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Interior and closure operators on generalized co-residuated lattices

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Abstract. In this paper, we introduce the notions and properties of generalized co-residuated lattices as a non-commutative algebraic structure. We define right and left distance functions. In particular, we study the relations between various operators and various connections. We give their examples.

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1. Introduction

Ward and Dilworth [1] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [2, 3] investigated the properties of fuzzy Galois connections and fuzzy closure operators on a residuated lattice which supports part of foundation of theoretic computer science. On the other hand, Georgescu and Popescue [4, 5] introduced fuzzy Galois connection in a generalized residuated lattice as a non-commutative algebraic structure which is induced by two implications.

As a dual sense of complete residuated lattice, Zheng and Wang [6] introduced a complete co-residuated lattice as the generalization of t-conorm. For an extension of Pawlak's rough sets [7, 8], Junsheng and Qing [9] investigated $(\odot, \&)$-generalized fuzzy rough set on $(L, \odot, \&)$, where $(L, \&)$ is a complete residuated lattice and $(L, \odot)$ is complete coresiduated lattice. Ko and Kim [10] introduced the concepts of fuzzy join and meet complete lattices using distance spaces instead of fuzzy partially ordered spaces in complete co-residuated lattices. Moreover, Oh and kim [11] investigated the properties of Alexandrov fuzzy topologies, distance functions, join preserving maps and join approximation maps in complete co-residuated lattices.

The aim of this paper is to study the notions and properties of generalized co-residuated lattices as a non-commutative algebraic structure. Using right (resp. left) distance spaces instead of fuzzy partially ordered spaces, we define various connections, right (resp. left) isotone (antitone) maps and rough sets on a generalized co-residuated lattice.

We investigate the properties of right and left closure on a generalized co-residuated lattice. In particular, we obtain right (left) closure (interior) operators and rough sets from various connections. We give their examples.

2. Preliminaries

As an extension of Zheng's co-residuated lattices [6], we define generalized co-residuated lattices as a non-commutative algebraic structure.

Definition 2.1. A structure $(L,\vee,\wedge,\oplus,\ominus,\oslash,\bot,\top)$ is called a *generalized co-residuated lattice*, if it satisfies the following conditions:

(GR1) $(L,\vee,\wedge,\bot,\top)$ is lattice, where $\top$ is the upper bound and $\top$ denotes the universal lower bound,

(GR2) $x\oplus\top=x$ and $x\oplus(y\oplus z)=(x\oplus y)\oplus z$ for all $x,y,z\in L$,

(GR3) it satisfies a co-residuation, i.e.,

$$a\oplus b\geq c \text{ iff } a\geq c\ominus b \text{ iff } b\geq c\oslash a.$$

A generalized co-residuated lattice is called *co-residuated lattice*, if $x\oplus y=y\oplus x$ for each $x,y\in L$.

For $\alpha\in L, A\in L^X$, we denote $(A\ominus\alpha),(\alpha\oplus A),\alpha_X\in L^X$ as

$$(A\ominus\alpha)(x)=A(x)\ominus\alpha,\ (\alpha\oplus A)(x)=\alpha\oplus A(x),\quad \alpha_X(x)=\alpha.$$

Put $n_1(x)=\top\ominus x$ and $n_2(x)=\top\oslash x$. The condition $n_1(n_2(x))=n_2(n_1(x))=x$ for each $x\in L$ is called a *double negative law*.

Example 2.2 ([10, 11])**.** (1) If a generalized co-residuated lattice $(L,\vee,\wedge,\oplus,\ominus,\oslash,\bot,\top)$ is a co-residuated lattice, then $\ominus=\oslash$ and $n_1=n_2$.

(2) An infinitely distributive lattice $(L,\vee,\wedge,\oplus=\vee,\bot,\top)$ is a co-residuated lattice. In particular, the unit interval $([0,1],\vee,\wedge,\oplus=\vee,0,1)$ is a co-residuated lattice, where

$$x\ominus y=\bigwedge\{z\in L\mid y\vee z\geq x\}=\begin{cases}0, & \text{if } y\geq x,\\ x, & \text{if } y\not\geq x.\end{cases}$$

Put $n(x)=1\ominus x=1$ for $x\neq 1$ and $n(1)=0$. Then $n(n(x))=0$ for $x\neq 1$ and $n(n(1))=1$. Thus n does not satisfy a double negative law.

(3) Let $([0,1],\vee,\wedge,\oplus,0,1)$ be a co-residuated lattice, where

$$\begin{aligned}x\oplus y&=(x^p+y^p)^{\frac{1}{p}}\wedge 1,\ \ 1\leq p<\infty,\\ x\ominus y&=\bigwedge\{z\in[0,1]\mid (z^p+y^p)^{\frac{1}{p}}\geq x\}\\ &=\bigwedge\{z\in[0,1]\mid z\geq (x^p-y^p)^{\frac{1}{p}}\}\\ &=(x^p-y^p)^{\frac{1}{p}}\vee 0.\end{aligned}$$

Put $n(x)=1\ominus x=(1-x^p)^{\frac{1}{p}}$ for $1\leq p<\infty$. Then $n(n(x))=x$ for $x\in[0,1]$. Thus n satisfies a double negative law.

(4) Let $L \subset \{(x,y) \in R^2 \mid x > 0\}$ be a set and for $(x_1,y_1),(x_2,y_2) \in L$, we define

$$(x_1,y_1) \leq (x_2,y_2) \text{if and only if } \; x_1 < x_2 \;\text{ or }\; x_1 = x_2, y_1 \leq y_2.$$

Then the structure $(L, \vee, \wedge, \oplus, \ominus, \oslash, (\frac{1}{2},1),(1,0))$ is a generalized co-residuated lattice with a double negative law where $\perp = (\frac{1}{2},1)$ is the least element and $\top = (1,0)$ is the greatest element from the following statements:

$$\begin{array}{ll} (x_1,y_1) \oplus (x_2,y_2) & = (2x_1x_2, 2x_2y_1 + y_2 - 2x_2) \wedge (1,0), \\ (x_1,y_1) \ominus (x_2,y_2) & = (\frac{x_1}{2x_2}, 1 + \frac{y_1-y_2}{2x_2}) \vee (\frac{1}{2},1), \\ (x_1,y_1) \oslash (x_2,y_2) & = (\frac{x_1}{2x_2}, y_1 + \frac{x_1}{x_2}(1-y_2)) \vee (\frac{1}{2},1). \end{array}$$

Furthermore, we have $(x,y) = n_2(n_1(x,y)) = n_1(n_2(x,y))$ from:

$$\begin{array}{l} n_1(x,y) = (1,0) \ominus (x,y) = (\frac{1}{2x}, 1 - \frac{y}{2x}), \\ n_2(x,y) = (1,0) \oslash (x,y) = (\frac{1}{2x}, \frac{1}{x}(1-y)), \\ n_2(n_1(x,y)) = (1,0) \oslash (\frac{1}{2x}, 1 - \frac{y}{2x}) = (x,y), \\ n_1(n_2(x,y)) = (1,0) \ominus (\frac{1}{2x}, \frac{1}{x}(1-y)) = (x,y). \end{array}$$

Let $A = \{(\frac{2}{3}, y) \mid y \in R\}$ be given. Then $\bigvee A$ and $\bigwedge A$ do not exist. Thus L is not complete.

In this paper, we assume $(L, \vee, \wedge, \oplus, \ominus, \oslash, \perp, \top)$ is a generalized co-residuated lattice with a double negative law and and if the family supremum or infumum exists, we denote $\bigvee$ and $\bigwedge$.

3. Residuated and Galois connections

In this section, we study notions of residuated and Galois connections on generalized co-residuated lattices. Moreover, we investigate the relations between various connections and operators.

