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Crossing octahedron sets and their applications to Q-algebras

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Abstract. A crossing octahedron set is composed of three components : interval-valued fuzzy set, intuitionistic fuzzy set and negative-valued function. In this paper, we first introduce the concept of crossing octahedron set and investigate its properties. Further, we define a crossing octahedron Q-ideal, Q-subalgebra and BCK-ideal of a Q-algebra, and we deal with their relationships. In addition, the homomorphic images and pre-images of a crossing octahedron Q-ideal are also studied. Related properties of crossing octahedron Q-ideals are explored and discussed. Finally, the Cartesian product of two crossing octahedron Q-ideals is investigated.

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1. Introduction

Iseki and Tanaka [1] studied the concept of BCK-algebras in 1966. As a generalization of the set theoretic difference and propositional calculus proposed by Imai and Iseki [2], Iseki [3] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other structures including lattices and Boolean algebras. There is a great deal of literature that has been produced on the theory of BCK/BCI-algebras. In particular, the emphasis seems to have been put on the ideal theory of BCK/BCI-algebras. Hu and Li [4, 5] introduce a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al. [6] introduced a new notion, called Q-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras and generalizes some theorems discussed in BCI-algebras. Fuzzy set theory is the concept and technique which lay a form of mathematical precision to human thought process that in many ways are imprecise and ambiguous by the standards of classical mathematics. Fuzzy sets, intuitionistic fuzzy set, interval-valued fuzzy set, bipolar fuzzy set and other mathematical tools are often useful approaches to dealing with uncertainties. In 1965, Zadeh [7] introduced the notion of fuzzy sets. At present, this concept has been applied to many mathematical branches. There are several kinds of fuzzy sets extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets, $\cdots$, etc. The idea of the "intuitionistic fuzzy set" was first published by Atanassov [8, 9] as a generalization of the notion of fuzzy sets. In 1991, Xi [10] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Zadeh [11] made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function) and he constructed a method of approximate inference using interval-valued fuzzy sets. Lee [12] and Zhang [13] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [-1, 1]. Ejegwa et al. [14] present a brief overview on intuitionistic fuzzy sets which cuts across some definitions, operations, algebra, modal operators and normalization on intuitionistic fuzzy sets. Mostafa et al. [15, 16, 17] discussed intuitionistic and interval-valued fuzzy Q-ideals in Q-algebra. Mostafa et al. [18] introduced the notion of crossing intuitionistic KU-ideals and investigated its several properties. Also, the relations between crossing intuitionistic KU-ideal and crossing intuitionistic BCK-ideal are given. The image and the pre-image of a bipolar intuitionistic KU-ideal in KU-algebras are defined and how the image and the pre-image of a crossing intuitionistic KU-ideal in KU-algebras become crossing intuitionistic KU-ideals are studied. Moreover, the Cartesian product of crossing intuitionistic KU-ideals in a KU-algebras is established. Kim et al. [19] introduced the concept of octahedron sets composed of three components: interval-valued fuzzy set, intuitionistic fuzzy set and fuzzy set, which will provide more information about ambiguity and uncertainty common in everyday life, and dealt with its various properties. Recently, Jun et al. [20, 21] introduced a new function which is called a *negative-valued function*, and constructed N-structures. They applied N-structures to BCK/BCI-algebras, and discussed N-subalgebras and N-ideals in BCK/BCI-algebras. Jun et al. [22, 23] established an extension of a bipolar-valued fuzzy set, which is introduced by Lee [12] and Zhang [13]. They called it a *crossing cubic structure* and studied its several properties. Also, they applied crossing cubic structures to BCK/BCI-algebras and studied crossing cubic subalgebras.

In this paper, we introduce the notion of a crossing octahedron set and a crossing octahedron Q-ideal of a Q-algebra and investigate its properties. Furthermore, we study the homomorphic image and pre-image of crossing octahedron Q-ideals under homomorphism of Q-algebras. Moreover, the Cartesian product of two crossing Octahedron Q-ideals in a Q-algebra is given.

2. Preliminaries

Now we review some definitions and properties that will be useful in our results.

Definition 2.1 ([3]). Let X be a set with a binary operation " $*$ " and a constant 0. Then $(X, *, 0)$ is called a BCI-algebra, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(BCI_1) $((x * y) * (x * z)) * (z * y)] = 0$,
(BCI_2) $(x * (x * y)) * y = 0$,
(BCI_3) $x * x = 0$,
(BCI_4) $x * y = 0$ and $y * x = 0$ implies $x = y$.
If a BCI-algebra X satisfies the identity $0 * x = 0$, then X is called a BCK-algebra. It is well-known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Definition 2.2 ([1]). Let $(X, *, 0)$ be a BCK-algebra and let S be a nonempty subset of X. Then S is called a *subalgebra* of X, if $x * y \in S$ for any $x,\ y \in S$, i.e., S is closed under the binary operation $*$ on X.

Definition 2.3 ([6]). An algebraic system $(X, *, 0)$ of type $(2, 0)$ is called a Q-*algebra*, if it satisfies the following axioms: for any $x,\ y,\ z \in X$,
(i) $x * x = 0$,
(ii) $x * 0 = x$,
(iii) $(x * y) * z = (x * z) * y$.
For brevity, we also call X a Q-algebra. In X, we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$.

Example 2.4 ([6]). Let $X = \{0, 1, 2\}$ be the set with Cayley Table 2.1:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Table 2.1

Then $(X, *, 0)$ is a Q-algebra.

Theorem 2.5 ([6]). *Every BCK-algebra is a Q-algebra but the converse is not true.*

Example 2.6 ([6]). Let $X = \{0, 1, 2, 3\}$ be the set with Cayley Table 2.2:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Table 2.2

Then $(X, *, 0)$ is a Q-algebra but not a BCK-algebra.

Theorem 2.7 ([6]). *Every Q-algebra X satisfying the conditions (BCI_1), (BCI_4) and (iv)*

(iv) $$(x * y) * x = 0 \text{ for any } x,\ y \in X$$

is a BCK-algebra.

Definition 2.8 ([1]). Let X be a BCK-algebra and let J be a nonempty subset of X. Then J is called an *ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(I_1) $0 \in J$,
(I_2) $y * x \in J$ implies $y \in J$.

Definition 2.9 ([17]). Let X be a Q-algebra and let J be a nonempty subset of X. Then J is called a *Q-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(I_1) $0 \in J$,
(Q) $(x * y) * z \in J$ and $y \in J$ imply $x * z \in J$.

For a nonempty set X, a mapping $\lambda : X \to I$ is called a *fuzzy set* in X (See [7]), where $I = [0, 1]$. We denote the set of all fuzzy sets in X as I^X, where $I = [0, 1]$.

Definition 2.10 ([10]). Let X be a BCK-algebra and let $\lambda \in I^X$. Then λ is called a *fuzzy BCK-ideal* of X, if it satisfies the following conditions: for any $x,\ y \in X$,
(FI_1) $\lambda(0) \geq \lambda(x)$,
(I_2) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y)$.

Definition 2.11 ([16]). Let X be a Q-algebra and let $\lambda \in I^X$. Then λ is called a *fuzzy Q-ideal* of X, if it satisfies the following conditions: for any $x,\ y,\ z \in X$,
(FI_1) $\lambda(0) \geq \lambda(x)$,
(FQ) $\lambda(x * z) \geq \lambda((x * y) * z) \wedge \lambda(y)$.

Lemma 2.12 ([16]). *Every fuzzy Q-ideal of a Q-algebra X is a fuzzy BCK-ideal of X.*

Each member of a set $I \oplus I = \{(a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$ is called an *intuitionistic fuzzy number* (briefly, IFN), and $(0, 1)$ and $(1, 0)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively (See [24]). We will denote intuitionistic fuzzy numbers $(a^{\in}, a^{\notin})$, $(b^{\in}, b^{\notin})$, $(c^{\in}, c^{\notin})$, etc. as $\bar{a}$, $\bar{b}$, $\bar{c}$, etc.

Definition 2.13 ([8, 9, 14]). For a nonempty set X, a mapping $\bar{A} : X \to I \oplus I$ is called an *intuitionistic fuzzy set* (briefly, IFS) in X, where for each $x \in X$, $\bar{A}(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of non-membership of an element x to $\bar{A}$ respectively. In particular, $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the intuitionistic fuzzy empty set and the intuitionistic fuzzy whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

We will denote the set of all IFSs in X as $IFS(X)$.

