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Hybrid approach for solving fuzzy fractional linear optimization problems

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Abstract. In this paper, a new method is proposed to solve fuzzy fractional linear problems with triangular fuzzy coefficients. It consists of a combination of two resolution approaches in order to deduce a more efficient hybrid method. Indeed, this hybridization combines Veeramani-Sumathi method and Dinckelbach's theorem. To solve fuzzy fractional linear problems, our new method proceeds in four stages. First, it transforms fuzzy fractional linear program into a deterministic fractional linear multiobjective program. Then, it transforms this second form into a linear mono-objective program. Thereafter, it solves the obtained last form of the problem by using the Danztig's simplex method. Finally, it uses the arithmetic fuzzy operations to bring back the obtained solution in the fuzzy set. For our new method, a theorem have been produced to highlight the justification of the theoretical mathematical foundations and two didactic examples are been solved to prove its numerical performances.

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1. Introduction

Areas such as financial planning, production planning, business management, marketing, media selection, university planning, students admission, health care and hospital planning, and so one, are often faced with decision problems that consist in optimizing the ratios such as department/equity, profit/cost, inventory/sales, real cost/standard cost, performance/employee, student/cost, nurse/patient ratio, and so one[1, 2, 3, 4, 5, 6]. The problems listed above are effectively solved through linear fractional modelling. In practice, all coefficients are not generally accurate because of measurement or variation errors due to the market conditions. Therefore, these situations are well modelled by fuzzy linear fractional programming.

Many researchers have worked on methods for solving fuzzy fractional linear problems, but it is difficult to find a successful and universal method in order to help the decision-maker. This is why there are several proposed methods, in the literature, about the single objective fractional linear programs resolution [1, 7, 8, 9, 10, 11].

For example, Veeramani and Sumathi [12] proposed a method to solve the problems of mono-objective blurred linear fractional programming. It starts with transforming the problem into an equivalent deterministic multiobjective linear fractional programming problem and then solves each objective function separately. Thereafter, based on the solutions obtained, they define a level of imprecision and aspiration for each objective, which allows them to back into a fuzzy set to provide the solution of the initial problem. But, this method used by Veeramani and Sumathi, which consists of separately optimizing the objective functions after defuzzification, is not optimal because it does not consider that the objectives are fractional and can be conflicting.

In the literature, there are many techniques of linearization of fractional objective functions. One of them is the theorem proposed by Dinckelbach. The principle of its transformation has been the subject of several works aimed at confirming its effectiveness. In the multiobjective context, the formula consists of linearization followed by aggregation.

In the work, we have developed a new approach to solve fuzzy fractional linear optimization problems, based on the work of Veeramani and Sumathi. It is a combination of the defuzzification technique proposed by Veeramani and Sumathi and the theorem proposed Dinckelbach [13, 14, 15]. Our new method can be summarized in four essential steps: the conversion of the fuzzy fractional linear program into a deterministic multiobjective program; the transformation of the problem of the obtained deterministic multiobjective program into a mono-objective linear problem; the resolution through Danzig's simplex and the conversion of the deterministic solution to the fuzzy solution by using fuzzy arithmetic operations. We formulated and demonstrated three theorems and gave two examples to justify the performance of the method. Our numerical results are greatly improved compared to those of two other methods taken from the literature [16, 17].

For a better presentation of this work, we will present the preliminaries in Section 2. Session 3 will be dedicated to the presentation of our results, and Session 4 will be devoted to conclusion.

2. Preliminaries

This section presents some notions on triangular fuzzy numbers that we will need for the rest of the work. The main elements of this section is taken from [2, 9, 11, 18, 19, 20, 21, 22].

2.1. **Definitions.**

Definition 2.1. Let X be a set. A *fuzzy subset* $\tilde{a}$ of X is characterized by a membership function $\mu_{\tilde{a}} : X \to [0,1]$ and represented by a set of ordered pairs. That is defined as follows:

$$\tilde{a} = \{(x, \mu_{\tilde{a}}(x))/x \in X\}, \tag{2.1}$$

where $\mu_{\tilde{a}}(x) \in [0,1]$ represents extent to which x belongs to $\tilde{a}$.

Remark 2.2. $\tilde{a}$ is the graph of the membership function $\mu_{\tilde{a}}$.

Definition 2.3. A fuzzy set $\tilde{a}$ of X is *convex*, if

$$\mu_{\tilde{a}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\left(\mu_{\tilde{a}}(x_1), \mu_{\tilde{a}}(x_2)\right)$$

for all $x_1,\ x_2 \in X$ and for all $\lambda \in [0,1]$.