Lemma 3.1. *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $(x \oplus y) \leq (x \oplus z)$, $x \ominus y \leq x \ominus z$ and $z \ominus x \leq y \ominus x$ for $\ominus \in \{\ominus, \oslash\}$.*
(2) *$y \oplus (x \oslash y) \geq x$ and $(x \ominus y) \oplus y \geq x$.*
(3) *$x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ for $\ominus \in \{\ominus, \oslash\}$.*
(4) *$x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigwedge_{i\in\Gamma}(x_i \ominus y)$ for $\ominus \in \{\ominus, \oslash\}$.*
(5) *$x \oplus (y \ominus z) \geq (x \oplus y) \ominus z$ and $(x \oslash y) \oplus z = (x \oplus z) \oslash y$.*
(6) *$x \ominus (y \oplus z) = (x \ominus z) \ominus y$ and $(x \oslash y) \oslash z = x \oslash (y \oplus z)$.*
(7) *$(x \ominus y) \oslash z = (x \oslash z) \ominus y$.*
(8) *$(y \oslash z) \oplus (x \oslash y) \geq x \oslash z$ and $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.*
(9) *$(x \oslash z) \geq (y \oplus x) \oslash (y \oplus z)$ and $(x \ominus z) \geq (x \oplus y) \ominus (z \oplus y)$.*
(10) *$y \oslash z \geq (x \oslash z) \ominus (x \oslash y)$ and $x \oslash y \geq (x \oslash z) \oslash (y \oslash z)$.*
(11) *$x \ominus y \geq (x \ominus z) \ominus (y \ominus z)$ and $y \ominus z \geq (x \ominus z) \oslash (x \ominus y)$.*
(12) *$x \ominus x = x \oslash x = \perp$, $x \ominus \perp = x \oslash \perp = x$ and $\perp \ominus x = \perp \oslash x = \perp$.*
(13) *$x \ominus y = \perp$ iff $x \leq y$ iff $x \oslash y = \perp$.*
(14) *$x \oplus y = \perp$ iff $x = \perp$ and $y = \perp$.*
(15) *$x \ominus y = n_1(y) \oslash n_1(x)$ and $x \oslash y = n_2(y) \ominus n_2(x)$.*
(16) *$n_1(y \oplus z) = n_1(z) \ominus y$ and $n_2(y \oplus z) = n_2(y) \oslash z$. Moreover, $n_2(x \ominus y) = y \oplus n_2(x)$ and $n_1(x \oslash y) = n_1(x) \oplus y$.*
(17) *For each $k = 1, 2$, $n_k(\bigwedge_{i\in\Gamma} x_i) = \bigvee_{i\in\Gamma} n_k(x_i)$ and $n_k(\bigvee_{i\in\Gamma} x_i) = n_k(\bigwedge_{i\in\Gamma} x_i)$.*

Proof. (1) Since $y = y \wedge z$, $x \oplus y = x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$. Then $(x \oplus y) \leq (x \oplus z)$. Since $y \leq z \leq (z \ominus x) \oplus x$, $y \ominus x \leq z \ominus x$. Since $x \leq (x \ominus y) \oplus y \leq (x \ominus y) \oplus z$, $x \ominus z \leq x \ominus y$. The cases of $\oslash$ are similarly proved.

(2) Since $x \oslash y \geq x \oslash y$, $y \oplus (x \oslash y) \geq x$. Since $x \ominus y \geq x \ominus y$, $(x \ominus y) \oplus y \geq x$.

(3) By (1), $(\bigvee_{i \in \Gamma} x_i) \ominus y \geq \bigvee_{i \in \Gamma}(x_i \ominus y)$. Since $(\bigvee_{i \in \Gamma}(x_i \ominus y)) \oplus y \geq \bigvee_{i \in \Gamma}((x_i \ominus y) \oplus y) \geq \bigvee_{i \in \Gamma} x_i$, $(\bigvee_{i \in \Gamma} x_i) \ominus y \leq \bigvee_{i \in \Gamma}(x_i \ominus y)$.

By (1), $x \ominus (\bigwedge_{i \in \Gamma} y_i) \geq \bigvee_{i \in \Gamma}(x \ominus y_i)$. Since $\bigvee_{i \in \Gamma}(x \ominus y_i) \oplus (\bigwedge_{i \in \Gamma} y_i) \geq \bigwedge_{i \in \Gamma}((x \ominus y_i) \oplus y_i) \geq x$, $\bigvee_{i \in \Gamma}(x \ominus y_i) \geq x \ominus (\bigwedge_{i \in \Gamma} y_i)$.

(4) It follows from (1).

(5) Since $x \oplus ((y \ominus z) \oplus z) \geq x \oplus y$, $x \oplus (y \ominus z) \geq (x \oplus y) \ominus z$.

Since $y \oplus ((x \oslash y) \oplus z) = (y \oplus (x \oslash y)) \oplus z \geq x \oplus z$, $(x \oslash y) \oplus z \geq (x \oplus z) \oslash y$.

(6) Since $(x \ominus (y \oplus z)) \oplus (y \oplus z) \geq x$ iff $(x \ominus (y \oplus z)) \oplus y \geq x \ominus z$, $x \ominus (y \oplus z) \geq (x \ominus z) \ominus y$. Since $((x \ominus z) \ominus y) \oplus y \oplus z \geq (x \ominus z) \oplus z \geq x$, $(x \ominus z) \ominus y \geq x \ominus (y \oplus z)$. Then $x \ominus (y \oplus z) = (x \ominus z) \ominus y$.

Since $y \oplus z \oplus (x \oslash (y \oplus z)) \geq x$ iff $z \oplus (x \oslash (y \oplus z)) \geq x \oslash y$, $x \oslash (y \oplus z) \geq (x \oslash y) \oslash z$. Since $y \oplus z \oplus ((x \oslash y) \oslash z) \geq y \oplus (x \oslash y) \geq x$, $(x \oslash y) \oslash z \geq x \oslash (y \oplus z)$. Then $(x \oslash y) \oslash z = x \oslash (y \oplus z)$

(7) Since $(z \oplus ((x \ominus y) \oslash z)) \oplus y \geq (x \ominus y) \oplus y \geq x$, $((x \ominus y) \oslash z) \oplus y \geq x \oslash z$. Then $(x \ominus y) \oslash z \geq (x \oslash z) \ominus y$. Since $z \oplus (((x \oslash z) \ominus y)) \oplus y) \geq z \oplus (x \oslash z) \geq x$, $z \oplus ((x \oslash z) \ominus y) \geq x \ominus y$. Thus $(x \oslash z) \ominus y \geq (x \ominus y) \oslash z$.

(8) Since $x \ominus y \geq x \ominus y$, $y \oplus (x \ominus y) \geq x$. Moreover, $y \geq x \ominus (x \ominus y)$. Since $(x \ominus y) \oplus (y \ominus z) \oplus z \geq (x \ominus y) \oplus y \geq x$, $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.

(9) Since $(z \oplus (y \oslash z)) \oplus (x \oslash y) \geq y \oplus (x \oslash y) \geq x$, $(y \oslash z) \oplus (x \oslash y) \geq x \oslash z$.

(10) Since $(y \oplus z) \oplus (x \oslash z) = y \oplus (z \oplus (x \oslash z)) \geq y \oplus x$, $(x \oslash z) \geq (y \oplus x) \oslash (y \oplus z)$.

Since $(x \ominus z) \oplus (z \oplus y) = ((x \ominus z) \oplus z) \oplus y) \geq x \oplus y$, $(x \ominus z) \geq (x \oplus y) \ominus (z \oplus y)$.

(11) Since $(y \oplus z) \oplus (x \ominus y) \geq x \oplus z$, $x \ominus y \geq (x \oplus z) \ominus (y \oplus z)$. Since $x \oplus (y \ominus x) \oplus (z \ominus y) \geq z$, $y \ominus x \geq (z \ominus x) \ominus (z \ominus y)$. Since $x \oplus y \leq z \oplus (x \ominus z) \oplus w \oplus (y \ominus w)$, $(x \oplus y) \ominus (z \oplus w) \leq (x \ominus z) \oplus (y \ominus w)$.

(12) Let $x \oplus \bot = \bot \oplus x = x$. Then $x \ominus x = x \oslash x = \bot$. $x \ominus \bot = \bigwedge\{z \in L \mid z \oplus \bot \geq x\} = x$, and $x \oslash \bot = \bigwedge\{z \in L \mid \bot \oplus z \leq x\} = x$.

(13) Let $x \ominus y = \bot$. Then $y = \bot \oplus y = (x \ominus y) \oplus y = \bigwedge\{z \in L \mid z \oplus y \geq x\} \oplus y = \bigwedge\{z \oplus y \in L \mid z \oplus y \geq x\} \geq x$. Thus $x \leq y$.

Let $x \leq y$. Then $x \ominus y = \bigwedge\{z \in L \mid z \oplus y \geq x\} = \bot$. Other cases are similarly proved.

(14) Let $x \oplus y = \bot$. Then $y = y \ominus (x \oplus y) = (y \ominus y) \ominus x = \bot \ominus x = \bot$ and $x = x \oslash (x \oplus y) = (x \oslash x) \oslash y = \bot \oslash y = \bot$. Conversely, $\bot \oplus \bot = \bot$.

(15) By (11), $x \ominus y \geq (\top \ominus y) \oslash (\top \ominus x) = n_1(y) \oslash n_1(x)$. By (10),

$$x \oslash y \geq (\top \oslash y) \ominus (\top \oslash x) = n_2(y) \ominus n_2(x).$$

Moreover, we have

$$x \ominus y = n_2(n_1(x)) \ominus n_2(n_1(y)) \leq n_1(y) \oslash n_1(x)$$

and

$$x \oslash y = n_1(n_2(x)) \oslash n_1(n_2(y)) \leq n_2(y) \ominus n_2(x).$$

Thus $x \ominus y = n_1(y) \oslash n_1(x)$ and $x \oslash y = n_2(y) \ominus n_2(x)$.

(16) By (6), $n_1(y \oplus z) = \top \ominus (y \oplus z) = (\top \ominus z) \ominus y = n_1(z) \ominus y$ and $n_2(y \oplus z) = \top \oslash (y \oplus z) = (\top \oslash y) \oslash z = n_2(y) \oslash z$.