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and

$0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$ (See [25]). Refer to [19] for the definitions of the order and the equality of two interval numbers, and the infimum and the supremum of any interval numbers.

Definition 2.14 ([11, 25])**.** For a nonempty set X, a mapping $\widetilde{A} : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an IVS) in X. Let $IVS(X)$ denote the set of all IVSs in X. For each $\widetilde{A} \in IVS(X)$ and $x \in X$, $\widetilde{A}(x) = [A^-(x), A^+(x)]$ is called the *degree of membership* of an element x to A, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For each $\widetilde{A} \in IVS(X)$, we write $\widetilde{A} = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in X defined by respectively: for each $x \in X$,

$$\widetilde{0}(x) = \mathbf{0} \text{ and } \widetilde{1}(x) = \mathbf{1}.$$

We define relations $\subset$ and $=$ on $IVS(X)$ as follows:

$$(\forall\ \widetilde{A},\ \widetilde{B} \in IVS(X))(\widetilde{A} \subset \widetilde{B} \Longleftrightarrow (x \in X)(\widetilde{A}(x) \leq \widetilde{B}(x)),$$

$$(\forall\ \widetilde{A},\ \widetilde{B} \in IVS(X))(\widetilde{A} = \widetilde{B} \Longleftrightarrow (x \in X)(\widetilde{A}(x) = \widetilde{B}(x)).$$

For each $\widetilde{A} \in IVS(X)$, the complement of $\widetilde{A}$, denoted by $\widetilde{A}^c$, is an IVS in X defined as follows:
for each $x \in X$,

$$\widetilde{A}^c(x) = [1 - A^+(x), 1 - A^-(x)].$$

For any $(\widetilde{A}_j)_{j \in J} \subset IVS(X)$, its *intersection* $\bigcap_{j \in J} \widetilde{A}_j$ and *union* $\bigcup_{j \in J} \widetilde{A}_j$ are IVSs in X defined respectively as follows: for each $x \in X$,

$$(\bigcap_{j \in J} \widetilde{A}_j)(x) = \bigwedge_{j \in J} \widetilde{A}_j(x) = [\bigwedge_{j \in J} A_j^-, \bigwedge_{j \in J} A_j^+],$$

$$(\bigcup_{j \in J} \widetilde{A}_j)(x) = \bigvee_{j \in J} \widetilde{A}_j(x) = [\bigvee_{j \in J} A_j^-, \bigvee_{j \in J} A_j^+].$$

3. Crossing octahedron sets

In this section, first of all, we recall the concept of an N-function on a nonempty set X and list some concepts related to it (for examples, the inclusion between two N-functions, the complement of an N-function, the intersection and the union of N-functions). Next, we define a crossing octahedron set and introduce some concepts related to crossing octahedron sets (for examples, the inclusion between two crossing octahedron sets, complement of a crossing octahedron set, the intersection and union of arbitrary crossing octahedron sets).

For a nonempty set X, a mapping $X \to [-1, 0]$ is called a *negative-valued function from X to* $[-1, 0]$ (briefly, *N-function on X*) (See [20, 21]). We will write N-functions on X by $A^N,\ B^N,\ C^N,\ \cdots$, etc. and denote the set of all N-functions on X as $N(X)$. In particular, the N-function $(-1)^N$ [resp. 0^N] on X defined by $(-1)^N(x) = -1$ [resp. $0^N(x) = 0$] for each $x \in X$ is called the *whole N-function* [resp. *empty N-function*] on X.

Definition 3.1. Let X be a nonempty set and let $A^N,\ B^N \in N(X)$.

(i) We say that A^N is a *subset* of B^N, denoted by $A^N \subset B^N$, if for each $x \in X$,

$$A^N(x) \geq B^N(x).$$

(ii) The *complement* of A^N, denoted by $c(A^N)$, is an N-function on X defined as:

$$c(A^N)(x) = -1 - A^N(x) \text{ for each } x \in X.$$

(iii) The *intersection* of A^N and B^N, denoted by $A^N \cap B^N$, is an N-function on X defined as: for each $x \in X$,

$$(A^N \cap B^N)(x) = A^N(x) \vee B^N(x).$$

(iv) The *union* of A^N and B^N, denoted by $A^N \cup B^N$, is an N-function on X defined as: for each $x \in X$,

$$(A^N \cup B^N)(x) = A^N(x) \wedge B^N(x).$$

From the above definition, we have the followings.

Proposition 3.2. *Let $A^N,\ B^N,\ C^N \in N(X)$. Then*

(1) (Idempotent laws)*:* $A^N \cap A^N = A^N,\ A^N \cup A^N = A^N,$

(2) (Commutative laws)*:* $A^N \cap B^N = B^N \cap A^N,\ A^N \cup B^N = B^N \cup A^N,$

(3) (Associative laws)*:* $(A^N \cap B^N) \cap C^N = A^N \cap (B^N \cap C^N),$
$(A^N \cup B^N) \cup C^N = A^N \cup (B^N \cup C^N),$

(4) (Distributive laws)*:* $A^N \cup (B^N \cap C^N) = (A^N \cup B^N) \cap (A^N \cup C^N),$
$A^N \cap (B^N \cup C^N) = (A^N \cap B^N) \cup (A^N \cap C^N),$

(5) (Absorption laws)*:* $A^N \cup (A^N \cap B^N) = A^N,\ A^N \cap (A^N \cup B^N) = A^N,$

(6) (DeMorgan's laws)*:* $c(A^N \cap B^N) = c(A^N) \cup c(B^N),$
$c(A^N \cup B^N) = c(A^N) \cap c(B^N),$

(7) $c(c(A^N)) = A^N,$

(8) $c((-1)^N) = 0^N,\ c(0^N) = (-1)^N,$

(9) $A^N \cap 0^N = 0^N,\ A^N \cup 0^N = A^N,\ A^N \cap (-1)^N = A^N,\ A^N \cup (-1)^N = (-1)^N,$

(10) *if $A^N \subset B^N$ and $B^N \subset C^N$, then $A^N \subset C^N$,*

(11) *if $A^N \subset B^N$, then $A^N \cap C^N \subset B^N \cap C^N,\ A^N \cup C^N \subset B^N \cup C^N$.*

Proof. We prove only (1) and (6), and the remainder's proofs are omitted.

(1) Let $x \in X$. Then by Definition 3.1 (iii) and (iv), we have

$$(A^N \cap A^N)(x) = A^N(x) \vee A^N(x) = A^N(x)$$

and

$$(A^N \cup A^N)(x) = A^N(x) \wedge A^N(x) = A^N(x).$$

(6) Let $x \in X$. Then by Definition 3.1 (ii), we get

$$\begin{aligned} c(A^N \cap B^N)(x) &= -1 - (A^N \cap B^N)(x) \text{ [By Definition 3.1 (ii)]} \\ &= -1 - (A^N(x) \vee B^N(x)) \text{ [By Definition 3.1 (iii)]} \\ &= -(1 - A^N(x)) \wedge (-1 - B^N(x)) \\ &= c(A^N)(x) \wedge c(B^N)(x) \text{ [By Definition 3.1 (ii)]} \\ &= (c(A^N) \cup c(B^N))(x). \text{ [By Definition 3.1 (iv)]} \end{aligned}$$

Thus $c(A^N \cap B^N) = c(A^N) \cup c(B^N)$. Similarly, we can prove that

$$c(A^N \cup B^N) = c(A^N) \cap c(B^N).$$

□

Remark 3.3. $A^N \cap c(A^N) \neq 0^N,\ A^N \cup c(A^N) \neq (-1)^N$ in general (See Example 3.4).

From Proposition 3.2, we can easily see that $(N(X), \cap, \cup, c, 0^N, (-1)^N)$ forms a Boolean algebra except the property of Remark 3.3.

Example 3.4. For any nonempty set X, consider the N-function $(-0.5)^N$ on X defined by $(-0.5)^N(x) = -0.5$ for each $x \in X$. Then we can easily check that

$$(-0.5)^N \cap c((-0.5)^N) \neq 0^N,\ (-0.5)^N \cup c((-0.5)^N) \neq (-1)^N.$$

Each member of $[I] \times (I \oplus I) \times [-1, 0]$ is called a *crossing octahedron number* and write $\widetilde{\bar{a}} = \left\langle \widetilde{a}, \bar{a}, a^N \right\rangle = \left\langle [a^-, a^+], (a^{\in}, a^{\notin}), a^N \right\rangle$.

