+1}_{hmin}[(1-\tau')^{\alpha+1}]^n.$$

Use the fact that the series on the right hand side of the above inequality converges towards

$$\frac{\tau\phi^{\alpha+1}_{hmin}}{1-(1-\tau')^{\alpha+1}}$$

and $\phi_{hmin} = (1 - \|\varphi_h\|_\infty)$, we get

$$T^{\Delta t}_h = \sum_{n=0}^{+\infty} \Delta t_n \leq \frac{\tau(1-\|\varphi_h\|_\infty)^{\alpha+1}}{1-(1-\tau')^{\alpha+1}}.$$

□

Remark 3.4. Using Taylor's expansion, we get

$$1-(1-\tau')^{\alpha+1} = (\alpha+1)\tau' + o(\tau'),$$

which implies that

$$\frac{\tau}{1-(1-\tau')^{\alpha+1}} = \frac{\tau}{\tau'}\frac{1}{(\alpha+1)+o(1)} \leq \frac{\tau}{\tau'}\frac{2}{(\alpha+1)}.$$

If we take $\tau = h^2$, we have

$$\frac{\tau}{\tau'} = \frac{1}{\lambda}\min\{2\phi^{\alpha+1}_{hmin}, 1\}.$$

Then

$$\frac{\tau}{1-(1-\tau')^{\alpha+1}} \leq \frac{2\tau}{\tau'(\alpha+1)} = \frac{2}{\lambda(\alpha+1)}\min\{2\phi^{\alpha+1}_{hmin}, 1\}.$$

We conclude that $\dfrac{\tau}{1-(1-\tau')^{\alpha+1}}$ is bounded.

Remark 3.5. From (3.8), we deduce by induction that

$$V^{(n)}_{hmin} \leq V^{(k)}_{hmin}(1-\tau')^{n-k}, \text{ for } n \geq k,$$

and we see that

$$T^{\Delta t}_h - t_k = \sum_{n=k}^{+\infty} \Delta t_n \leq \sum_{n=k}^{+\infty} \tau(V^{(k)}_{hmin})^{\alpha+1}[(1-\tau')^{\alpha+1}]^{n-k},$$

which implies that

$$T^{\Delta t}_h - t_k \leq \frac{\tau(V^{(k)}_{hmin})^{\alpha+1}}{1-(1-\tau')^{\alpha+1}}.$$

Since $V_{hmin}^{(k)} = (1 - \|U_h^k\|_\infty)$, we get

$$T_h^{\Delta t} - t_k \leq \frac{\tau(1 - \|U_h^k\|_\infty)^{\alpha+1}}{1 - (1 - \tau')^{\alpha+1}}.$$

In the sequel, we take $\tau = h^2$.

4. Convergence of the discrete quennching time

In this section, under some assumptions, we show that the numerical quenching time of the discrete solution converges to the real one when the mesh size goes to zero. We denote by

$$u_h(t_n) = (u(x_0,t_n), u(x_1,t_n), ..., u(x_I,t_n))^T \text{ and } \|U_h^{(n)}\|_\infty = \max_{0\leq i\leq I} |U_i^{(n)}|.$$

In order to obtain the convergence of the numerical quenching time, we firstly prove the following theorem about the convergence of the discrete scheme.

Theorem 4.1. *Assume that the continuous problem (1.4)–(1.6) has a solution* $u \in C^{4,2}([0,1]\times[0,T])$ *such that* $\sup_{t\in[0,T]} \|u(.,t)\|_\infty = \iota$, $(0 < \iota < 1)$. *Suppose the initial condition at (2.3) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \longrightarrow 0. \tag{4.1}$$

Then, for h sufficiently small, the discrete problem (2.1)–(2.3) has a solution $U_h^{(n)}$, $0 \leq n \leq J$, *and we have the following relation*

$$\max_{0\leq n\leq J}(\|U_h^{(n)} - u_h(t_n)\|_\infty) = O(\|\varphi_h - u_h(0)\|_\infty + h) \quad as \quad h \longrightarrow 0.$$

Where J is such that $\sum_{j=0}^{J-1} \Delta t_j \leq T$ *and* $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

Proof. For each h, the discrete problem (2.1)–(2.3) has a solution $U_h^{(n)}$. Let $N \leq J$, the greatest value of n such that there exists a positive constant β (with $\iota < \beta < 1$) such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\beta - \iota}{2}, \quad n < N. \tag{4.2}$$

We know that $N \geq 1$ because of (4.1). Using the triangular inequality, for $n < N$, we have