(17) By (3), $n_k(\bigwedge_i x_i) = \bigvee_i n_k(x_i)$ for each $k = 1, 2$. Since $\bigwedge_i x_i = n_2(n_1(\bigwedge_i x_i)) = n_2(\bigvee_i n_1(x_i))$, $\bigwedge_i n_2(x_i) = n_2(\bigvee_i n_1(n_2(x_i))) = n_2(\bigvee_i x_i)$. Other cases are similarly proved. □

Definition 3.2. Let X be a set. A function $d^r_X : X \times X \to L$ is called a *right distance function*, if it satisfies the following conditions:

(D1) $d^r_X(x, x) = \perp$ for all $x \in X$,

(D2) If $d^r_X(x, y) = d^r_X(y, x) = \top$, then $x = y$,

(R) $d^r_X(x, y) \oplus d^r_X(y, z) \geq d^r_X(x, z)$, for all $x, y, z \in X$.

A function $d^l_X : X \times X \to L$ is called a *left distance function*, if it satisfies (D1), (D2) and

(L) $d^l_X(y, z) \oplus d^l_X(x, y) \geq d^l_X(x, z)$, for all $x, y, z \in X$.

The triple (X, d^r_X, d^l_X) is a bi-distance space.

Example 3.3. (1) We define a function $d^r_L, d^l_L : L \times L \to L$ as

$$d^r_L(x, y) = x \ominus y,\ d^l_L(x, y) = x \oslash y.$$

By Lemma 3.1 (8), (L, d^r_L, d^l_L) is a bi-distance space.

(2) We define a function $d^r_{L^X}, d^l_{L^X} : L^X \times L^X \to L$ as

$$d^r_{L^X}(A, B) = \bigvee_{x\in X}(A(x) \ominus B(x)),\ \ d^l_{L^X}(A, B) = \bigvee_{x\in X}(A(x) \oslash B(x)).$$

By Lemma 3.1 (8), $(L^X, d^r_{L^X}, d^l_{L^X})$ is a bi-distance space.

Definition 3.4. Let X and Y be two sets. Let $F, H : L^X \to L^Y$ and $G, K : L^Y \to L^X$ be operators.

(1) The pair (F, G) is called a *residuated connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $F(A) \leq B$ iff $A \leq G(B)$.

(2) The pair (H, K) is called a *Galois connection* between X and Y, if for $A \in L^X$ and $B \in L^Y$, $B \leq H(A)$ iff $A \leq K(B)$.

(3) The pair (H, K) is called a *dual Galois connection* between X and Y, if for $A \in L^X$ and $B \in L^Y$, $H(A) \leq B$ iff $K(B) \leq A$.

(4) A map $F : L^X \to L^Y$ is a *right isotone map*, if for all $A,\ B \in L^X$, $d^r_{L^X}(A, B) \geq d^r_{L^Y}(F(A), F(B))$.

(5) A map $F : L^X \to L^Y$ is a *left isotone map*, if for all $A,\ B \in L^X$, $d^l_{L^X}(A, B) \geq d^l_{L^Y}(F(A), F(B))$.

(6) A map $F : L^X \to L^Y$ is a *right antitone map*, if for all $A,\ B \in L^X$, $d^l_{L^X}(A, B) \geq d^r_{L^Y}(F(B), F(A))$.

(7) A map $F : L^X \to L^Y$ is a *left antitone map*, if for all $A,\ B \in L^X$, $d^r_{L^X}(A, B) \geq d^l_{L^Y}(F(B), F(A))$.

Theorem 3.5. *Let $G : L^X \to L^Y$ be a map.*

(1) *A map $G : L^X \to L^Y$ is a right isotone map iff $\alpha \oplus G(A) \geq G(\alpha \oplus A)$ and $G(A) \leq G(B)$ for $A \leq B$ iff $G(A \oslash \alpha) \geq G(A) \oslash \alpha$ and $G(A) \leq G(B)$ for $A \leq B$.*

(2) *A map $G : L^X \to L^Y$ is a left isotone map iff $G(A) \oplus \alpha \geq G(A \oplus \alpha)$ and $G(A) \leq G(B)$ for $A \leq B$ iff $G(A \ominus \alpha) \geq G(A) \ominus \alpha$ and $G(A) \leq G(B)$ for $A \leq B$.*

(3) *A map $G : L^X \to L^Y$ is a left antitone map iff $G(\alpha \oplus A) \geq G(A) \ominus \alpha$ and $G(B) \leq G(A)$ for $A \leq B$ iff $G(A) \oplus \alpha \geq G(A \oslash \alpha)$ and $G(B) \leq G(A)$ for $A \leq B$.*

(4) *A map $G : L^X \to L^Y$ is a right antitone map iff $G(A \oplus \alpha) \geq G(A) \oslash \alpha$ and $G(B) \leq G(A)$ for $A \leq B$ iff $\alpha \oplus G(A) \geq G(A \ominus \alpha)$ and $G(B) \leq G(A)$ for $A \leq B$.*

(5) *If $G : L^X \to L^Y$ is a left isotone map, then $n_2 G : L^X \to L^Y$ is a right antitone map.*

(6) *If $G : L^X \to L^Y$ is a right isotone map, then $n_1 G : L^X \to L^Y$ is a left antitone map.*

(7) *If $G : L^X \to L^Y$ is a right antitone map, then $n_1 G : L^X \to L^Y$ is a left isotone map.*

(8) *If $G : L^X \to L^Y$ is a left antitone map, then $n_2 G : L^X \to L^Y$ is a right isotone map.*

Proof. (1) Let $d^r_{L^X}(A, B) \geq d^r_{L^Y}(G(A), G(B))$. Put $A = \alpha \oplus B$. Then

$$\alpha \geq d^r_{L^X}(\alpha \oplus B, B) \geq d^r_{L^Y}(G(\alpha \oplus B), G(B)).$$

Thus $\alpha \oplus G(B) \geq G(\alpha \oplus B)$.

Conversely, put $\alpha = d^r_{L^X}(A, B)$. Since $d^r_{L^X}(A, B) \oplus B \geq A$,

$$d^r_{L^X}(A, B) \oplus G(B) \geq G(d^r_{L^X}(A, B) \oplus B) \geq G(A).$$

So $d^r_{L^X}(A, B) \geq d^r_{L^Y}(G(A), G(B))$.

Second, let $\alpha \oplus G(A) \geq G(\alpha \oplus A)$ and $G(A) \leq G(B)$ for $A \leq B$. Since $\alpha \oplus G(A \oslash \alpha) \geq G(\alpha \oplus (A \oslash \alpha)) \geq G(A)$, $G(A \oslash \alpha) \geq G(A) \oslash \alpha$.

Conversely, let $G(A \oslash \alpha) \geq A \oslash \alpha$ and $G(A) \leq G(B)$ for $A \leq B$. Since $G((\alpha \oplus A) \oslash \alpha) \geq G(\alpha \oplus A) \oslash \alpha$ iff $\alpha \oplus G((\alpha \oplus A) \oslash \alpha) \geq G(\alpha \oplus A)$, we have

$$G(\alpha \oplus A) \leq \alpha \oplus G((\alpha \oplus A) \oslash \alpha) \leq \alpha \oplus G(A).$$

(3) Let $G : L^X \to L^Y$ be a left antitone map. Then $d^r_{L^X}(A, B) \geq d^l_{L^Y}(G(B), G(A))$. Put $A = \alpha \oplus B$. Then $\alpha \geq d^r_{L^X}(\alpha \oplus B, B) \geq d^l_{L^Y}(G(B), G(\alpha \oplus B))$. Thus $G(\alpha \oplus B) \geq G(B) \ominus \alpha$.

Conversely, since $G(d^r_{L^X}(A, B) \oplus B) \geq G(B) \ominus d^r_{L^X}(A, B)$ and $G(d^r_{L^X}(A, B) \oplus B) \leq G(A)$ for $d^r_{L^X}(A, B) \oplus B \geq A$, we have

$$d^r_{L^X}(A, B) \geq G(B) \oslash G(d^r_{L^X}(A, B) \oplus B) \geq G(B) \oslash G(A).$$

Second, we show that $G(\alpha \oplus A) \geq G(A) \ominus \alpha$ and $G(B) \leq G(A)$ for $A \leq B$ iff $G(A) \oplus \alpha \geq G(A \oslash \alpha)$ and $G(B) \leq G(A)$ for $A \leq B$.

Let $G(\alpha \oplus A) \geq G(A) \ominus \alpha$ and $G(B) \leq G(A)$ for $A \leq B$. Then $G(\alpha \oplus A) \oplus \alpha \geq G(A)$. Thus we get

$$G(A) \oplus \alpha \geq G(\alpha \oplus (A \oslash \alpha)) \oplus \alpha \geq G(A \oslash \alpha).$$

Let $G(A) \oplus \alpha \geq G(A \oslash \alpha)$ and $G(B) \leq G(A)$ for $A \leq B$. Then $G(A) \geq G(A \oslash \alpha) \ominus \alpha$. Put $A = \alpha \oplus B$. Then $G(\alpha \oplus B) \geq G((\alpha \oplus B) \oslash \alpha) \ominus \alpha \geq G(B) \ominus \alpha$.