Definition 2.4. A *triangular fuzzy number* $\tilde{a}$ is denoted by $(a^{(1)}, a^{(2)}, a^{(3)})$ with $a^{(1)} < a^{(2)} < a^{(3)}$ is a fuzzy set where the membership function $\mu_{\tilde{a}}(x)$ can be defined as:

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a^{(1)}}{a^{(2)} - a^{(1)}} & \text{if } \ a^{(1)} \leq x < a^{(2)} \\ \dfrac{a^{(3)} - x}{a^{(3)} - a^{(2)}} & \text{if } \ a^{(2)} \leq x < a^{(3)} \\ 0 & \text{else.} \end{cases} \tag{2.2}$$

Definition 2.5. Let $N(\mathbb{R})$ be the set of all triangular fuzzy numbers with values in $\mathbb{R}$. A ranking is a function F: $N(\mathbb{R}) \to \mathbb{R}$, which maps each fuzzy number into the real line, where a natural order exists. Let $\tilde{a} = (a^{(1)}, a^{(2)}, a^{(3)})$ is a triangular fuzzy number. Then $F(\tilde{a}) = \dfrac{a^{(1)} + 2a^{(2)} + a^{(3)}}{4}$.

The ranking function allows to compare two triangular fuzzy numbers. Let us consider two triangular fuzzy numbers $\tilde{a}$ and $\tilde{b}$ with $\tilde{a} = (a^{(1)}, a^{(2)}, a^{(3)})$ and $\tilde{b} = (b^{(1)}, b^{(2)}, b^{(3)})$. Then, we are define relations as follow:

- ⋄ $\tilde{a} \approx \tilde{b}$ if and only if $F(\tilde{a}) = F(\tilde{b})$,
- ⋄ $\tilde{a} \succeq \tilde{b}$ if and only if $F(\tilde{a}) \geq F(\tilde{b})$,
- ⋄ $\tilde{a} \preceq \tilde{b}$ if and only if $F(\tilde{a}) \leq F(\tilde{b})$,
- ⋄ $\tilde{a} \succ \tilde{b}$ if and only if $F(\tilde{a}) > F(\tilde{b})$.

Remark 2.6. Let $\tilde{a} = (a^{(1)}, a^{(2)}, a^{(3)})$ and $\tilde{b} = (b^{(1)}, b^{(2)}, b^{(3)})$ be any two positive triangular fuzzy numbers. $\tilde{a} \preceq \tilde{b}$ if and only if $a^{(1)} \leq b^{(1)}$, $a^{(2)} \leq b^{(2)}$ and $a^{(3)} \leq b^{(3)}$.

2.2. **Operations.** Let $\tilde{a} = (a^{(1)}, a^{(2)}, a^{(3)})$ and $\tilde{b} = (b^{(1)}, b^{(2)}, b^{(3)})$ be two triangular fuzzy numbers, where $a^{(1)},\ a^{(2)},\ a^{(3)},\ b^{(1)},\ b^{(2)},\ b^{(3)} \in \mathbb{R}_{+}^{*}$. Then the arithmetic operations are defined by:

- ⋄ $\tilde{a} \oplus \tilde{b} = (a^{(1)} + b^{(1)}, a^{(2)} + b^{(2)}, a^{(3)} + b^{(3)})$,
- ⋄ $\tilde{a} \ominus \tilde{b} = (a^{(1)} - b^{(3)}, a^{(2)} - b^{(2)}, a^{(3)} - b^{(1)})$,
- ⋄ $k\tilde{a} = \begin{cases} (ka^{(1)}, ka^{(2)}, ka^{(3)}) \ if \ k \geq 0, \\ \\ (ka^{(3)}, ka^{(2)}, ka^{(1)}) \ if \ k < 0, \end{cases}$

◇ $\frac{\tilde{a}}{\tilde{b}} = \left(\frac{a^{(1)}}{b^{(3)}}, \frac{a^{(2)}}{b^{(2)}}, \frac{a^{(3)}}{b^{(1)}}\right)$.