Definition 3.5. Let X be a non-empty set. Then a mapping $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle : X \to [I] \times (I \oplus I) \times [-1, 0]$ is called a *crossing octahedron set* (briefly, COS) in X. In particular, the *crossing octahedron empty set* and the *crossing octahedron whole set* are denoted by $\breve{0}$ and $\breve{1}$ respectively and are defined as respectively: for each $x \in X$,

$$\breve{0}(x) = \langle [0, 0], (0, 1), 0 \rangle\,,\ \ \breve{1}(x) = \langle [1, 1], (1, 0), -1 \rangle\,.$$

The set of all COSs in X is denoted by $COS(X)$.

Example 3.6. (1) Let $X = \{0, a, b, c\}$ and let $\mathcal{A}$ be given by:

$$\mathcal{A}(0) = \langle [0.1, 0.8], (0.6, 0.4), -0.7 \rangle\,,\ \mathcal{A}(a) = \langle [0.2, 0.4], (0.5, 0.4), -0.5 \rangle\,,$$

$$\mathcal{A}(b) = \langle [0.1, 0.3], (0.4, 0.3), -0.3 \rangle\,,\ \mathcal{A}(c) = \langle [0.1, 0.2], (0.3, 0.2), -0.1 \rangle\,.$$

Then clearly, $\mathcal{A} \in COS(X)$.

(2) Let $\bar{A} \in IFS(X)$. Then we can easily check that

$$\left\langle [A^{\in}, 1 - A^{\notin}], \bar{A}, -1 + A^{\in} \right\rangle \in COS(X).$$

(3) Let $A = (A^P, A^N)$ be a bipolar fuzzy set in a set X. Then clearly,

$$\left\langle [A^P, A^P], (A^P, 1 - A^P), A^N \right\rangle \in COS(X),$$

$$\left\langle [A^P, A^P], (A^P, 1 - A^P), -A^P \right\rangle \in COS(X),$$

$$\left\langle [-A^N, -A^N], (A^P, 1 - A^P), A^N \right\rangle \in COS(X),$$

where $(-A^P)(x) = -A^P(x) \in [-1, 0],\ (-A^N)(x) = -A^N(x) \in I$ for each $x \in X$.

(4) Let $\widetilde{A} \in IVS(X)$. Then we can easily show that

$$\left\langle \widetilde{A}, (A^-, 1 - A^+), -1 + A^- \right\rangle \in COS(X).$$

From (2), (3) and (4) in Example 3.2, it is obvious that a COS is a generalization of an IVFS, an IFS and a bipolar fuzzy set.

Definition 3.7. Let X be a non-empty set and let $\mathcal{A},\ \mathcal{B} \in COS(X)$.

(i) We say that $\mathcal{A}$ is a *subset* of $\mathcal{B}$, denoted by $\mathcal{A} \subset \mathcal{B}$, if $\widetilde{A} \subset \widetilde{B}$, $\bar{A} \subset \bar{B}$ and $A^N \supset B^N$, i.e., for each $x \in X$,

$$A^-(x) \leq B^-(x),\ A^+(x) \leq B^+(x),\ A^{\in}(x) \leq B^{\in}(x),\ A^{\notin}(x) \geq B^{\notin}(x),\ A^N(x) \geq B^N(x).$$

(ii) We say that $\mathcal{A}$ is *equal to* $\mathcal{B}$, denoted by $\mathcal{A} = \mathcal{B}$, if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{A}$.

(iii) The *complement* of $\mathcal{A}$, denoted by $\mathcal{A}^c$, is a COS in X defined as: for each $x \in X$,

$$\mathcal{A}^c(x) = \left\langle \widetilde{A}^c(x), \bar{A}^c(x), c(A^N)(x) \right\rangle.$$

(iv) The *intersection* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cap \mathcal{B}$ is a COS in X, defined as: for each $x \in X$,

$$(\mathcal{A} \cap \mathcal{B}) = \left\langle \widetilde{A}(x) \wedge \widetilde{B}(x), \bar{A}(x) \wedge \bar{B}(x), A^N(x) \vee B^N(x) \right\rangle.$$

(v) The *union* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \cup \mathcal{B}$ is a COS in X, defined as: for each $x \in X$,

$$(\mathcal{A} \cup \mathcal{B}) = \left\langle \widetilde{A}(x) \vee \widetilde{B}(x), \bar{A}(x) \vee \bar{B}(x), A^N(x) \wedge B^N(x) \right\rangle.$$

From Definitions 3.5 and 3.7 (i), it is obvious that $\breve{0} \subset \mathcal{A} \subset \breve{1}$ for each $\mathcal{A} \in COS(X)$.

Example 3.8. (1) Let X be a non-empt set and Consider two COSs $\widetilde{A}$ and $\widetilde{B}$ given by respectively: for each $x \in X$,

$$\mathcal{A}(x) = \langle [0.1, 0.2], (0.5, 0.3), -0.4 \rangle, \ \mathcal{B}(x) = \langle [0.1, 0.8], (0.7, 0.2), -0.5 \rangle.$$

Then clearly, $\mathcal{A} \subset \mathcal{B}$, $\mathcal{A}^c = \langle [0.8, 0.9], (0.3, 0.5), -0.6 \rangle$.

(2) Let X be a non-empt set and Consider two COSs $\widetilde{A}$ and $\widetilde{B}$ given by respectively: for each $x \in X$,

$$\mathcal{A}(x) = \langle [0.1, 0.4], (0.4, 0.5), -0.3 \rangle, \ \mathcal{B}(x) = \langle [0.2, 0.3], (0.3, 0.2), -0.4 \rangle.$$

Then clearly, $\mathcal{A} \cap \mathcal{B} = \langle [0.1, 0.3], (0.3, 0.5), -0.3 \rangle$, $\mathcal{A} \cup \mathcal{B} = \langle [0.2, 0.4], (0.4, 0.2), -0.4 \rangle$.

From Definitions 3.5 and 3.7, we have the similar result to Proposition 3.2.

Proposition 3.9. *Let X be a non-empty set, let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C} \in COS(X)$. Then*

(1) (Idempotent laws) $\mathcal{A} \cap \mathcal{A} = \mathcal{A}$, $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$,
(2) (Commutative laws) $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$, $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$,
(3) (Associative laws) $\mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C}$,
$\mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C}$,
(4) (Distributive laws) $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$,
$\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$,
(5) (Absorption laws) $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{B}) = \mathcal{A}$, $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{B}) = \mathcal{A}$,
(6) (DeMorgan's laws) $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$, $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$,
(7) $(\mathcal{A}^c)^c = \mathcal{A}$,
(8) $\mathcal{A} \cap \mathcal{B} \subset \mathcal{A}$, $\mathcal{A} \cap \mathcal{B} \subset \mathcal{B}$,
(9) $\mathcal{A} \subset \mathcal{A} \cup \mathcal{B}$, $\mathcal{B} \subset \mathcal{A} \cup \mathcal{B}$,
(10) *if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B} \subset \mathcal{C}$, then $\mathcal{A} \subset \mathcal{C}$,*
(11) *if $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{A} \cap \mathcal{C} \subset \mathcal{B} \cap \mathcal{C}$, $\mathcal{A} \cup \mathcal{C} \subset \mathcal{B} \cup \mathcal{C}$,*
(12) (12_a) $\mathcal{A} \cup \breve{0} = \mathcal{A}$, $\mathcal{A} \cap \breve{0} = \breve{0}$,
(12_b) $\mathcal{A} \cup \breve{1} = \breve{1}$, $\mathcal{A} \cap \breve{1} = \mathcal{A}$,
(12_c) $\breve{1}^c = \breve{0}$, $\breve{0}^c = \breve{1}$,
(12_d) $\mathcal{A} \cup \mathcal{A}^c \neq \breve{1}$, $\mathcal{A} \cap \mathcal{A}^c \neq \breve{0}$ *in general* (See Example 3.10).

Proof. We show only (1) and the remainder's proofs are omitted.