(5) Let $G : L^X \to L^Y$ be a left isotone map. Then by Lemma 3.1 (15), we have

$$d^l_{L^X}(A, B) \leq d^l_{L^Y}(G(A), G(B)) = d^r_{L^Y}(n_2 G(B), n_2 G(A)).$$

Thus $n_2 G : L^X \to L^Y$ is a right antitone map.

(6) Let $G : L^X \to L^Y$ be a right isotone map. Then by Lemma 3.1 (15), we get

$$d^r_{L^X}(A, B) \geq d^r_{L^Y}(G(A), G(B)) = e^l_{L^Y}(n_1 G(B), n_1 G(A)).$$

Thus $n_1G : L^X \to L^Y$ is a left antitone map.

(2), (4), (7) and (8) are proved similar methods as (1), (3), (5) and (6) respectively. □

Definition 3.6. A map $C : L^X \to L^X$ is called a *right* (resp. *left*) *closure operator, if it satisfies the following conditions:*

(C1) $A \leq C(A)$, for all $A \in L^X$.

(C2) $C(C(A)) = C(A)$, for all $A \in L^X$.

(C3) C is a right (resp. left) isotone map.

A map $I : L^X \to L^X$ is called a right (resp. left) interior operator, if it satisfies the following conditions:

(I1) $I(A) \leq A$ for all $A \in L^X$,

(I2) $I(I(A)) = I(A)$ for all $A \in L^X$

(I3) I is a right (resp. left) isotone map.

The pair $(I(A), C(A))$ ic called a *rough set* of A.

Theorem 3.7. (1) *Let $C : L^X \to L^X$ be a right closure operator. Define a map $I : L^X \to L^X$ as $I(A) = n_1(C(n_2(A)))$. Then I is a left interior operator where $(I(A), C(A))$ is a rough set of A.*

(2) *Let $C : L^X \to L^X$ be a left closure operator. Define a map $I : L^X \to L^X$ as $I(A) = n_2(C(n_1(A)))$. Then I is a right interior operator where $(I(A), C(A))$ is a rough set of A.*

Proof. (1) (I1) Since $n_2(A) \leq C(n_2(A))$, $I(A) = n_1(C(n_2(A))) \leq n_1(n_2(A)) = A$.

(I2) b$I(I(A)) = n_1(C(n_2(n_1(C(n_2(A)))))) = n_1(C(C(n_2(A)))) = n_1(n_2(A)) = A$.

(I3) I is a left isotone map from:

$$\begin{array}{l} d^l_{L^X}(A, B) = d^r_{L^X}(n_2(B), n_2(A)) \geq d^r_{L^X}(C(n_2(B)), C(n_2(A))) \\ = d^l_{L^X}(n_1(C(n_2(A))), n_1(C(n_2(B)))) = d^l_{L^X}(I(A), I(B)). \end{array}$$

(2) It is proved by a similar method as (1). □

Theorem 3.8. *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be two maps.*

(1) *A pair (G, H) is a residuated connection with two right isotone maps G and H iff for all $A \in L^X$ and $B \in L^Y$, $d^r_{L^Y}(G(A), B) = d^r_{L^X}(A, H(B))$.*

(2) *A pair (G, H) is a residuated connection with two left isotone maps G and H iff for all $A \in L^X$ and $B \in L^Y$, $d^l_{L^Y}(G(A), B) = d^l_{L^X}(A, H(B))$.*

(3) *A pair (G, H) is a Galois connection with right antitone map G and left antitone map H iff for all $A \in L^X$ and $B \in L^Y$, $d^l_{L^X}(A, H(B)) = d^r_{L^Y}(B, G(A))$.*

(4) *A pair (G, H) is a Galois connection with left antitone map G and right antitone map H iff for all $A \in L^X$ and $B \in L^Y$, $d^r_{L^X}(A, H(B)) = d^l_{L^Y}(B, G(A))$.*

(5) *A pair (G, H) is a dual Galois connection with right antitone map G and left antitone map H iff for all $A \in L^X$ and $B \in L^Y$, $d^l_{L^X}(H(B), A) = d^r_{L^Y}(G(A), B)$.*

(6) *A pair (G, H) is a dual Galois connection with left antitone map G and right antitone map H iff for all $A \in L^X$ and $B \in L^Y$, $d^r_{L^X}(H(B), A) = d^l_{L^Y}(G(A), B)$.*

Proof. (1) Let (G, H) be a residuated connection. Then $G(A) \leq G(A)$ iff $A \leq H(G(A))$ and $H(B) \leq H(B)$ iff $G(H(B)) \leq B$. Thus $d^r_{L^Y}(G(A), B) = d^r_{L^X}(A, H(B))$

from

$$d^r_{L^Y}(G(A),B) \geq d^r_{L^X}(A,H(G(A))) \oplus d^r_{L^X}((H(G(A)),H(B)) \geq d^r_{L^X}(A,H(B)),$$
$$d^r_{L^X}(A,H(B)) \geq d^r_{L^Y}(G(A),G(H(B))) \oplus d^r_{L^Y}(G(H(B)),B) \geq e^r_{L^Y}(G(A),B).$$

Conversely, since $d^r_{L^Y}(G(A),B) = d^r_{L^X}(A,H(B))$ for all $A \in L^X$ and $B \in L^Y$, $d^r_{L^X}(A,H(B)) = \perp$ $A \leq H(B)$ iff $G(A) \leq B$ iff $d^r_{L^Y}(G(A),B) = \perp$. Then (G,H) is a residuated connection. Put $B = G(A)$. Then $\perp = d^r_{L^Y}(G(A),G(A)) = d^r_{L^X}(A,H(G(A)))$. Thus $A \leq H(G(A))$. Put $A = H(B)$. Then we similarly obtain $G(H(B)) \leq B$. Thus we obtain two right isotone maps G and H from:

$$\begin{aligned}&d^r_{L^X}(A,B) = d^r_{L^X}(A,B) \oplus d^r_{L^X}(B,H(G(B)))\\ &\geq d^r_{L^X}(A,H(G(B))) = d^r_{L^Y}(G(A),G(B)),\\ &d^r_{L^Y}(A,B) = d^r_{L^Y}(G(H(A)),A) \oplus d^r_{L^Y}(A,B)\\ &\geq d^r_{L^Y}(G(H(A),B) = d^r_{L^X}(H(A),H(B)).\end{aligned}$$

(3) Let (G,H) be a Galois connection. Then $G(A) \leq G(A)$ iff $A \leq H(G(A))$ and $H(B) \leq H(B)$ iff $B \leq G(H(B))$. Moreover, since G is a right antitone map and H is a left antitone map, we have

$$d^r_{L^Y}(B,G(A)) \geq d^l_{L^X}(H(G(A)),H(B)) \geq d^l_{L^X}(A,H(B)),$$
$$d^l_{L^X}(A,H(B)) \geq d^r_{L^Y}(G(H(B)),G(A)) \geq d^r_{L^Y}(B,G(A)).$$

Thus $d^l_{L^X}(A,H(B)) = d^r_{L^Y}(B,G(A))$.

Conversely, since $d^l_{L^X}(A,H(B)) = d^r_{L^Y}(B,G(A))$, $A \leq H(B)$ iff $B \leq G(A)$. Moreover,

$$d^r_{L^Y}(G(A),G(B)) = d^l_{L^X}(B,H(G(A))) \leq d^l_{L^X}(B,A),$$
$$d^l_{L^X}(H(A),H(B)) = d^r_{L^Y}(B,G(H(A))) \leq d^r_{L^Y}(B,A).$$

(5) Let (G,H) be a dual Galois connection. Then $G(A) \leq G(A)$ iff $H(G(A)) \leq A$ and $H(B) \leq H(B)$ iff $G(H(B)) \leq B$. Moreover, since G is a right antitone map and H is a left antitone map, we have

$$d^r_{L^Y}(G(A),B) \geq d^l_{L^X}(H(B),H(G(A))) \geq d^l_{L^X}(H(B),A),$$
$$d^l_{L^X}(H(B),A) \geq d^r_{L^Y}(G(A),G(H(B))) \geq d^r_{L^Y}(G(A),B).$$

Thus $d^l_{L^X}(H(B),A) = d^r_{L^Y}(G(A),B)$.

Conversely, since $d^l_{L^X}(A,H(B)) = d^r_{L^Y}(B,G(A))$, $H(B) \leq A$ iff $G(A) \leq B$. Moreover,

$$d^r_{L^Y}(G(A),G(B)) = d^l_{L^X}(H(G(B)),A) \leq d^l_{L^X}(B,A),$$
$$d^l_{L^X}(H(A),H(B)) = d^r_{L^Y}(G(H(B)),A) \leq d^r_{L^Y}(B,A).$$

(2), (4) and (6) are similarly proved as (1), (3) and (5) respectively. □

Theorem 3.9. *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be right isotone maps with a residuated connection (G,H). Then the following statements hold:*

(1) *$H \circ G : L^X \to L^X$ is a right closure operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a right interior operator,*

(3) *if $X = Y$, then $(G(H(A)), H(G(A)))$ is a rough set of A.*

Proof. (1) Since $A \leq H(G(A))$, $H(G(A)) \leq H(G(H(G(A))))$ for all $A \in L^X$. Since $B \geq G(H(B))$, $G(A) \geq G(H(G(A)))$ and $H(G(A)) \geq H(G(H(G(A))))$. Then $H(G(A)) = H(G(H(G(A))))$. Since G and H are right isotone maps, $d^r_{L^X}(A, B) \geq d^r_{L^X}(H(G(A)), H(G(B)))$.