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq \iota + \frac{\beta - \iota}{2} = \frac{\beta + \iota}{2} < 1. \tag{4.3}$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization, for $n < N$. Using Taylor's expansion, we have

$$\begin{aligned}\delta_t e_0^{(n)} - \gamma(u(x_0,t_n))\delta^2 e_0^{(n)} = [&\alpha\gamma(u(x_0,t_n))(1 - \theta_0^{(n)})^{-\alpha-1} \\ &+\gamma'(\eta_0^{(n)})(1 - U_0^{(n)})^{-\alpha} + \gamma'(\eta_0^{(n)})\delta^2 U_0^{(n)}]e_0^{(n)} \\ &+\gamma(u(x_0,t_n))\left(\frac{h}{12}u_{xxxx}(\tilde{x}_0,t_n) + \frac{2}{3}u_{xxx}(x_0,t_n)\right)h\end{aligned}$$

$$-\gamma(u(x_0,t_n))\frac{\Delta t_n}{2}u_{tt}(x_0,t_n),$$

$$\begin{aligned}\delta_t e_i^{(n)} - \gamma(u(x_i,t_n))\delta^2 e_i^{(n)} = {}& [\alpha\gamma(u(x_i,t_n))(1-\theta_i^{(n)})^{-\alpha-1} \\ &+\gamma'(\eta_i^{(n)})(1-U_i^{(n)})^{-\alpha} + \gamma'(\eta_i^{(n)})\delta^2 U_i^{(n)}]e_i^{(n)} \\ &+\gamma(u(x_i,t_n))\frac{h^2}{12}u_{xxxx}(\tilde{x}_i,t_n) \\ &-\gamma(u(x_i,t_n))\frac{\Delta t_n}{2}u_{tt}(x_i,t_n),\end{aligned}$$

$$\begin{aligned}\delta_t e_I^{(n)} - \gamma(u(x_I,t_n))\delta^2 e_I^{(n)} = {}& [\alpha\gamma(u(x_I,t_n))(1-\theta_I^{(n)})^{-\alpha-1} \\ &+\gamma'(\eta_I^{(n)})(1-U_I^{(n)})^{-\alpha} \\ &+\gamma'(\eta_I^{(n)})\delta^2 U_I^{(n)} - \frac{2}{h}\gamma'(\eta_I^{(n)})B(U_I^{(n)}) \\ &-\frac{2}{h}\gamma(u(x_I,t_n))B'(\sigma_I^{(n)})]e_I^{(n)} \\ &+\gamma(u(x_I,t_n))\left(\frac{h}{12}u_{xxxx}(\tilde{x}_I,t_n) - \frac{2}{3}u_{xxx}(x_I,t_n)\right)h \\ &-\gamma(u(x_I,t_n))\frac{\Delta t_n}{2}u_{tt}(x_I,t_n),\end{aligned}$$

where $\theta_i^{(n)}$, $\eta_i^{(n)}$ are intermediate values between $U_i^{(n)}$ and $u(x_i,t_n)$, $0 \le i \le I$, and $\sigma_I^{(n)}$ is an intermediate value between $U_I^{(n)}$ and $u(x_I,t_n)$. Since $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$ and $u_{tt}(x,t)$ are bounded and $\Delta t_n = O(h^2)$, there exist a positive constant $L > 0$ such that

$$\delta_t e_0^{(n)} - \delta^2 e_0^{(n)} \le C_0^{(n)} e_0^{(n)} + Lh,$$

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \le C_i^{(n)} e_i^{(n)} + Lh^2, \quad 1 \le i \le I-1,$$

$$\delta_t e_I^{(n)} - \delta^2 e_I^{(n)} \le C_I^{(n)} e_I^{(n)} + Lh,$$

where

$$C_0^{(n)} = \alpha\gamma(u(x_0,t_n))(1-\theta_0^{(n)})^{-\alpha-1} + \gamma'(\eta_0^{(n)})(1-U_0^{(n)})^{-\alpha} + \gamma'(\eta_0^{(n)})\delta^2 U_0^{(n)},$$

$$C_i^{(n)} = \alpha\gamma(u(x_i,t_n))(1-\theta_i^{(n)})^{-\alpha-1} + \gamma'(\eta_i^{(n)})(1-U_i^{(n)})^{-\alpha} + \gamma'(\eta_i^{(n)})\delta^2 U_i^{(n)},$$
$1 \le i \le I-1$,

$$\begin{aligned}C_I^{(n)} = {}& \alpha\gamma(u(x_I,t_n))(1-\theta_I^{(n)})^{-\alpha-1} + \gamma'(\eta_I^{(n)})(1-U_I^{(n)})^{-\alpha} + \gamma'(\eta_I^{(n)})\delta^2 U_I^{(n)} \\ &-\frac{2}{h}\gamma'(\eta_I^{(n)})B(U_I^{(n)}) - \frac{2}{h}\gamma(u(x_I,t_n))B'(\sigma_I^{(n)}).\end{aligned}$$