(1) From Definition 3.7 (i) and (ii), it is sufficient to prove that $\widetilde{A} \cap \widetilde{A} = \widetilde{A}$, $\bar{A} \cap \bar{A} = \bar{A}$ and $A^N \cap A^N = A^N$. It is obvious that $A^N \cap A^N = A^N$ by Proposition 3.2 (1). Let $x \in X$. Then we have

$$(\widetilde{A} \cap \widetilde{A})(x) = [A^-(x) \wedge A^-(x), A^+(x) \wedge A^+(x)] = [A^-(x), A^+(x)] = \widetilde{A}(x),$$
$$(\bar{A} \cap \bar{A})(x) = (A^{\in}(x) \wedge A^{\in}(x), A^{\notin}(x) \vee A^{\notin}(x)) = (A^{\in}(x), A^{\notin}(x)).$$

Thus $\mathcal{A} \cap \mathcal{A} = \mathcal{A}$. Similarly, we can prove that $\mathcal{A} \cup \mathcal{A} = \mathcal{A}$. □

Example 3.10. Let X be a non-empt set and Consider the COS $\widetilde{A}$ given by: for each $x \in X$,

$$\mathcal{A}(x) = \langle [0.5, 0.5], (0.5, 0.5), -0.5 \rangle .$$

Then clearly, $\mathcal{A} \cup \mathcal{A}^c \neq \breve{1}$ and $\mathcal{A} \cap \mathcal{A}^c \neq \breve{0}$.

From Proposition 3.9 and Example 3.10, we can see that $(COS(X), \cap, \cup, ^c, \breve{0}, \breve{1})$ forms a Boolean algebra except (12_d).

Now we define the image and pre-image of a crossing octahedron set under a mapping and study some of its properties.

Definition 3.11. Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle \in COS(X)$, $\mathcal{B} = \left\langle \widetilde{B}, \bar{B}, B^N \right\rangle \in COS(Y)$.

(i) The *pre-image of* $\mathcal{B}$ *under* f, denoted by $f^{-1}(\mathcal{B}) = \left\langle f^{-1}(\widetilde{B}), f^{-1}(\bar{B}), f^{-1}(B^N) \right\rangle$, is a crossing octahedron set in X defined as follows: for each $x \in X$,

$$f^{-1}(\widetilde{B})(x) = \langle [(B^- \circ f)(x), (B^+ \circ f)(x))], ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)), (B^N \circ f)(x) \rangle .$$

(ii) The *image of* $\mathcal{A}$ *under* f, denoted by $f(\mathcal{A}) = \left\langle f(\widetilde{A}), f(\bar{A}), f(A^N) \right\rangle$, is a crossing octahedron set in Y defined as follows: for each $y \in Y$,

$$f(\widetilde{A})(y) = \begin{cases} [\bigvee_{x \in f^{-1}(y)} A^-(x), \bigvee_{x \in f^{-1}(y)} A^+(x)] & \text{if } f^{-1}(y) \neq \phi \\ [0, 0] & \text{otherwise,} \end{cases}$$

$$f(\bar{A})(y) = \begin{cases} (\bigvee_{x \in f^{-1}(y)} A^{\in}(x), \bigwedge_{x \in f^{-1}(y)} A^{\notin}(x)) & \text{if } f^{-1}(y) \neq \phi \\ (0, 1) & \text{otherwise,} \end{cases}$$

$$f(A^N)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} A^N(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.12. Let $X = \{x, y, z\}$, $Y = \{a, b, c, d\}$ and let $f : X \to Y$ be the mapping defined by: $f(x) = f(y) = a$, $f(z) = c$.

Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ be the crossing octahedron set in X defined by Table 3.1:

X	$\widetilde{A}(t)$	$\bar{A}(t)$	$A^N(t)$
x	$[0.2, 0.6]$	$(0.6, 0.3)$	-0.7
y	$[0.3, 0.5]$	$(0.5, 0.2)$	-0.6
z	$[0.4, 0.7]$	$(0.7, 0.2)$	-0.8

Table 3.1

Y	$f(\widetilde{A})(x)$	$f(\bar{A})(x)$	$f(A^N)(x)$
a	$[0.3, 0.6]$	$(0.6, 0.2)$	-0.7
b	$[0, 0]$	$(0, 1)$	0
c	$[0.4, 0.7]$	$(0.7, 0.2)$	-0.8
d	$[0, 0]$	$(0, 1)$	0

Table 3.2

Then we have easily the following Table 3.2 for $f(\mathcal{A})$:

Now let $\mathcal{B} = \left\langle \widetilde{B}, \bar{B}, B^N \right\rangle$ be the crossing octahedron set in Y defined by Table 3.3:

Y	$\widetilde{B}(x)$	$\bar{B}(x)$	B^N
a	$([0.3, 0.5]$	$(0.5, 0.4)$	-0.6
b	$([0.2, 0.6]$	$(0.7, 0.2)$	-0.8
c	$([0.4, 0.7]$	$(0.6, 0.3)$	-0.7
d	$([0.2, 0.5]$	$(0.4, 0.5)$	-0.5

Table 3.3

Then we have easily the following Table 3.4 for $f^{-1}(\mathcal{B})$:

X	$f^{-1}(\widetilde{B})(t)$	$f^{-1}(\bar{B})(t)$	$f^{-1}(B^N)(t)$
x	$[0.3, 0.5]$	$(0.5, 0.4)$	-0.6
y	$[0.3, 0.5]$	$(0.5, 0.4)$	-0.6
z	$[0.4, 0.7]$	$(0.6, 0.3)$	-0.7

Table 3.4

Proposition 3.13. *Let* $\mathcal{A},\ \mathcal{A}_1,\ \mathcal{A}_2 \in COS(X)$, $(\mathcal{A}_j)_{j\in J} \subset COS(X)$, *let* $\mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2 \in COS(Y)$, $(\mathcal{B}_j)_{j\in J} \subset COS(Y)$ *and let* $f : X \to Y$ *be a mapping. Then*
(1) *if* $\mathcal{A}_1\mathcal{A}_2$, *then* $f(\mathcal{A}_1) \subset f(\mathcal{A}_2)$,
(2) *if* $\mathcal{B}_1 \subset \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \subset f^{-1}(\mathcal{B}_2)$,
(3) $\mathcal{A} \subset f^{-1}(f(\mathcal{A}))$ *and if* f *is injective, then* $\mathcal{A} = f^{-1}(f(\mathcal{A}))$,
(4) $f(f^{-1}(\mathcal{B})) \subset \mathcal{B}$ *and if* f *is surjective,* $f(f^{-1}(\mathcal{B})) = \mathcal{B}$,
(5) $f^{-1}(\bigcup_{j\in J} \mathcal{B}_j) = \bigcup_{j\in J} f^{-1}(\mathcal{B}_j)$,
(6) $f^{-1}(\bigcap_{j\in J} \mathcal{B}_j) = \bigcap_{j\in J} f^{-1}(\mathcal{B}_j)$,
(7) $f(\bigcup_{j\in J} \mathcal{A}_j) = \bigcup_{j\in J} f(\mathcal{A}_j)$,
(8) $f(\bigcap_{j\in J} \mathcal{A}_j) \subset \bigcap_{j\in J} f(\mathcal{A}_j)$ *and if* f *is injective, then* $f(\bigcap_{j\in J} \mathcal{A}_j) = \bigcap_{j\in J} f(\mathcal{A}_j)$,
(9) *if* f *is surjective, then* $f(\mathcal{A})^c \subset f(\mathcal{A}^c)$,
(10) $f^{-1}(\mathcal{B}^c) = f^{-1}(\mathcal{B})^c$,
(11) $f^{-1}(\breve{0}) = \breve{0}$, $f^{-1}(\breve{1}) = \breve{1}$,
(12) $f(\breve{0}) = \breve{0}$ *and if* f *is surjective, then* $f(\breve{1}) = \breve{1}$.