(2) and (3) are similarly proved as (1) and the definition of rough set. □

Corollary 3.10. *Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be left isotone maps with a residuated connection (G, H). Then the following statements hold:*

(1) *$H \circ G : L^X \to L^X$ is a left closure operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a left interior operator,*

(3) *If $X = Y$, then $(G(H(A)), H(G(A)))$ is a rough set of A.*

Theorem 3.11. *Let $G : L^X \to L^Y$ be a right antitone map and $H : L^Y \to L^X$ be a left antitone map with a Galois connection (G, H). Then*

(1) *$H \circ G : L^X \to L^X$ is a left closure operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a right closure operator.*

Proof. (1) Since $A \leq H(G(A))$, $H(G(A)) \leq H(G(H(G(A))))$ for all $A \in L^X$. Since $B \leq G(H(B))$ then $G(A) \leq G(H(G(A)))$ and $H(G(A)) \geq H(G(H(G(A))))$, because H is a left antitone map. Then $H(G(A)) = H(G(H(G(A))))$.

Since G is a right antitone map and H is a left antitone map, $d^l_{L^X}(A, B) \geq d^r_{L^X}(G(B), G(A)) \geq d^l_{L^X}(H(G(A)), H(G(B)))$.

(2) is similarly proved as (1). □

Corollary 3.12. *Let $G : L^X \to L^Y$ be a left antitone map and $H : L^Y \to L^X$ be a right antitone map with a Galois connection (G, H). Then*

(1) *$H \circ G : L^X \to L^X$ is a right closure operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a left closure operator.*

Theorem 3.13. *Let $G : L^X \to L^Y$ be a right antitone map and $H : L^Y \to L^X$ be a left antitone map with a dual Galois connection (G, H). Then*

(1) *$H \circ G : L^X \to L^X$ is a left interior operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a right interior operator.*

Proof. (1) Since $H(G(A)) \leq A$, $H(G(H(G(A)))) \leq H(G(A))$ for all $A \in L^X$. Since $G(H(B)) \leq B$, $G(H(G(A))) \leq G(A)$ and $H(G(A)) \leq H(G(H(G(A))))$, because H is a left antitone map. Then $H(G(A)) = H(G(H(G(A))))$.

Since G is a right antitone map and H is a left antitone map, $d^l_{L^X}(A, B) \geq d^r_{L^X}(G(B), G(A)) \geq d^l_{L^X}(H(G(A)), H(G(B)))$.

(2) is similarly proved as (1). □

Corollary 3.14. *Let $G : L^X \to L^Y$ be a left antitone map and $H : L^Y \to L^X$ be a right antitone map with a Galois connection (G, H). Then*

(1) *$H \circ G : L^X \to L^X$ is a right interior operator,*

(2) *$G \circ H : L^Y \to L^Y$ is a left interior operator.*

Theorem 3.15. *Let $F, G : L^X \to L^X$ be maps such that $d^r_{L^X}(F(A), B) = d^r_{L^X}(A, G(B))$. Then the following statements are equivalent.*

(1) *F is a right interior operator.*

(2) *G is a right closure operator.*

(3) $F \circ G = F$.
(4) $G \circ F = G$.

Proof. Since $d^r_{L^X}(F(A), B) = d^r_{L^X}(A, G(B))$, by Theorem 3.8 (1), F and G are right isotone maps.

(1) ⇒(2). Since $\perp = d^r_{L^X}(F(A), A) = d^r_{L^X}(A, G(A))$, $A \le G(A)$. Then we have

$$\begin{aligned} d^r_{L^X}(G(G(A)), G(A)) &= d^r_{L^X}(F(G(G(A))), A) \\ &= d^r_{L^X}(F(F(G(G(A)))), A) \\ &= d^r_{L^X}(F(G(G(A))), G(A)) \\ &= d^r_{L^X}(G(G(A)), G(G(A))) \\ &= \perp. \end{aligned}$$

Thus G is a right closure operator.

(2) ⇒(3). Since F is a right isotone map, $\perp = d^r_{L^X}(A, G(A)) \ge d^r_{L^X}(F(A), F(G(A)))$. Then $F(A) \le F(G(A))$. Moreover, $F(A) = F(G(A))$ from:

$$\begin{aligned} d^r_{L^X}(F(G(A)), F(A)) &= d^r_{L^X}(G(A), G(F(A))) \\ &= d^r_{L^X}(G(A), G(G(F(A)))) \\ &\le d^r_{L^X}(A, G(F(A))) \\ &= \perp. \text{ [Since } G \text{ is a right isotone map]} \end{aligned}$$

(3) ⇒(4). Let $F \circ G = F$. Then $G \circ F \circ G = G \circ F$. Since

$$\perp = d^r_{L^X}(F(G(A)), F(G(A))) = d^r_{L^X}(G(A), G(F(G(A))))$$

and

$$\perp = d^r_{L^X}(G(A), G(A)) = d^r_{L^X}(F(G(A)), A),$$

$G \circ F \circ G \ge G$ and $F \circ G(A) \le A$. Thus $G \circ F \circ G(A) \le G(A)$. So $G \circ F = G \circ F \circ G = G$.

(4) ⇒(3). It follows from $F \circ G \circ F = F$.

(3) and (4) ⇒(1).

$$\begin{aligned} &d^r_{L^X}(F(A), A) \le d^r_{L^X}(F(A), F(G(A))) \oplus d^r_{L^X}(F(G(A)), A) = \perp, \\ &d^r_{L^X}(F(A), F(F(A))) = d^r_{L^X}(A, G(F(F(A)))) = d^r_{L^X}(A, G(F(A))) = \perp. \end{aligned}$$

□

Corollary 3.16. *Let $F, G : L^X \to L^X$ be maps such that $d^l_{L^X}(F(A), B) = d^l_{L^X}(A, G(B))$. Then the following statements are equivalent.*
(1) *F is a left interior operator.*
(2) *G is a left closure operator.*
(3) $F \circ G = F$.
(4) $G \circ F = G$.

Corollary 3.17. *Let $F, G : L^X \to L^X$ be maps such that $d^l_{L^X}(F(A), B) = d^l_{L^X}(A, G(B))$. Then the following statements are equivalent.*
(1) *F is a left closure operator.*
(2) *G is a left interior operator.*
(3) $G \circ F = F$.
(4) $F \circ G = G$.

Corollary 3.18. *Let $F, G : L^X \to L^X$ be maps such that $d^r_{L^X}(F(A), B) = d^r_{L^X}(A, G(B))$. Then the following statements are equivalent.*
(1) *F is a right closure operator.*
(2) *G is a right interior operator.*

(3) $G \circ F = F$.
(4) $F \circ G = G$.

4. Examples of interior and closure operators

In this section, we investigate interior, closure operators and rough sets for an information $(X, Y, R \in L^{X\times Y})$, where X is a set of objects and Y is a set of attributes.

Definition 4.1. For each $A \in L^X$ and $B \in L^Y$ and $R \in L^{X\times Y}$, $R^{\oplus}, {}^{\oplus}R, R^{\ominus}, R^{\oslash}, {}^{\ominus}R, \oslash R : L^X \to L^Y$ are defined as:

$R^{\oplus}(A)(y) = \bigwedge_{x\in X}(R(x,y) \oplus A(x)),\ {}^{\oplus}R(A)(y) = \bigwedge_{x\in X}(A(x) \oplus R(x,y)),$
$R^{\ominus}(A)(y) = \bigvee_{x\in X}(R(x,y) \ominus A(x)),\ R^{\oslash}(A)(y) = \bigvee_{x\in X}(R(x,y) \oslash A(x)),$
${}^{\ominus}R(A)(y) = \bigvee_{x\in X}(A(x) \ominus R(x,y)),\ {}^{\oslash}R(A)(y) = \bigvee_{x\in X}(A(x) \oslash R(x,y)),$
$R^{1\oplus}(A)(y) = \bigwedge_{x\in X}(n_1R(x,y) \oplus n_1A(x)),\ {}^{2\oplus}R(A)(y) = \bigwedge_{x\in X}(n_2A(x) \oplus n_2R(x,y)).$