Set $M = \max_{0\le i\le I}\{C_i^{(n)}\}$ and introduce the vector $Z_h^{(n)}$ defined as follows

$$Z_i^{(n)} = e^{(M+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Lh), \quad 0 \le i \le I, \quad n < N.$$

By a straightforward computations, we have

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} > C_0^{(n)} Z_0^{(n)} + Lh,$$

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} > C_i^{(n)} Z_i^{(n)} + Lh^2, \quad 1 \le i \le I-1,$$

$$\delta_t Z_I^{(n)} - \delta^2 Z_I^{(n)} > C_I^{(n)} Z_I^{(n)} + Lh,$$

$$Z_i^{(0)} > e_i^{(0)}, \quad 0 \le i \le I.$$

It follows from Lemma (2.3) that

$$Z_i^{(n)} > e_i^{(n)}, \quad 0 \le i \le I.$$

By the same way, we also prove that

$$Z_i^{(n)} > -e_i^{(n)}, \quad 0 \le i \le I,$$

which implies that

$$Z_i^{(n)} > |e_i^{(n)}|, \quad 0 \le i \le I.$$

we deduce that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \le e^{(M+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Lh), \quad n < N. \tag{4.4}$$

Now, let us show that $N = J$. Suppose that $N < J$. If we replace n by N in (4.4), and taking into account the inequality (4.2), we obtain

$$\frac{\beta - \iota}{2} \le \|U_h^{(N)} - u_h(t_N)\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Lh). \tag{4.5}$$

Since $e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + Lh) \longrightarrow 0$ as $h \longrightarrow 0$, we deduce from (4.5) that $\frac{\beta - \iota}{2} \le 0$, which is impossible. Consequently, $N = J$, and we conclude the proof. □

Theorem 4.2. *Suppose that the solution u of problem (1.4)–(1.6) quenches in a finite time T_q such that $u \in C^{4,2}([0,1] \times [0,T_q))$ and the iniatial data at (2.3) satisfies*

$$||\varphi_h - u_h(0)||_\infty = o(1) \ \text{as} \ h \longrightarrow 0.$$

Under the hypothesis of Theorem (3.3), the problem (2.1)–(2.3) has a discrete solution $U_h^{(n)}$ which quenches in a finite time $T_h^{\Delta t}$ and we have

$$\lim_{h \to 0} T_h^{\Delta t} = T_q.$$

Proof. We know from Remark (3.4) that $\frac{\tau}{1 - (1 - \tau')^{p+1}}$ is bounded.

Let $0 < \varepsilon < \frac{T_q}{2}$, there exists a constant $\eta = \beta - \iota$ $(0 < \iota < \beta < 1)$ such that

$$\frac{\tau(1 - \varrho)^{\alpha+1}}{1 - (1 - \tau')^{\alpha+1}} < \frac{\varepsilon}{2}, \quad \varrho \in [1 - \eta, 1) \tag{4.6}$$

Since u quenches, there exists $T_1 \in (T_q - \frac{\varepsilon}{2}, T_q)$ and $h_0(\varepsilon) > 0$ such that $1 - \frac{\eta}{2} \le \|u(.,t_n)\|_\infty < 1$ for $t_n \in [T_1, T_q)$. Let k be a positive integer such that $t_k = \sum_{n=0}^{k-1} \Delta t_n \in [T_1, T_q)$ for $h \le h_0(\varepsilon)$. It follows from Theorem (4.1) that the problem (2.1)–(2.3) has a solution $U_h^{(n)}$ which verifies $\|U_h^{(n)} - u_h(t_n)\|_\infty < \frac{\eta}{2}$ for $n \leq k$, $h \leq h_0(\varepsilon)$. This fact implies that

$$\|U_h^{(k)}\|_\infty \geq \|u(.,t_k)\|_\infty - \|U_h^{(k)} - u_h(t_k)\|_\infty \geq 1 - \frac{\eta}{2} - \frac{\eta}{2} = 1 - \eta, \quad h \leq h_0(\varepsilon).$$