Proof. We prove only (2) and the remainder's proofs are omitted.
(2) Suppose $\mathcal{B}_1 \subset \mathcal{B}_2$. Then by Definition 3.7 (i), we have

$$\widetilde{B_1} \subset \widetilde{B_2},\ \bar{B}_1 \subset \bar{B}_2,\ B_1^N \supset B_2^N.$$

Let $x \in X$. Then by Definitions 3.11 (i) and 3.7 (i), we get

$$f^{-1}(\widetilde{B_1})(x) = [B_1^-(f(x)), B_1^+(f)(x))] \leq [B_2^-(f(x)), B_2^+(f)(x))] = f^{-1}(\widetilde{B_2})(x),$$
$$f^{-1}(\bar{B}_1)(x) = (B_1^{\in}(f(x)), B_1^{\notin}(f)(x))) \leq [B_2^{\in}(f(x)), B_2^{\notin}(f)(x))] = f^{-1}(\bar{B}_2)(x),$$
$$f^{-1}(B_1^N)(x) = B_1^N(f(x)) \geq B_2^N(f(x)) = f^{-1}(B_2^N)(x).$$

Thus $f^{-1}(\mathcal{B}_1) \subset f^{-1}(\mathcal{B}_2)$. □

Remark 3.14. If f is not surjective, then Proposition 3.13 (9) does nor hold in general (See Example 3.15).

Example 3.15. Let X, Y be sets, $f : X \to Y$ be the mapping and $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ be the crossing octahedron set in X given in Example 3.12. Then clearly, f is not sujective. Moreover, form Tables 3.1 and 3.2, we have the following Tables:

X	$\widetilde{A}^c(t)$	$\bar{A}^c(t)$	$c(A^N)(t)$
x	$[0.4, 0.8]$	$(0.3, 0.6)$	-0.3
y	$[0.5, 0.7]$	$(0.2, 0.5)$	-0.4
z	$[0.3, 0.6]$	$(0.2, 0.7)$	-0.2

Table 3.5

Y	$f(\widetilde{A})^c(x)$	$f(\bar{A})^c(x)$	$c(f(A^N))(x)$
a	$[0.4, 0.7]$	$(0.2, 0.6)$	-0.3
b	$[1, 1]$	$(1, 0)$	-1
c	$[0.3, 0.6]$	$(0.2, 0.7)$	-0.2
d	$[1, 1]$	$(1, 0)$	-1

Table 3.6

Thus we the following Table:

Y	$f(\widetilde{A}^c)(x)$	$f(\bar{A}^c)(x)$	$f(c(A^N))(x)$
a	$[0.5, 0.8]$	$(0.3, 0.5)$	-0.4
b	$[0, 0]$	$(0, 1)$	0
c	$[0.3, 0.6]$	$(0.2, 0.7)$	-0.2
d	$[0, 0]$	$(0, 1)$	0

Table 3.6

So $f(\widetilde{A}^c)(a) \nleq f(\widetilde{A})^c(a)$, $f(\bar{A}^c)(a) \nleq f(\bar{A})^c(a)$, $f(c(A^N))(a) \nleq c(f(A^N))(a)$. Hence $f(\mathcal{A}^c) \not\subset f(\mathcal{A})^c$.

The following is an immediate result of Definition 3.11 (i).

Proposition 3.16. *If $g : Y \to Z$ is a mapping, then $(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C}))$, for each $\mathcal{C} \in COS(X)$, where $g \circ f$ is the composition of f and g.*

4. A crossing octahedron Q-(subalgebra) ideal of Q-algebras

In this section, we introduce the concepts of a crossing octahedron subalgebra [resp. BCK-ideal and Q-ideal] of a Q-algebra X and relations among them. In particular, we give a sufficient condition that the preimage under a homomorphism is a crossing octahedron Q-ideal (See Proposition 4.15).

In what follows, let X denotes a Q-algebra unless otherwise specified.

Definition 4.1. Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle \in COS(X)$. Then $\mathcal{A}$ is called a *crossing octahedron Q-subalgebra* of X, it satisfies the following conditions: for all $x,\ y,\ z \in X$,

(IVSA) $\widetilde{A}(x*z) \geq \widetilde{A}(x) \wedge \widetilde{A}(z)$, i.e.,

$$A^-(x*z) \geq A^-(x) \wedge A^-(z),\ A^+(x*z) \geq A^+(x) \wedge A^+(z),$$

(IFSA) $\bar{A}(x*z) \geq \bar{A}(x) \wedge \bar{A}(z)$, i.e.,

$$A^{\in}(x*z) \geq A^{\in}(x) \wedge A^{\in}(z),\ A^{\notin}(x*z) \leq A^{\notin}(x) \vee A^{\notin}(z),$$

(NSA) $A^N(y*z) \leq A^N(x) \vee A^N(z)$.

Definition 4.2. Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle \in COS(X)$. Then $\mathcal{A}$ is called a *crossing octahedron BCK-ideal* of X, it satisfies the following conditions: for all $x,\ y \in X$,

(IVBCKI) $\widetilde{A}(0) \geq \widetilde{A}(x)$ and $\widetilde{A}(x) \geq \widetilde{A}(x*y) \wedge \widetilde{A}(y)$, i.e.,

$$A^-(0) \geq A^-(x),\ A^+(0) \geq A^+(x)$$

and

$$A^-(x) \geq A^-(x*y) \wedge A^-(y),\ A^+(x) \geq A^+(x*y) \wedge A^+(y),$$

(IFBCKI) $\bar{A}(0) \geq \bar{A}(x)$ and $\bar{A}(x) \geq \bar{A}(x*y) \wedge \bar{A}(y)$, i.e.,

$$A^{\in}(0) \geq A^{\in}(x),\ A^{\notin}(0) \leq A^{\notin}(x),$$

and

$$A^{\in}(x) \geq A^{\in}(x*y) \wedge A^{\in}(y),\ A^{\notin}(x) \leq A^{\notin}(x*y) \vee A^{\notin}(y),$$

(NBCKI) $A^N(0) \leq A^N(x)$, $A^N(x) \leq A^N(x*y) \vee A^N(y)$.

Definition 4.3. Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle \in COS(X)$. Then $\mathcal{A}$ is called a *crossing octahedron Q-ideal* of X, it satisfies the following conditions: for all $x,\ y \in X$,

(IVQI) $\widetilde{A}(0) \geq \widetilde{A}(x)$ and $\widetilde{A}(x*z) \geq \widetilde{A}((x*y)*z) \wedge \widetilde{A}(y)$, i.e.,

$$A^-(0) \geq A^-(x),\ A^+(0) \geq A^+(x)$$

and

$$A^-(x*z) \geq A^-((x*y)*z) \wedge A^-(y),\ A^+(x*z) \geq A^+((x*y)*z) \wedge A^+(y),$$

(IFQI) $\bar{A}(0) \geq \bar{A}(x)$ and $\bar{A}(x*z) \geq \bar{A}((x*y)*z) \wedge \bar{A}(y)$, i.e.,

$$A^{\in}(0) \geq A^{\in}(x),\ A^{\notin}(0) \leq A^{\notin}(x),$$

and

$$A^{\in}(x*z) \geq A^{\in}((x*y)*z) \wedge A^{\in}(y),\ A^{\notin}(x*z) \leq A^{\notin}((x*y)*z) \vee A^{\notin}(y),$$

(NQI) $A^N(0) \leq A^N(x)$, $A^N(x*z) \leq A^N((x*y)*z) \vee A^N(y)$.

Example 4.4. Let $X = \{0, 1, 2, 3\}$ be the set with Cayley Table 4.1:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

Table 4.1

Then clearly, $(X, *, 0)$ is a Q-algebra. Consider the COS $\mathcal{A}$ given by:

$$\mathcal{A}(0) = \langle [0.3, 0.5], (0.6, 0.4), -0.5 \rangle\,,\ \mathcal{A}(1) = \langle [0.2, 0.3], (0.5, 0.3), -0.3 \rangle\,,$$
$$\mathcal{A}(2) = \langle [0.1, 0.2], (0.4, 0.2), -0.2 \rangle\,,\ \mathcal{A}(3) = \langle [0.01, 0.09], (0.3, 0.2), -0.1 \rangle\,.$$

Then we can easily check that $\mathcal{A}$ is a crossing octahedron Q-ideal of X.