Theorem 4.2. (1) *$R^{\oplus}$ and ${}^{\oslash}R^{-1}$ are left isotone maps with a residuated connection $({}^{\oslash}R^{-1}, R^{\oplus})$.*
(2) *$R^{\oplus} \circ^{\oslash} R^{-1} : L^Y \to L^Y$ is a left closure operator.*
(3) *${}^{\oslash}R^{-1} \circ R^{\oplus} : L^X \to L^X$ is a left interior operator.*
(4) *$({}^{\oslash}R^{-1} \circ R^{\oplus}(A), R^{-1\oplus} \circ^{\oslash} R(A))$ is a rough set of $A \in L^X$.*
(5) *If $X = Y$, then $({}^{\oslash}R^{-1} \circ R^{\oplus}(A), R^{\oplus} \circ^{\oslash} R^{-1}(A))$ is a rough set of $A \in L^X$.*

Proof. (1) Since $R(x,y) \oplus B(x) \oplus (A(x) \oslash B(x)) \geq R(x,y) \oplus A(x)$, $d^l_{L^X}(A,B) \geq d^l_{L^Y}(R^{\oplus}(A), R^{\oplus}(B))$. Since $(B(y) \oslash R(x,y)) \oplus (A(y) \oslash B(y)) \leq A(y) \oslash R(x,y)$, $d^l_{L^Y}(A,B) \geq d^l_{L^X}({}^{\oslash}R^{-1}(A), {}^{\oslash}R^{-1}(B))$. Then by Theorem 3.8 (2), we only show the following statement:

$$\begin{aligned} d^l_{L^Y}(B, R^{\oplus}(A)) &= \bigvee_{y\in Y}(B(y) \oslash R^{\oplus}(A)(y)) \\ &= \bigvee_{y\in Y}\Big(B(y) \oslash \bigwedge_{x\in X}(R(x,y) \oplus A(x))\Big) \\ &= \bigvee_{y\in Y}\bigvee_{x\in X}\Big((B(y) \oslash R(x,y)) \oslash A(x)\Big) \\ &= \bigvee_{x\in X}\Big(\bigvee_{y\in Y}(B(y) \oslash R(x,y)) \oslash A(x)\Big) \\ &= \bigvee_{x\in X}\Big({}^{\oslash}R^{-1}(B)(x) \oslash A(x)\Big) \\ &= d^l_{L^X}({}^{\oslash}R^{-1}(B), A). \end{aligned}$$

(2), (3) and (4), (5) follow from Corollary 3.10 and the definition of a rough set. □

Theorem 4.3. (1) *${}^{\oplus}R$ and ${}^{\ominus}R^{-1}$ are right isotone maps with a residuated connection $({}^{\ominus}R^{-1}, {}^{\oplus}R)$.*
(2) *${}^{\oplus}R \circ^{\ominus} R^{-1} : L^Y \to L^Y$ is a right closure operator.*
(3) *${}^{\ominus}R^{-1} \circ^{\oplus} R : L^X \to L^X$ is a right interior operator.*
(4) *$({}^{\ominus}R^{-1} \circ^{\oplus} R(A), {}^{-1\oplus}R \circ^{\ominus} R(A))$ is a rough set of A.*
(5) *If $X = Y$, then $({}^{\ominus}R^{-1} \circ^{\oplus} R(A), {}^{\oplus}R \circ^{\ominus} R^{-1}(A))$ is a rough set of A.*

Proof. (1) Since $(A(x) \ominus B(x)) \oplus B(x) \oplus R(x,y) \geq A(x) \oplus R(x,y)$, $d^r_{L^X}(A,B) \geq d^r_{L^Y}({}^{\oplus}R(A), {}^{\oplus}R(B))$. Since $(A(y) \ominus B(y)) \oplus (B(x) \ominus R(x,y)) \geq A(x) \ominus R(x,y)$, $d^r_{L^Y}(A,B) \geq d^r_{L^X}({}^{\ominus}R^{-1}(A), {}^{\ominus}R^{-1}(B))$. Then by Theorem 3.8 (1), we only show the following statement:

$$\begin{aligned} d^r_{L^Y}(B,^{\oplus} R(A)) &= \bigvee_{y\in Y}(B(y)\ominus^{\oplus} R(A)(y)) \\ &= \bigvee_{y\in Y}\Big(B(y)\ominus \bigwedge_{x\in X}(A(x)\oplus R(x,y))\Big) \\ &= \bigvee_{y\in Y}\bigvee_{x\in X}\Big(B(y)\ominus (A(x)\oplus (R(x,y))\Big) \\ &= \bigvee_{x\in X}\Big(\bigvee_{y\in Y}(B(y)\ominus R(x,y))\ominus A(x)\Big) \\ &= d^r_{L^X}(^{\ominus}R^{-1}(B),A). \end{aligned}$$

(2), (3) and (4), (5) follow from Theorem 3.9 and the definition of a rough set. □

Theorem 4.4. *Let (X, d^l_X) be a left distance function. Let $^{\oslash}(d^l_X)^{-1}, (d^l_X)^{\oplus}, : L^X \to L^X$ be maps in above theorem with $R^{-1} = (d^l_X)^{-1}$. Then the following statements hold.*

(1) *$^{\oslash}(d^l_X)^{-1}$ is a left closure operator.*

(2) *$(d^l_X)^{\oplus}$ is a left interior operator with a rough set $((d^l_X)^{\oplus}(A),^{\oslash}(d^l_X)^{-1}(A))$ for each $A \in L^X$.*

(3) *$(d^l_X)^{\oplus} =^{\oslash} (d^l_X)^{-1} \circ (d^l_X)^{\oplus}$.*

(4) *$(d^l_X)^{\oplus} \circ^{\oslash} (d^l_X)^{-1} =^{\oslash} (d^l_X)^{-1}$.*

(5) *Define $I : L^X \to L^X$ as $I(A) = n_2(^{\oslash}(d^l_X)^{-1}(n_1(A)))$. Then I is a right interior operator such that $I(A) = \bigwedge_{z\in X}(A(z)\oplus n_2(n_2(d^l_X(y,z))))$.*

(6) *Define $C : L^X \to L^X$ as $C(A) = n_2((d^l_X)^{\oplus}(n_1(A)))$. Then C is a right closure operator such that $C(A)(y) = \bigvee_{x\in X}(A(x)\ominus n_2(n_2(d^l_X(x,y))))$.*

Proof. (1) Since $(B(y)\oslash d^l_X(x,y))\oplus(A(y)\oslash B(y)) \geq A(y)\oslash d^l_X(x,y)$,

$$d^l_{L^X}(A,B) \geq d^l_{L^X}(^{\oslash}(d^l_X)^{-1}(A),^{\oslash}(d^l_X)^{-1}(B)).$$

Since d^l_X is a left distance function, $\bigvee_{y\in X}((d^l_X(y,z)\oplus d^l_X(x,y)) = d^l_X(x,z)$. Thus

$$^{\oslash}(d^l_X)^{-1}(A)(y) = \bigvee_{z\in X}(B(z)\oslash d^l_X(y,z)) \geq B(y)\oslash d^l_X(y,y) = A(y)$$

and

$$\begin{aligned} ^{\oslash}(d^l_X)^{-1}(^{\oslash}(d^l_X)^{-1}(B))(x) &= \bigvee_{y\in X}(^{\oslash}(d^l_X)^{-1}(B)(y)\oslash d^l_X(x,y)) \\ &= \bigvee_{y\in X}\Big(\bigvee_{z\in X}(B(z)\oslash d^l_X(y,z))\oslash d^l_X(x,y)\Big) \\ &= \bigvee_{z\in X}\Big(B(z)\oslash \bigwedge_{y\in X}(d^l_X(y,z)\oplus d^l_X(x,y))\Big) \\ &= \bigvee_{z\in X}(B(z)\oslash d^l_X(x,z)) \\ &=^{\oslash}(d^l_X)^{-1}(B)(x). \end{aligned}$$

Thus $^{\oslash}(d^l_X)^{-1}$ is a left closure operator.

(2) Since $d^l_X(x,y)\oplus B(x)\oplus(A(x)\oslash B(x)) \geq d^l_X(x,y)\oplus A(y)$,

$$d^l_{L^X}(A,B) \geq d^l_{L^X}((d^l_X)^{\oplus}(A),(d^l_X)^{\oplus}(B)).$$

Since d^l_X is a left isotone map, $\bigwedge_{y\in X}((d^l_X(y,z)\oplus d^l_X(x,y)) = d^l_X(x,z)$. Then

$$(d^l_X)^{\oplus}(A) \leq A$$

and

$$(d^l_X)^{\oplus}((d^l_X)^{\oplus}(A))(y) = \bigwedge_{z\in X}(d^l_X(z,y)\oplus (d^l_X)^{\oplus}(A)(z))$$

$$\begin{aligned} &= \bigwedge_{z\in X}((d^l_X(z,y) \oplus \bigwedge_{x\in X}(d^l_X(x,z) \oplus A(x))) \\ &= \bigwedge_{z\in X}(\bigwedge_{x\in X}((d^l_X(z,y) \oplus d^l_X(x,z)) \oplus A(x))) \\ &= \bigwedge_{x\in X}(\bigwedge_{z\in X}((d^l_X(z,y) \oplus d^l_X(x,z)) \oplus A(x))) \\ &= \bigwedge_{x\in X}((d^l_X(x,y) \oplus A(x)) \\ &= (d^l_X)^{\oplus}(A)(y). \end{aligned}$$

(3) Since $(d^l_X)^{\oplus} \leq^{\oslash} (d^l_X)^{-1} \circ (d^l_X)^{\oplus}$, by (1), we only show

$$(d^l_X)^{\oplus} \geq^{\oslash} (d^l_X)^{-1} \circ (d^l_X)^{\oplus}$$

from:

$$\begin{aligned} \perp \quad &= d^r_{L^X}((d^l_X)^{\oplus}(A), (d^l_X)^{\oplus}(A)) = d^r_{L^X}((d^l_X)^{\oplus}(A), (d^l_X)^{\oplus}((d^l_X)^{\oplus}(A))) \\ &= d^r_{L^X}((^{\oslash}(d^l_X)^{-1}((d^l_X)^{\oplus}(A)), (d^l_X)^{\oplus}(A)). \end{aligned}$$

(2) and (4) are similarly proved as (1) and (3).