2.3. **Fuzzy fractional linear optimization program.** The fractional linear program with triangular fuzzy coefficients can be written as follows:

$$(2.3)\quad \begin{cases} \max\ \tilde{Z} = \dfrac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{p}}{\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{q}}, \\ \text{subject to} \\ \sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq \tilde{b}_i, i = \overline{1,m}, \\ x_j \geq 0, j = \overline{1,n}, \end{cases}$$

where $\tilde{c}_j = (c_j^{(1)}, c_j^{(2)}, c_j^{(3)})$, $\tilde{d}_j = (d_j^{(1)}, d_j^{(2)}, d_j^{(3)})$, $\tilde{a}_i = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)})$, $\tilde{b}_i = (b_i^{(1)}, b_i^{(2)}, b_i^{(3)})$, $\tilde{p} = (p^{(1)}, p^{(2)}, p^{(3)})$, $\tilde{q} = (q^{(1)}, q^{(2)}, q^{(3)})$ for $i = \overline{1,m}$ and $j = \overline{1,n}$. Let

$$S = \{x \in \mathbb{R}_+^n / \sum \tilde{a}_{ij} x_j \preceq \tilde{b}_i, i = \overline{1,m}\ j = \overline{1,n}\}.$$

We assume that the feasible region S is nonempty and bounded and the denominator

$$\sum_{j=1}^{n} \tilde{d}_j x_j + \tilde{q} \succ \tilde{0}.$$

Definition 2.7. Any vector $x \in \mathbb{R}_+^n$ that satisfies the constraints, i.e., $x \in S$ is a feasible solution of the fuzzy fractional linear optimization problem.

Remark 2.8. Let $x \in S$. Then $\tilde{Z}(x)$ is called a *fuzzy objective value* of the fuzzy fractional linear optimization problem.

Definition 2.9. Let F be a ranking function defined on $N(\mathbb{R})$. A vector $\overline{x} \in S$ is said to be the *optimal solution* of the fuzzy fractional linear optimization problem,if there does not exist an $x \in$ S such as $F(\tilde{Z}(\overline{x})) \leq F(\tilde{Z}(x))$.

2.4. **Deterministic fractional multi-objective optimization program.** A fractional linear multiobjective optimization program is presented as follows:

$$(2.4)\quad \begin{cases} \max \left(Z_k(x) = \dfrac{\sum_{j=1}^{n} c_j^k x_j + p^k}{\sum_{j=1}^{n} d_j^k x_j + q^k} = \dfrac{P_k(x)}{Q_k(x)}, \quad k = \overline{1,K} \right), \\ \text{subject to} \\ Ax \leq b, \\ x \geq 0, \end{cases}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, c_j^k , d_j^k, $x \in \mathbb{R}^n$, p^k and $q^k \in \mathbb{R}$, $\forall k = 1, \cdots, K$, $\forall j = 1, \cdots, n$.

Definition 2.10 ([23])**.** The problem (2.4) is said to be: (i) *concave-convex*, if the functions $P_k, k=\overline{1,3}$ are concave and the functions $Q_k, k=\overline{1,3}$ and constraints are convex,

(ii) *convex-concave*, if the functions $P_k, k=\overline{1,3}$ and the constrained functions are convex, and the functions $Q_k, k=\overline{1,3}$ are concave.

In the sequel, let $\Omega=\{x\in\mathbb{R}^n_+/Ax\leq b\}$ be the set of feasible solutions. It is compact, convex and non-empty.

Definition 2.11 ([24])**.** A point $\overline{x}\in\Omega$ is said to be *strongly efficient* (S-efficient), if there is no $x\in\Omega$ such as $Z_k(x)\geq Z_k(\overline{x})$, $\forall k=\overline{1,K}$ and $Z_j(x)>Z_j(\overline{x})$ for at least one $j\neq k$. This solution also said to be *Pareto optimal*.

Definition 2.12 ([24])**.** A point $\overline{x}\in\Omega$ is said to be *weakly efficient* (W-efficient), if there is no $x\in\Omega$ such that $Z_i(x)>Z_i(\overline{x})$ $\forall i=\overline{1,K}$. This solution also called weakly Pareto optimal.

Remark 2.13. Let P be the set of Pareto optimal solutions, and $\overline{P}$ the set of weakly Pareto optimal solutions. We have $P\subseteq\overline{P}$.

Definition 2.14. The *ideal point* $y^*=(y_1^*,y_2^*,...,y_p^*)$ is the vector whose each coordinates y_k^* correspond to the optimum solution of the objective function Z_k under the constraints of initial problem. That is $y_k^*=\max\limits_{x\in\Omega}(Z_k(x)),\ k=\overline{1,p}$.

Theorem 2.15 ([13, 14, 15])**.** *(Dinkelbach's theorem)*
$Z_k^*=\dfrac{P_k(x^*)}{Q_k(x^*)}=\max\left\{\dfrac{P_k(x)}{Q_k(x)}/x\in\Omega\right\}$ *if and only if* $f(Z_k^*)=f(Z_k^*,x^*)=\max\{P_k(x)-Z_k^*Q_k(x)/x\in\Omega\}=0$.