Lemma 4.5. *If $\mathcal{A}$ is a crossing octahedron Q-subalgebra of X, then $\mathcal{A}(0) \geq \mathcal{A}(x)$ for each $x \in X$.*

Proof. Put $z = x$ in the Definition 4.1. Then from (SVSA) and Definition 2.3 (i),

$$A^-(0) = A^-(x * x) \geq A^-(x) \wedge A^-(x) = A^-(x),$$
$$A^+(0) = A^+(x * x) \geq A^+(x) \wedge A^+(x) = A^+(x).$$

Thus $\widetilde{A}(0) \geq \widetilde{A}(x)$. Also from (IFSA) and Definition 2.3 ((i),

$$A^{\in}(0) = A^{\in}(x * x) \geq A^{\in}(x) \wedge A^{\in}(x) = A^{\in}(x),$$
$$A^{\notin}(0) = A^{\notin}(x * x) \leq A^{\notin}(x) \vee A^{\notin}(x) = A^{\notin}(x).$$

So $\bar{A}(0) \geq \bar{A}(x)$. Finally from (NSA) and Definition 2.3 (i), we have $A^N(0) \leq A^N(x)$. Hence $\mathcal{A}(0) \geq \mathcal{A}(x)$. □

Proposition 4.6. *Every crossing octahedron BCK-ideal of a BCK-algebra X is a crossing octahedron Q-ideal of X.*

Proof. Let X be a BCK-algebra, let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ be a crossing octahedron BCK-ideal of X and let $x,\ y,\ z \in X$. Then it is obvious that

$$\widetilde{A}(0) \geq \widetilde{A}(x),\ \bar{A}(0) \geq \bar{A}(x),\ A^N(0) \leq A^N(x).$$

Moreover, we have

$$\widetilde{A}(x * z) \geq \widetilde{A}((x * z) * y) \wedge \widetilde{A}(y) \text{ [By (IVBCKI)]}$$
$$= \widetilde{A}((x * y) * z) \wedge \widetilde{A}(y) \text{ [By Definition 2.3 (iii)]},$$
$$\bar{A}(x * z) \geq \bar{A}((x * z) * y) \wedge \bar{A}(y) \text{ [By (IFBCKI)]}$$
$$= \bar{A}((x * y) * z) \wedge \bar{A}(y),$$
$$A^N(x * z) \leq A^N((x * z) * y) \wedge A^N(y) \text{ [By (NBCKI)]}$$
$$= A^N((x * y) * z) \wedge A^N(y).$$

Thus the conditions (IVQI),(IFQI) and (NQI) hold. So the result holds. □

Proposition 4.7. *Let X be a Q-algebra satisfying the conditions (BCI_1), (BCI_4) and (vi). Then every crossing octahedron BCK-ideal of X is a crossing octahedron Q-subalgebra of X.*

Proof. Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ be a crossing octahedron BCK-ideal of X and let $x,\ y \in X$. Then we have

$$\begin{aligned}\widetilde{A}((x*y)*x) &= \widetilde{A}(0) \text{ [By the condition (vi)]}\\ &\geq \widetilde{A}(x) \text{ [By (IVBCKI)]},\\ \bar{A}((x*y)*x) &= \bar{A}(0) \text{ [By the condition (vi)]}\\ &\geq \bar{A}(x) \text{ [By (IFBCKI)]},\\ A^N((x*y)*x) &= A^N(0) \text{ [By the condition (vi)]}\\ &\leq A^N(x) \text{ [By (NBCKI)]}.\end{aligned}$$

Thus we have

$$\widetilde{A}(x*y) \geq \widetilde{A}((x*y)*x) \wedge \widetilde{A}(x) = \widetilde{A}(x) \geq \widetilde{A}(x) \wedge \widetilde{A}(y),$$

$$\bar{A}(x*y) \geq \bar{A}((x*y)*x) \wedge \bar{A}(x) = \bar{A}(x) \geq \bar{A}(x) \wedge \bar{A}(y),$$

$$A^N(x*y) \leq A^N((x*y)*x) \vee A^N(x) = A^N(x) \leq A^N(x) \vee A^N(y).$$

So $\mathcal{A}$ is a crossing octahedron Q-subalgebra of X. □

Proposition 4.8. *If $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ is a crossing octahedron Q-ideal of X, then the subset $X_{\mathcal{A}(0)}$ is a Q-ideal of X, where*

$$X_{\mathcal{A}(0)} = \{x \in X : \widetilde{A}(x) = \widetilde{A}(0), \bar{A}(x) = \bar{A}(0), A^N(x) = A^N(0)\}.$$

Proof. Suppose $(x*y)*z,\ y \in X_{\mathcal{A}(0)}$ for any $x,\ y,\ z \in X$. Then clearly,

$$\widetilde{A}((x*y)*z) = \widetilde{A}(0) = \widetilde{A}(y),\ \bar{A}((x*y)*z) = \bar{A}(0) = \bar{A}(y),$$

$$A^N((x*y)*z) = A^N(0) = A^N(y).$$

Thus we get

$$\widetilde{A}(x*y) \geq \widetilde{A}((x*y)*z) \wedge \widetilde{A}(y) = \widetilde{A}(0) \wedge \widetilde{A}(0) = \widetilde{A}(0),$$

$$\bar{A}(x*y) \geq \bar{A}((x*y)*z) \wedge \bar{A}(y) = \bar{A}(0) \wedge \bar{A}(0) = \bar{A}(0),$$

$$A^N(x*y) \leq A^N((x*y)*z) \vee A^N(y) = A^N(0) \vee A^N(0) = A^N(0).$$

Since $\mathcal{A}$ is a crossing octahedron Q-ideal of X, we obtain

$$\widetilde{A}(0) \geq \widetilde{A}(x),\ \bar{A}(0) \geq \bar{A}(x),\ A^N(0) \leq A^N(x).$$

So we have

$$\widetilde{A}(x*y) = \widetilde{A}(0),\ \bar{A}(x*y) = \bar{A}(0),\ A^N(x*y) = A^N(0).$$

Hence $x*y \in X_{\mathcal{A}(0)}$. Therefore $X_{\mathcal{A}(0)}$ is a Q-ideal of X. □

Lemma 4.9. *Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ be a crossing octahedron Q-ideal of X. If $x \leq y \in X$, then $\mathcal{A}(x) \geq \mathcal{A}(y)$, i.e.,*

$$\widetilde{A}(x) \geq \widetilde{A}(y),\ \bar{A}(x) \geq \bar{A}(y),\ A^N(x) \leq A^N(y).$$

Proof. Let $x,\ y \in X$ such that $x \leq y$. Then clearly, $x * y = 0$. Thus

$$\begin{aligned}\widetilde{A}(x) &= \widetilde{A}(x*0) \text{ [By Definition 2.3 (ii)]}\\ &\geq \widetilde{A}((x*y)*0) \wedge \widetilde{A}(y) \text{ [By (IVQI)]}\\ &\geq \widetilde{A}(x*y) \wedge \widetilde{A}(y) \text{ [By Definition 2.3 (ii)]}\\ &= \widetilde{A}(0) \wedge \widetilde{A}(y) \text{ [Since } x*y=0]\\ &= \widetilde{A}(y).\end{aligned}$$

Similarly, we have $\bar{A}(x) \geq \bar{A}(y)$ and $A^N(x) \leq A^N(y)$. Thus $\mathcal{A}(x) \geq \mathcal{A}(y)$. □

Lemma 4.10. *Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle$ be a crossing octahedron BCK-ideal of X. If $x * y \leq z \in X$, then we get*

$$\widetilde{A}(x) \geq \widetilde{A}(z) \wedge \widetilde{A}(y),\ \bar{A}(x) \geq \bar{A}(z) \wedge \bar{A}(y),\ A^N(x) \leq A^N(z) \vee A^N(y).$$

Proof. Let $x,\ y,\ \in X$ such that $x * y \leq z$. Then by Lemma 4.9, $\widetilde{A}(x*y) \geq \widetilde{A}(z)$. Thus we get

$$\begin{aligned}\widetilde{A}(x) &= \widetilde{A}(x*0) \text{ [By Definition 2.3 (ii)]}\\ &\geq \widetilde{A}((x*y)*0) \wedge \widetilde{A}(y) \text{ [By (IVQI)]}\\ &= \widetilde{A}(x*y) \wedge \widetilde{A}(y) \text{ [By Definition 2.3 (ii)]}\\ &\geq \widetilde{A}(z) \wedge \widetilde{A}(y). \text{ [Since } \widetilde{A}(x*y) \geq \widetilde{A}(z)]\end{aligned}$$

So $\widetilde{A}(x) \geq \widetilde{A}(z) \wedge \widetilde{A}(y)$. Similarly, we can prove that

$$\bar{A}(x) \geq \bar{A}(z) \wedge \bar{A}(y),\ A^N(x) \leq A^N(z) \vee A^N(y).$$

Hence the result holds. □

Proposition 4.11. *Every crossing octahedron Q-ideal of a Q-algebra X is a crossing octahedron BCK-ideal of X.*