From Theorem (3.3), $U_h^{(n)}$ quenches at the time $T_h^{\Delta t}$. It follows from Remark (3.5) and (4.6) that $|T_h^{\Delta t} - t_k| \leq \frac{\tau(1-\|U_h^{(k)}\|_\infty)^{\alpha+1}}{1-(1-\tau')^{\alpha+1}} < \frac{\varepsilon}{2}$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T_q - T_h^{\Delta t}| \leq |T_q - t_k| + |t_k - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Which leads us to the result. □

5. Numerical experiments

In this section, we present some numerical approximations of the quenching time of the problem (1.4)–(1.6) in the case where $u_0(x) = 0.7 - \frac{1}{2}x^4$,
$\gamma(U_i^{(n)}) = \frac{(U_i^{(n)})^{(1-p)}}{p}$, $B(U_i^{(n)}) = (U_i^{(n)})^{-q}, 0 \leq i \leq I$ where $0 < p \leq 1$ and $q > 0$.

We consider the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n^e} = (U_i^{(n)})^{(1-p)} \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{ph^2} + \frac{(U_i^{(n)})^{(1-p)}}{p}(1 - U_i^{(n)})^{-\alpha},$$
$$1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n^e} = (U_0^{(n)})^{(1-p)} \frac{2U_1^{(n)} - 2U_0^{(n)}}{ph^2} + \frac{(U_0^{(n)})^{(1-p)}}{p}(1 - U_0^{(n)})^{-\alpha},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n^e} = (U_I^{(n)})^{(1-p)} \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{ph^2} - \frac{2(U_I^{(n)})^{(1-p)}}{ph}(U_I^{(n)})^{-q}$$
$$+ \frac{(U_I^{(n)})^{(1-p)}}{p}(1 - U_I^{(n)})^{-\alpha},$$

$$U_i^{(0)} = \varphi_i,\ 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n^e = \min\left\{\frac{h^2}{2}, h^2(1 - \|U_h^{(n)}\|_\infty)^{\alpha+1}\right\}$. We also consider the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = (U_i^{(n)})^{(1-p)} \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{ph^2}$$
$$+ \frac{(U_i^{(n)})^{(1-p)}}{p}(1 - U_i^{(n)})^{-\alpha},\ 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = (U_0^{(n)})^{(1-p)} \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{ph^2}$$
$$+ \frac{(U_0^{(n)})^{(1-p)}}{p}(1 - U_0^{(n)})^{-\alpha},$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = (U_I^{(n)})^{(1-p)} \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{ph^2} - \frac{2(U_I^{(n)})^{(1-p)}}{ph} (U_I^{(n)})^{-q} + \frac{(U_I^{(n)})^{(1-p)}}{p} (1 - U_I^{(n)})^{-\alpha},$$

$$U_i^{(0)} = \varphi_i,\ 0 \leq i \leq I,$$

where $n \geq 0$, $\Delta t_n = h^2(1 - \|U_h^{(n)}\|_\infty)^{\alpha+1}$.

In the following tables, in rows, we present the numerical quenching times T^n, the numbers of iterations n and the orders $s = \dfrac{log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{log(2)}$ of the approximations corresponding to meshes $16, 32, 64, 128, 256, 512$. The numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when

$$|T^{n+1} - T^n| \leq 10^{-16}.$$

Table 1. Numerical quenching times, the numbers of iterations and the orders obtained with the explicit Euler method $\alpha = 4$, $p = 0.5$ and $q = 0.1$

I	T^n	n	s
16	0,0002849086616	667	-
32	0,0002819104743	2531	-
64	0,0002811654024	9562	2.01
128	0,0002809794133	35977	2.00
256	0,0002809329330	134803	2.00
512	0,0002809213121	502755	1.99

Table 2. Numerical quenching times, the numbers of iterations and the orders obtained with the implicit Euler method $\alpha = 4$, $p = 0.5$ and $q = 0.1$

I	T^n	n	s
16	0,0002849095211	667	-
32	0,0002819106436	2531	-
64	0,0002811654418	9562	2.01
128	0,0002809794230	35977	2.00
256	0,0002809329354	134803	2.00
512	0.0002809213128	502755	1.99

We also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 64$ and $(\alpha; p; q) = (4; 0.5; 0.1)$. Figures 1, 2 show that the discrete solution quenches. In figures 3, 4, we can appreciate that the discrete solution is nonincreasing and quenches at the first node. For figures 5, 6, we see that the discrete solution is increasing with respect to time and quenches at finite time 2.8×10^{-4}.