This theorem proposed by Dinkelbach [14] converts fractional linear multiobjective optimization problem in nonfractional linear multiobjective problem.

3. Main results

This section is devoted to the presentation of our new method, the didactic examples dealt with and the result analysis.

3.1. **New method.** Our method can be summarized in four main step as follows :

Step 1:	`Defuzzification`. Allows to transform the fuzzy fractional linear problem into deterministic multiobjective optimization problem,
Step 2:	`Agregation`. Consists in transforming the multiobjective optimization problem into single objective optimization problem,
Step 3:	`Resolution`. Consists in finding the solution of final single-objective optimization problem,
Step 4:	`Solution initialization`. Corresponds to the transformation of the obtained solution from deterministic form to fuzzy form.

Defuzzification.: Let us consider the problem (2.3).

Using the addition on fuzzy numbers, we can rewrite the problem (2.3) as follows:

$$(3.1)\qquad \begin{cases} \max\ \tilde{Z} = \dfrac{(\sum c_j^{(1)}x_j + p^{(1)}, \sum c_j^{(2)}x_j + p^{(2)}, \sum c_j^{(3)}x_j + p^{(3)})}{(\sum d_j^{(1)}x_j + q^{(1)}, \sum d_j^{(2)}x_j + q^{(2)}, \sum d_j^{(3)}x_j + q^{(3)})} \\ \qquad\ = \dfrac{\big(P_1(x), P_2(x), P_3(x)\big)}{\big(Q_1(x), Q_2(x), Q_3(x)\big)}; \\ \textit{subject to} \\ \sum a_{ij}^{(1)}x_j \leq b_i^{(1)}, \quad i = \overline{1,m}; \\ \sum a_{ij}^{(2)}x_j \leq b_i^{(2)}, \quad i = \overline{1,m}; \\ \sum a_{ij}^{(3)}x_j \leq b_i^{(3)}, \quad i = \overline{1,m}; \\ x_j \geq 0, j = \overline{1,n}. \end{cases}$$

Using the division on fuzzy numbers, we can also rewrite the problem (3.1):

$$(3.2)\qquad \begin{cases} \max\ Z_1 = \dfrac{\sum c_j^{(1)}x_j + p^{(1)}}{\sum d_j^{(3)}x_j + q^{(3)}} = \dfrac{P_1(x)}{Q_3(x)}; \\ \max\ Z_2 = \dfrac{\sum c_j^{(2)}x_j + p^{(2)}}{\sum d_j^{(2)}x_j + q^{(2)}} = \dfrac{P_2(x)}{Q_2(x)}; \\ \max\ Z_3 = \dfrac{\sum c_j^{(3)}x_j + p^{(3)}}{\sum d_j^{(1)}x_j + q^{(1)}} = \dfrac{P_3(x)}{Q_1(x)}; \\ \textit{subject to} \\ \sum a_{ij}^{(1)}x_j \leq b_i^{(1)}, \quad i = \overline{1,m}; \\ \sum a_{ij}^{(2)}x_j \leq b_i^{(2)}, \quad i = \overline{1,m}; \\ \sum a_{ij}^{(3)}x_j \leq b_i^{(3)}, \quad i = \overline{1,m}; \\ x_j \geq 0,\ j = \overline{1,n}. \end{cases}$$

Problem (3.2) is a deterministic fractional linear multi-objective optimization program.

Let us denote by Ω the deteministic set associated to the feasible set S.

Remark 3.1. We have $\Omega \subset S$.

Proof. Let $x \in \Omega$, i.e., $\sum a_{ij}^{(1)}x_j \leq b_i^{(1)}$, $\sum a_{ij}^{(2)}x_j \leq b_i^{(2)}$, $\sum a_{ij}^{(3)}x_j \leq b_i^{(3)}$ $i = \overline{1,m}$, $j = \overline{1,n}$. We have $\big(\sum a_{ij}^{(1)}x_j \leq b_i^{(1)}, \sum a_{ij}^{(2)}x_j \leq b_i^{(2)}, \sum a_{ij}^{(3)}x_j \leq b_i^{(3)}\big)$. This means that $\big(\sum a_{ij}^{(1)}x_j, \sum a_{ij}^{(2)}x_j, \sum a_{ij}^{(3)}x_j\big) \leq \big(b_i^{(1)}, b_i^{(2)}, b_i^{(3)}\big)$ $i = \overline{1,m}$, $j = \overline{1,n}$, which is also equal to $\sum \big(a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}\big)x_j \leq \big(b_i^{(1)}, b_i^{(2)}, b_i^{(3)}\big)$, $i = \overline{1,m}$, $j = \overline{1,n}$. This implies that $\sum \tilde{a}_{ij}x_j \preceq \tilde{b}_i$, $i = \overline{1,m}$, $j = \overline{1,n}$. Then $x \in S$. Thus $\Omega \subset S$. □

We formulate the following theorem.