(5) By Theorem 3.8 (2), I is a right interior operator. Moreover,

$$\begin{aligned} &I(A)(x) = n_2(^{\oslash}(d^l_X)^{-1}(n_1(A))(x) = n_2(\bigvee_{y\in X}(n_1(A)(y) \oslash d^l_X(x,y))) \\ &= \bigwedge_{y\in X} n_2((n_1(A)(y) \oslash d^l_X(x,y))) = \bigwedge_{y\in X} n_2(n_2(d^l_X(x,y)) \ominus A(y)) \\ &= \bigwedge_{y\in X}(A(y) \oplus n_2(n_2(d^l_X(x,y)))). \text{ [By Lemma 3.1 (15), 16)]} \end{aligned}$$

(6) It is similarly proved as (5). □

Corollary 4.5. *Let (X, d^r_X) be a right distance function. Let $^{\ominus}(d^r_X)^{-1}, ^{\oplus}(d^r_X), : L^X \to L^X$ be maps in above theorem with $R^{-1} = (d^r_X)^{-1}$. Then the following statements hold.*

(1) *$^{\ominus}(d^r_X)^{-1}$ is a right closure operator.*

(2) *$^{\oplus}(d^r_X)$ is a right interior operator with a rough set $(^{\oplus}(d^r_X)(A), ^{\ominus}(d^r_X)^{-1}(A))$ for each $A \in L^X$.*

(3) *$^{\oplus}(d^r_X) = ^{\ominus}(d^r_X)^{-1} \circ ^{\oplus}(d^r_X)$.*

(4) *$^{\oplus}(d^r_X) \circ ^{\ominus}(d^r_X)^{-1} = ^{\ominus}(d^r_X)^{-1}$.*

Theorem 4.6. (1) *$R^{\oslash}$ is a right antitone map and $R^{-1\ominus}$ is a left antitone map.*

(2) *$R^{\ominus}$ is a left antitone map and $R^{-1\oslash}$ is a right antitone map.*

(3) *The pair $(R^{\oslash}, R^{-1\ominus})$ is a dual Galois connection.*

(4) *The pair $(R^{\ominus}, R^{-1\oslash})$ is a dual Galois connection.*

(5) *$R^{-1\ominus} \circ R^{\oslash} : L^X \to L^X$ is a left interior operator and $R^{\oslash} \circ R^{-1\ominus} : L^Y \to L^Y$ is a right interior operator.*

(6) *$R^{-1\oslash} \circ R^{\ominus} : L^X \to L^X$ is a right interior operator and $R^{\ominus} \circ R^{-1\oslash} : L^Y \to L^Y$ is a left interior operator.*

Proof. (1) Since $(A(x) \oslash B(x)) \oplus (R(x,y) \oslash A(x)) \geq R(x,y) \oslash B(x)$,

$$d^l_{L^X}(A,B) \geq d^r_{L^Y}(R^{\oslash}(B), R^{\oslash}(A)).$$

Since $(R(x,y) \ominus A(y)) \oplus (A(y) \ominus B(y)) \geq R(x,y) \ominus B(y)$,

$$d^r_{L^X}(A,B) \geq d^l_{L^Y}(R^{-1\ominus}(B), R^{-1\ominus}(A)).$$

(2) It is similarly proved as (1).

(3) From Theorem 3.8 (1), $d^l_{L^X}(R^{-1\ominus}(B), A) = d^r_{L^Y}(R^{\oslash}(A), B)$ from:

$$\begin{aligned}
& d^l_{L^X}(R^{-1\ominus}(B), A) \\
& = \bigvee_{x\in X}(R^{-1\ominus}(B)(x) \oslash A(x)) \\
& = \bigvee_{x\in X}(\bigvee_{y\in X}(R(x,y) \ominus B(y)) \oslash A(x)) \\
& = \bigvee_{x\in X}\bigvee_{y\in X}((R(x,y) \oslash A(x)) \ominus B(y)) \\
& = \bigvee_{y\in X}(R^{\oslash}(A) \ominus B(y)) \\
& = d^r_{L^Y}(R^{\oslash}(A), B).
\end{aligned}$$

(2) and (4) are similarly proved as (1) and (3) respectively.
(5) and (6) are proved from Theorem 3.11 and Corollary 3.12 respectively. □

Theorem 4.7. (1) *$R^{1\oplus}$ is a right antitone map and $^{2\oplus}R^{-1}$ is a left antitone map.*
(2) *The pair $(R^{1\oplus}, {}^{2\oplus}R^{-1})$ is a Galois connection.*
(3) *$^{2\oplus}R^{-1} \circ R^{1\oplus} : L^X \to L^X$ is a left closure operator and $R^{1\oplus} \circ {}^{2\oplus}R^{-1} : L^Y \to L^Y$ is a right closure operator.*

Proof. (1) Since $(n_1R(x,y) \oplus n_1A(x)) \oplus (n_1B(x) \oslash n_1A(x)) \geq R(x,y) \oplus n_1B(x)$, $(A(x) \ominus B(x)) = (n_1B(x) \oslash n_1A(x)) \geq (n_1R(x,y) \oplus n_1B(x)) \oslash (n_1R(x,y) \oplus n_1A(x))$. Then we have

$$d^r_{L^X}(A, B) \geq d^l_{L^Y}(R^{1\oplus}(B), R^{1\oplus}(B)).$$

Since $(n_2B(y) \ominus n_2A(y)) \oplus (n_2A(y) \oplus n_2R^{-1}(x,y) \geq n_2B(y) \oplus n_2R^{-1}(x,y)$,

$(A(y) \oslash B(y)) = (n_2B(y) \ominus n_2A(y)) \geq (n_2B(y) \oplus n_2R^{-1}(x,y)) \ominus (n_2A(y) \oplus n_2R^{-1}(x,y))$.

Thus we get

$$d^l_{L^Y}(A, B) \geq d^r_{L^X}({}^{2\oplus}R^{-1}(B), {}^{2\oplus}R^{-1}(A)).$$

(2) From Theorem 3.8 (4), $d^l_{L^X}(R^{-1\ominus}(B), A) = d^r_{L^Y}(R^{\oslash}(A), B)$ from:

$$\begin{aligned}
& d^l_{L^Y}(B, R^{1\oplus}(A)) = \bigvee_{y\in Y}(B(y) \oslash R^{1\oplus}(A)(y)) \\
& = \bigvee_{x\in X}(B(y) \oslash \bigwedge_{y\in X}(n_1R(x,y) \oplus n_1(A)(y))) \\
& = \bigvee_{x\in X}\bigvee_{y\in X}((B(y) \oslash n_1R(x,y)) \oslash n_1(A)(y)) \\
& = \bigvee_{x\in X}\bigvee_{y\in X}(A(y) \ominus n_2((B(y) \oslash n_1R(x,y)))) \\
& = \bigvee_{x\in X}\bigvee_{y\in X}(A(y) \ominus n_2(R(x,y) \ominus n_2B(y))) \\
& = \bigvee_{x\in X}\bigvee_{y\in X}(A(y) \ominus (n_2(B)(y) \oplus n_2R(x,y))) \\
& = d^r_{L^Y}(A, {}^{2\oplus}R^{-1}(B)).
\end{aligned}$$

(3) It follows from Theorem 3.11. □

Example 4.8. Let $(L, \oplus, \oplus, \ominus, \oslash, (\frac{1}{2}, 1), (1, 0))$ be a generalized co-residuated lattice with a double negative law, where $\perp = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element in Example 2.2 (7).