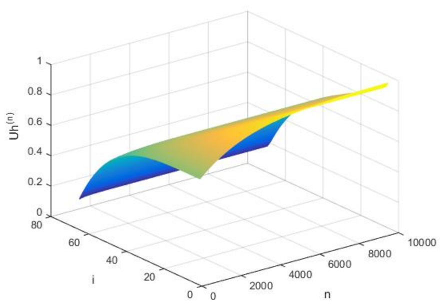

FIGURE 1. Evolution of the numerical solution (explicit scheme).

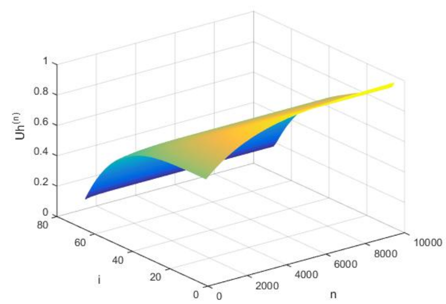

FIGURE 2. Evolution of the numerical solution (implicit scheme).

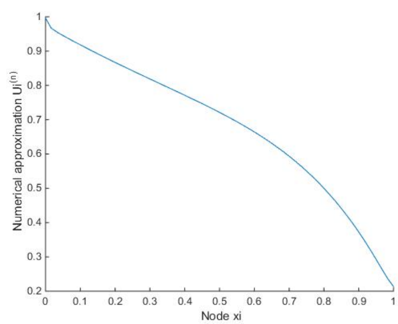

FIGURE 3. The profil of the approximation of u(x,T) where, T is the quenching time (explicit scheme).

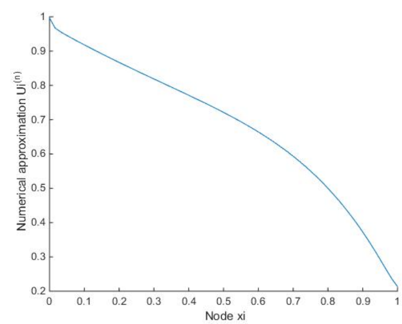

FIGURE 4. The profil of the approximation of u(x,T) where, T is the quenching time (implicit scheme).

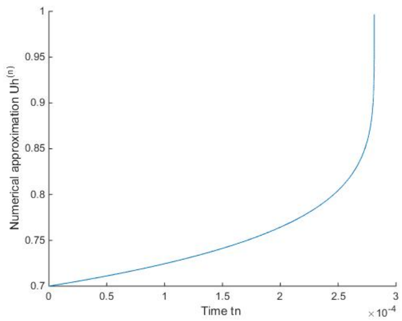

Figure 5. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (explicit scheme).

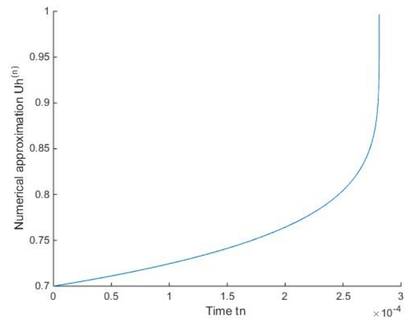

Figure 6. The profil of the approximation of $\|U_h^{(n)}\|_\infty$ (implicit scheme).

6. Conclusion

In this paper, we have studied the numerical quenching of the solution of the nonlinear diffusion equation with nonlinear source and singular boundary flux (1.1)–(1.3) and we have obtained good approximations of its quenching time.

We have constructed, by the finite difference method, the discrete problem (2.1)–(2.3) associated to the continuous problem (1.4)–(1.6). We have shown that under some conditions, the solution of the discrete problem (2.1)–(2.3) quenches in finite time and we have estimated its discrete quenching time. We have also established the convergence of the discrete time towards the theoretical time when the spatial and temporal discretionary steps tend towards zero. Finally, we have given some numerical experiments to illustrate our analysis.

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N'Guessan Koffi (nkrasoft@yahoo.fr)
Université Alassane Ouattara de Bouaké, UFR-SED, 01 BP V 18 Bouaké 01, Côte d'Ivoire

Anoh Assiedou Rodrigue (r1992anoh@gmail.com)
Université Félix Houphouët Boigny d'Abidjan, UFR-MI, 22 BP 582 Abidjan 22, Côte d'Ivoire

Coulibaly Adama (couliba@yahoo.fr)
Université Félix Houphouët Boigny d'Abidjan, UFR-MI, 22 BP 582 Abidjan 22, Côte d'Ivoire

Touré Kidjégbo Augustin (latoureci@gmail.com)
Institut National Polytechnique Houphouët-Boigny de Yamoussoukro, BP 2444 Yamoussoukro, Côte d'Ivoire