Proof. Straightforward. □

Proposition 4.12. *Let X be a Q-algebra satisfying the conditions (BCI_1), (BCI_4) and (vi). Then every crossing octahedron BCK-ideal of X is a crossing octahedron Q-subalgebra of X.*

Proof. Let $\mathcal{A}$ be a crossing octahedron BCK-ideal of a Q-algebra X and let $x,\ y \in X$. Then by (vi), $\widetilde{A}((x*y)*x) = \widetilde{A}(0) \geq \widetilde{A}(x)$. Similarly, we have

$$\bar{A}((x*y)*x) \geq \bar{A}(x),\ A^N((x*y)*x) \leq A^N(x).$$

Thus we get

$$\begin{aligned}\widetilde{A}(x*y) &\geq \widetilde{A}((x*y)*x) \wedge \widetilde{A}(x) \text{ [By (IVBCKI)]}\\ &= \widetilde{A}(0) \wedge \widetilde{A}(x) \text{ [By (vi)]}\\ &= \widetilde{A}(x).\\ &\geq \widetilde{A}(x) \wedge \widetilde{A}(y).\end{aligned}$$

Similarly, we have $\bar{A}(x*y) \geq \bar{A}(x) \wedge \bar{A}(y),\ A^N(x*y) \leq A^N(x) \vee A^N(y)$. So $\mathcal{A}$ is a crossing octahedron Q-subalgebra of X. □

Proposition 4.13. *Let $(\mathcal{A}_j)_{j\in J} = (\left\langle \widetilde{A}_j, \bar{A}_j, A^N_j \right\rangle)_{j\in J}$ be a family of crossing octahedron Q-ideals of X, where J denotes a index set. Then $\bigcap_{j\in J} \mathcal{A}_j$ is a crossing octahedron Q-ideal of X.*

Proof. Let $\mathcal{A} = \bigcap_{j\in J} \mathcal{A}_j$ and let $x,\ y,\ z \in X$. Then we get

$$\mathcal{A}(0) = \bigwedge_{j\in J} \mathcal{A}_j(0) = \left\langle \bigwedge_{j\in J} \widetilde{A}_j(0), \bigwedge_{j\in J} \bar{A}_j(0), \bigvee_{j\in J} A_j^N(0) \right\rangle.$$

Thus we have $(\bigcap_{j\in J} \widetilde{A}_j)(0) = \bigwedge_{j\in J} \widetilde{A}_j(0) \geq \bigwedge_{j\in J} \widetilde{A}_j(x) = (\bigcap_{j\in J} \widetilde{A}_j)(x)$. Similarly, we get

$$(\bigcap_{j\in J} \widetilde{A}_j)(0) \geq (\bigcap_{j\in J} \widetilde{A}_j)(x),\ (\bigcup_{j\in J} A_j^N)(0) \leq (\bigcup_{j\in J} A_j^N)(x).$$

On the other hand,

$$\begin{aligned}\widetilde{A}(x*z) &= \textstyle\bigwedge_{j\in J} \widetilde{A}_j(x*z)\\ &\geq \textstyle\bigwedge_{j\in J}(\widetilde{A}_j((x*y)*z) \wedge \widetilde{A}_j(y)) \text{ [By the hypothesis]}\\ &= (\textstyle\bigwedge_{j\in J} \widetilde{A}_j((x*y)*z)) \wedge (\textstyle\bigwedge_{j\in J} \widetilde{A}_j)(y)\\ &= (\textstyle\bigcap_{j\in J} \widetilde{A}_j)((x*y)*z) \wedge (\textstyle\bigcap_{j\in J} \widetilde{A}_j)(y).\end{aligned}$$

Similarly, we have

$$\bar{A}(x*z) \geq (\bigcap_{j\in J} \bar{A}_j)((x*y)*z) \wedge (\bigcap_{j\in J} \bar{A}_j)(y),$$

$$A^N(x*z) \leq (\bigcup_{j\in J} A_j^N)((x*y)*z) \vee (\bigcup_{j\in J} A_j^N)(y).$$

So $\bigcap_{j\in J} \mathcal{A}_j$ is a crossing octahedron Q-ideal of X. □

Definition 4.14. Let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be two Q-algebras. Then a mapping $f : X \to Y$ is called a *homomorphism*, if $f(x*y) = f(x) *^{'} f(y)$ for any $x,\ y \in X$.

It is obvious that if $f : X \to Y$ is a homomorphism of Q-algebras, then $f(0) = 0^{'}$.

Proposition 4.15. *Let $(X, *, 0),\ (Y, *^{'}, 0^{'})$ be two Q-algebras and let $f : X \to Y$ be a homomorphism. If $\mathcal{A}$ is a crossing octahedron Q-ideal of Y, then $f^{-1}(\mathcal{A})$ is a crossing octahedron Q-ideal of X.*

Proof. Suppose $\mathcal{A}$ is a crossing octahedron Q-ideal of Y and let $x \in X$. Then

$$f^{-1}(\widetilde{A})(x) = \widetilde{A}(f(x)) \leq \widetilde{A}(0^{'}) = \widetilde{A}(f(0)) = f^{-1}(\widetilde{A})(0).$$

Similarly, we have $f^{-1}(\bar{A})(x) \geq f^{-1}(\bar{A})(0),\ f^{-1}(A^N)(x) \leq f^{-1}(A^N)(0)$.

Now let $x,\ y,\ z \in X$. Then

$$\begin{aligned}f^{-1}(\widetilde{A})(x*z) &= \widetilde{A}(f(x*z)) \text{ [By Definition 3.11]}\\ &= \widetilde{A}(f(x) *^{'} f(y))\\ &\quad \text{[Since } f \text{ is a homomorphism]}\\ &\geq \widetilde{A}(f(z) *^{'} (f(y) *^{'} f(x))) \wedge \widetilde{A}(f(y)) \text{ [By the hypothesis]}\\ &= \widetilde{A}(f(z*(y*x))) \wedge \widetilde{A}(f(y))\\ &\quad \text{[Since } f \text{ is a homomorphism]}\\ &= f^{-1}(\widetilde{A})(z*(y*x)) \wedge f^{-1}(\widetilde{A})(y),\end{aligned}$$

$$\begin{aligned}f^{-1}(A^N)(z*x) &= A^N(f(z*x))\\ &= A^N(f(x) *^{'} f(y))\\ &\leq A^N(f(z) *^{'} (f(y) *^{'} f(x))) \vee A^N(f(y))\end{aligned}$$

$$= A^N(f(z*(y*x))) \vee A^N(f(y))$$
$$= f^{-1}(A^N)(z*(y*x)) \vee f^{-1}(A^N)(y).$$

Similarly, we can prove that the following inequality holds:

$$f^{-1}(\bar{A})(z*x) \geq f^{-1}(\bar{A})(z*(y*x)) \wedge f^{-1}(\bar{A})(y).$$

Thus $f^{-1}(\mathcal{A})$ is a crossing octahedron Q-ideal of X. □

The following provides a sufficient condition which the converse of Proposition 4.3 holds.

Proposition 4.16. *$f : X \to Y$ be an epimorphism of Q-algebras and let $\mathcal{A} \in COS(Y)$. If $f^{-1}(\mathcal{A})$ is a crossing octahedron Q-ideal of X, then $\mathcal{A}$ is a crossing octahedron Q-ideal of Y.*

Proof. Suppose $f^{-1}(\mathcal{A})$ is a crossing octahedron Q-ideal of X and let $a \in Y$. Since f is surjective, there is $x \in X$ such that $a = f(x)$. Then

$$\begin{aligned}\widetilde{A}(a) &= \widetilde{A}(f(x)) \\ &= f^{-1}(\widetilde{A})(x) \text{ [By Definition 3.11]} \\ &\leq f^{-1}(\widetilde{A})(0) \text{ [By the hypothesis]} \\ &= \widetilde{A}(f(0)) \\ &= \widetilde{A}(0^{'}). \text{ [Since } f \text{ is a homomorphism]}\end{aligned}$$