Let $X = \{a, b, c\}$ be a set. Define $d_X, d^l_X : X \times X \to L$ as

$$d_X = \begin{pmatrix} (\frac{1}{2}, 1) & (\frac{4}{5}, -1) & (\frac{3}{5}, 0) \\ (\frac{7}{10}, -2) & (\frac{1}{2}, 1) & (\frac{4}{5}, 0) \\ (\frac{1}{2}, 3) & (\frac{7}{10}, -\frac{4}{3}) & (\frac{1}{2}, 1) \end{pmatrix}$$

$$d^l_X = \begin{pmatrix} (\frac{1}{2}, 1) & (\frac{3}{4}, \frac{1}{4}) & (\frac{3}{5}, \frac{2}{5}) \\ (\frac{7}{10}, -\frac{11}{10}) & (\frac{1}{2}, 1) & (\frac{7}{12}, -\frac{4}{3}) \\ (\frac{3}{5}, \frac{8}{5}) & (\frac{2}{3}, -\frac{1}{3}) & (\frac{1}{2}, 1) \end{pmatrix}$$

Then we easily show that d_X is a right and left distance function and d^l_X is a left distance function. But d^l_X is not a right distance function, because

$$d^l_X(b,c) \oplus d^l_X(c,a) = (\tfrac{7}{12}, -\tfrac{4}{3}) \oplus (\tfrac{3}{5}, \tfrac{8}{5}) = (\tfrac{7}{10}, -\tfrac{6}{5}) \not\geq d^l_X(b,a) = (\tfrac{7}{10}, -\tfrac{11}{10}) \ .$$

By Theorem 4.4 and Corollary 4.5, we have various rough sets as follows, for each $A \in L^X$,

$$(d^{\oplus}_X(A), ^{\oslash}(d_X)^{-1}(A)), (^{\oplus}d_X(A), ^{\ominus}d^{-1}_X(A)), ((d^l_X)^{\oplus}(A), ^{\oslash}(d^l_X)^{-1}(A)).$$

Since

$$\begin{array}{l} ^{\oplus}(d^l_X)(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d^l_X(x,y)) \\ ^{\ominus}(d^l_X)^{-1}(A)(x) = \bigvee_{y\in X}(A(y) \ominus d^l_X(x,y)) \end{array}$$

for $D = ((\frac{3}{4}, \frac{1}{4}), (\frac{5}{6}, \frac{11}{6}), (\frac{1}{2}, \frac{3}{2}))$,

$$\begin{array}{l} ^{\oplus}(d^l_X)(D) = ((\frac{3}{5}, \frac{8}{5}), (\frac{2}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{3}{2})) \\ ^{\ominus}(d^l_X)^{-1}(D) = ((\frac{3}{4}, \frac{1}{4}), (\frac{5}{6}, \frac{11}{6}), (\frac{5}{8}, \frac{21}{8})) \\ ^{\ominus}(d^l_X)^{-1}(^{\oplus}(d^l_X)(D)) = ((\frac{3}{5}, \frac{8}{5}), (\frac{4}{5}, -\frac{1}{2}), (\frac{1}{2}, \frac{3}{2})) \\ ^{\oplus}(d^l_X)(^{\ominus}(d^l_X)^{-1}(D)) = ((\frac{3}{4}, \frac{1}{4}), (\frac{5}{6}, -\frac{25}{12}), (\frac{5}{8}, \frac{21}{8})) \end{array}$$

Since d^l_X is not a right distance function, in general, $^{\oplus}(d^l_X)(D) \neq ^{\ominus}(d^l_X)^{-1}(^{\oplus}(d^l_X)(D))$ and $^{\ominus}(d^l_X)^{-1}(D) \neq ^{\oplus}(d^l_X)(^{\ominus}(d^l_X)^{-1}(D))$. Moreover, $^{\oplus}(d^l_X)$ is not a right interior operator because , for $\perp_b$ with $\perp_b(b) = \perp$ and $\perp_b(x) = \top$ for $x \neq b$,

$$\begin{array}{l} ^{\oplus}(d^l_X)(\perp_b)(-) = \bigwedge_{x\in X}(\perp_b(x) \oplus d^l_X(x,-)) = d^l_X(b,-) = ((\frac{7}{10}, -\frac{11}{10}), (\frac{1}{2}, 1), (\frac{7}{12}, -\frac{4}{3})) \\ ^{\oplus}(d^l_X)((-)^{\oplus}(d^l_X)(\perp_b))(-) = ((\frac{7}{10}, -\frac{6}{5}), (\frac{1}{2}, 1), (\frac{7}{12}, -\frac{4}{3})). \end{array}$$

As a information system $(X, Y, R \in L^{X\times Y})$, let $X = \{a, b, c\}$ be a set of objects and $Y = \{u, v\}$ be a set of attributes with an information $R \in L^{X\times Y}$ as

$$R = \begin{pmatrix} (1,0) & (\frac{5}{8}, \frac{5}{2}) \\ (\frac{2}{3}, -1) & (\frac{3}{4}, \frac{1}{4}) \\ (\frac{1}{2}, 2) & (\frac{1}{2}, 1) \end{pmatrix}$$

For $A = ((\frac{2}{3}, 1), (\frac{1}{2}, 2), (\frac{3}{4}, -1))$,

$$\begin{array}{rl} R^{\oplus}(A) = ((\frac{2}{3}, 0), (\frac{3}{4}, -1)), & ^{\oplus}R(A) = ((\frac{2}{3}, -\frac{7}{3}), (\frac{3}{4}, -1)), \\ R^{\ominus}(A) = ((\frac{3}{4}, \frac{1}{4}), (\frac{3}{4}, -\frac{3}{4})), & R^{\oslash}(A) = ((\frac{3}{4}, 0), (\frac{3}{4}, -\frac{5}{4})), \\ ^{\ominus}R(A) = ((\frac{3}{4}, -2), (\frac{3}{4}, -1)), & ^{\oslash}R(A) = ((\frac{3}{4}, -\frac{5}{2}), (\frac{3}{4}, -1)), \\ R^{1\oplus}(A) = ((\frac{3}{4}, \frac{1}{4}), (1, 0)), & ^{2\oplus}R(A) = ((\frac{3}{4}, 0), (1, 0)). \end{array}$$

For $B = ((\frac{3}{5}, 2), (\frac{2}{3}, -1))$,

$$\begin{array}{rl} R^{-1\oplus}(B) = ((\frac{5}{6}, \frac{5}{6}), (\frac{4}{5}, \frac{14}{5}), (\frac{3}{5}, \frac{16}{5})), & ^{\oplus}R^{-1}(B) = ((\frac{5}{6}, 0), (\frac{4}{5}, \frac{1}{3}), (\frac{3}{5}, 3)), \\ R^{-1\ominus}(B) = ((\frac{5}{6}, -\frac{2}{3}), (\frac{5}{9}, -\frac{3}{2}), (\frac{1}{2}, 1)), & R^{-1\oslash}(B) = ((\frac{5}{6}, -\frac{5}{3}), (\frac{5}{9}, -\frac{19}{9}), (\frac{1}{2}, 1)), \\ ^{\ominus}R^{-1}(B) = ((\frac{8}{15}, -\frac{9}{5}), (\frac{1}{2}, 1), (\frac{2}{3}, -1)), & ^{\oslash}R^{-1}(B) = ((\frac{8}{15}, -\frac{13}{5}), (\frac{1}{2}, 1), (\frac{2}{3}, -1)), \\ (R^{-1})^{1\oplus}(B) = ((\frac{5}{6}, -\frac{2}{3}), (1, 0), (1, 0)), & ^{2\oplus}R^{-1}(B) = ((\frac{5}{6}, -\frac{5}{3}), (1, 0), (1, 0)). \end{array}$$

Rough sets of $A \in L^X$ are

$$\begin{array}{l} (^{\oslash}R^{-1} \circ R^{\oplus}(A), R^{-1\oplus} \circ ^{\oslash} R(A)), \\ (^{\ominus}R^{-1} \circ ^{\oplus} R(A), ^{-1\oplus}R \circ ^{\ominus} R(A)), \\ (^{\oslash}R^{-1}(R^{\oplus}(A)), ^{2\oplus} R^{-1}(R^{1\oplus}(A))), \end{array}$$

where

$$\begin{aligned} R^{-1\oplus}(^{\oslash}R(A)) &= ((\tfrac{15}{16},\tfrac{5}{4}),(1,-\tfrac{11}{2}),(\tfrac{3}{4},-1)),\\ ^{\oplus}R^{-1}(^{\ominus}R(A)) &= ((\tfrac{15}{16},0),(1,-5),(\tfrac{3}{4},-1)),\\ ^{\ominus}R^{-1}(^{\oplus}R(A)) &= ((\tfrac{3}{5},-\tfrac{9}{5}),(\tfrac{1}{2},1),(\tfrac{3}{4},-1)),\\ ^{\oslash}R^{-1}(R^{\oplus}(A)) &= ((\tfrac{3}{5},-\tfrac{29}{20}),(\tfrac{1}{2},2),(\tfrac{3}{4},-1)),\\ ^{2\oplus}R^{-1}(R^{1\oplus}(A)) &= ((\tfrac{2}{3},1),(\tfrac{2}{3},\tfrac{9}{8}),(1,0)). \end{aligned}$$

5. Conclusion

In this paper, we are interested distance spaces instead of fuzzy partially ordered sets on generalized co-residuated lattices as a non-commutative algebraic structure. Using distance functions, we have investigated the relations between various closure (interior) operators and various connections. Moreover, as an application, we give various rough sets for an information system in Section 4.

In the future, we plan to investigate fuzzy concepts, information systems and decision rules by using the concepts of distance spaces in generalized co-residuated lattices.

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