Similarly, we have

$$\bar{A}(a) \geq \bar{A}(0^{'}),\ A^N(a) \leq A^N(0^{'}).$$

Now let $a,\ b,\ c \in Y$. Then there are $x,\ y,\ z \in X$ such that

$$a = f(x),\ b = f(y),\ c = f(z).$$

Thus we get

$$\begin{aligned}\widetilde{A}(c*^{'}a) &= \widetilde{A}(f(z)*^{'}f(x)) \\ &= \widetilde{A}(f(z*x)) \text{ [Since } f \text{ is a homomorphism]} \\ &= f^{-1}(\widetilde{A})(z*x) \text{ [By Definition 3.11]} \\ &\geq f^{-1}(\widetilde{A})(z*(y*x)) \wedge f^{-1}(\widetilde{A})(y) \text{ [By the hypothesis]} \\ &= \widetilde{A}(f(z*(y*x))) \wedge \widetilde{A}(f(y)) \\ &= \widetilde{A}(f(z)*^{'}(f(y)*^{'}f(x))) \wedge \widetilde{A}(f(y)) \\ &= \widetilde{A}(c*^{'}(b*^{'}a)) \wedge \widetilde{A}(b),\end{aligned}$$

$$\begin{aligned}A^N(c*^{'}a) &= A^N(f(z)*^{'}f(x)) \\ &= A^N(f(z*x)) \\ &= f^{-1}(A^N)(z*x) \\ &\leq f^{-1}(A^N)(z*(y*x)) \vee f^{-1}(A^N)(y) \\ &= A^N(f(z*(y*x))) \vee A^N(f(y)) \\ &= A^N(f(z)*^{'}(f(y)*^{'}f(x))) \vee A^N(f(y)) \\ &= A^N(c*^{'}(b*^{'}a)) \vee A^N(b).\end{aligned}$$

Similarly, we can show that the following inequality holds:

$$\bar{A}(c*^{'}a) \geq \bar{A}(c*^{'}(b*^{'}a)) \wedge \bar{A}(b).$$

So $\mathcal{A}$ is a crossing octahedron Q-ideal of Y. □

Remark 4.17. If f is not surjective, then Proposition 4.16 does not hold.

Let $\mathcal{A} = \left\langle \widetilde{A}, \bar{A}, A^N \right\rangle \in COS(Y)$. In Case "$A^{\in}$", we define $X_{x'} = f^{-1}(x')$ for each $x' \in Y$. Since f is a homomorphism, we have

$$X_{x'} * X_{y'} * X_{z'} \subset X_{(x' * y') * z'}. \tag{4.1}$$

Let x', y', $z' \in Y$ and suppose $(x' * y') * z' \notin f(X)$. Then clearly, $\beta((x' * y') * z') = 0$, where for each $y \in Y$,

$$f(A^{\in})(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} A^{\in}(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

Thus $X_{(x' * y') * z'} = \varnothing$. By (4.1), $x' \notin f(X)$, $y' \notin f(X)$ or $z' \notin f(X)$. So we get

$$\beta(x' * z') \geq 0 = \beta((x' * y') * z') \wedge \beta(y').$$

Hence $\mathcal{A}$ is not a crossing octahedron Q-ideal of Y.

5. The product of crossing octahedron Q-ideals

Definition 5.1. Let $\mathcal{A}$, $\mathcal{B} \in COS(X)$. Then the *Cartesian product* of $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \times \mathcal{B} = \left\langle \widetilde{A} \times \widetilde{B}, \bar{A} \times \bar{B}, A^N \times B^N \right\rangle$, is a COS in $X \times X$ defined as: for each $(x, y) \in X \times X$,

$$(\widetilde{A} \times \widetilde{B})(x, y) = \widetilde{A}(x) \wedge \widetilde{B}(y) = [A^-(x) \wedge B^-(x), A^+(x) \wedge B^+(x)],$$
$$(\bar{A} \times \bar{B})(x, y) = \bar{A}(x) \wedge \bar{y} = (A^{\in}(x) \wedge A^{\in}(y), A^{\notin}(x) \vee A^{\notin}(y)),$$
$$A^N \times B^N(x, y) = A^N(x) \vee B^N(y).$$

Remark 5.2. Let X and Y be two Q-algebras. We define an binary operation $*$ on $X \times Y$ as follows: for any (x_1, y_1), $(x_2, y_2) \in X \times Y$,

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2).$$

Then we can easily check that $(X \times Y, *, (0, 0))$ is a Q-algebra.

Proposition 5.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be two crossing octahedron Q-ideals of a Q-algebra X. Then $\mathcal{A} \times \mathcal{B}$ is a crossing octahedron Q-ideal of $X \times X$.*

Proof. Let $(x, y) \in X \times X$. Then we get

$$(\widetilde{A} \times \widetilde{B})(0, 0) = \widetilde{A}(0) \wedge \widetilde{B}(0) \geq \widetilde{A}(x) \wedge \widetilde{B}(y) = (\widetilde{A} \times \widetilde{B})(x, y).$$

Similarly, we have the following inequalities:

$$(\bar{A} \times \bar{B})(0, 0) \geq (\bar{A} \times \bar{B})(x, y),\ (A^N \times A^N)(0, 0) \leq (A^N \times A^N)(x, y).$$

Now let (x_1, x_2), (y_1, y_2), $(z_1, z_2) \in X \times X$. Then

$$\begin{aligned} &(\widetilde{A} \times \widetilde{B})[((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)] \wedge (\widetilde{A} \times \widetilde{B})(y_1, y_2) \\ &= (\widetilde{A} \times \widetilde{B})[(x_1 * y_1, x_2 * y_2) * (z_1, z_2)] \wedge (\widetilde{A} \times \widetilde{B})(y_1, y_2) \\ &= (\widetilde{A} \times \widetilde{B})[((x_1 * y_1) * z_1), ((x_2 * y_2) * z_2)] \wedge (\widetilde{A} \times \widetilde{B})(y_1, y_2) \\ &= [\widetilde{A}((x_1 * y_1) * z_1) \wedge \widetilde{B}((x_2 * y_2) * x_2)] \wedge [\widetilde{A}(y_1) \wedge \widetilde{B}(y_2)] \\ &= [\widetilde{A}((x_1 * y_1) * z_1) \wedge \widetilde{A}(y_1)] \wedge [\widetilde{B}((x_2 * y_2) * x_2) \wedge \widetilde{B}(y_2)] \\ &\leq \widetilde{A}(x_1 * z_1) \wedge \widetilde{B}(x_2 * z_2) \end{aligned}$$

$$= (\widetilde{A} \times \widetilde{B})(x_1 * z_1, x_2 * z_2),$$

$$\begin{aligned}
&(A^N \times B^N)[((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)] \vee (A^N \times B^N)(y_1, y_2) \\
&= (A^N \times B^N)[(x_1 * y_1, x_2 * y_2) * (z_1, z_2)] \vee (A^N \times B^N)(y_1, y_2) \\
&= (A^N \times B^N)[((x_1 * y_1) * z_1, (x_2 * y_2) * z_2)] \vee (A^N \times B^N)(y_1, y_2) \\
&= [A^N((x_1 * (y_1 * z_1) \vee B^N((x_2 * y_2) * z_2)] \vee [A^N(y_1) \vee B^N(y_2)] \\
&= [A^N((x_1 * (y_1 * z_1) \vee A^N(y_1)] \vee [B^N((x_2 * y_2) * z_2) \vee B^N(y_2)] \\
&\geq A^N(x_1 * z_1) \vee B^N(x_2 * z_2) \\
&= (A^N \times B^N)(x_1 * z_1, x_2 * z_2).
\end{aligned}$$

Similarly, we can show that the following inequality holds:

$$(\bar{A} \times \bar{B})[(((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)] \wedge (\bar{A} \times \bar{B})(y_1, y_2) \leq (\bar{A} \times \bar{B})(x_1 * z_1, x_2 * z_2).$$

Thus $\mathcal{A} \times \mathcal{B}$ is a crossing octahedron Q-ideal of $X \times X$. □

6. Conclusions

We have studied crossing octahedron Q-ideals of a Q-algebra. Also, we discussed few results of the crossing octahedron Q-ideal in Q-algebras. The image and the pre-images of a crossing octahedron Q-ideal in Q-algebras under homomorphism are defined and how the image and the pre-image of a crossing octahedron Q-ideal in Q-algebras become a crossing octahedron Q-ideal are studied. Moreover, the product of two crossing octahedron Q-ideals is established. The main purpose of our future work is to investigate topological structures based on crossing octahedron sets, group structures via crossing octahedron sets cubic, crossing octahedron BCC-ideals in BCC-algebras and so on.

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