Theorem 3.2. *Let $\overline{x} \in \Omega$. If $\overline{x}$ is a Pareto optimal solution of (3.2), then $\overline{x}$ is an optimal solution of the problem (3.1).*

Proof. Let $\overline{x}$ be a Pareto optimal solution of problem (3.2). Then $\forall x \in \Omega$, we have

$$\big(Z_1(x), Z_2(x), Z_3(x)\big) \leq \big(Z_1(\overline{x}), Z_2(\overline{x}), Z_3(\overline{x})\big).$$

By replacing each on by its expression, we obtain

$$\left(\frac{P_1(x)}{Q_3(x)}, \frac{P_2(x)}{Q_2(x)}, \frac{P_3(x)}{Q_1(x)}\right) \leq \left(\frac{P_1(\overline{x})}{Q_3(\overline{x})}, \frac{P_2(\overline{x})}{Q_2(\overline{x})}, \frac{P_3(\overline{x})}{Q_1(\overline{x})}\right).$$

Using the operations rules defined above, We get

$$\frac{\big(P_1(x), P_2(x), P_3(x)\big)}{\big(Q_1(x), Q_2(x), Q_3(x)\big)} \leq \frac{\big(P_1(\overline{x}), P_2(\overline{x}), P_3(\overline{x})\big)}{\big(Q_1(\overline{x}), Q_2(\overline{x}), Q_3(\overline{x})\big)}.$$

That is equivalent to :

$$\tilde{Z}(x) \leq \tilde{Z}(\overline{x}).$$

Thus $\overline{x}$ is an optimal solution of (3.1). □

Aggregation.: Here, we apply the theorem of Dinckelbach. That allows us to linearise each objective function in order to transform them in globlal objective. Thus, the fractional linear multiobjective problem (3.2) is transformed into the following linear problem:

$$(3.3)\qquad \begin{cases} \max \left(Z^{\omega} = \displaystyle\sum_{i=1}^{3} \omega_i (P_i(x) - Z_i^* Q_i(x))\right) \\ \textit{subject to} \\ \sum a_{ij}^{(1)} x_j \leq b_i^{(1)}, \quad i = \overline{1, m}; \\ \sum a_{ij}^{(2)} x_j \leq b_i^{(2)}, \quad i = \overline{1, m}; \\ \sum a_{ij}^{(3)} x_j \leq b_i^{(3)}, \quad i = \overline{1, m}; \\ x_j \geq 0, \quad j = \overline{1, n}; \end{cases}$$

where $\omega = (\omega_1, \omega_2, \omega_3) \in]0,1[^3$ are the weights of the objective functions such as $\omega_1 + \omega_2 + \omega_3 = 1$. The linear optimization problem is solved by all methods of solving linear problems.

Theorem 3.3. *Let $\overline{x}_\omega \in \Omega$ and ω_1, ω_2, $\omega_3 \in]0;1[$ such as $\omega_1 + \omega_2 + \omega_3 = 1$. If $\overline{x}_\omega$ is a Pareto optimal solution of problem (3.3), then $\overline{x}_\omega$ is also a Pareto optimal solution of problem (3.2).*

Proof. Assume that $\overline{x}_\omega$ is a Pareto optimal solution of problem (3.3). Then $\forall k \in \{1, 2, 3\}$, $Z_k^*(\overline{x}_\omega) = \dfrac{P_k(\overline{x}_\omega)}{Q_k(\overline{x}_\omega)}$ and we have $P_k(\overline{x}_\omega) - Z_k^* Q_k(\overline{x}_\omega) = 0$. As $w_i \geq 0$, $\forall i = \overline{1, 3}$, later on, we obtain $\displaystyle\sum_{k=1}^{3} \omega_k \big(P_k(\overline{x}_\omega) - Z_k^* Q_k(\overline{x}_\omega)\big) = 0$. Suppose that $\overline{x}_\omega$ is not a Pareto optimal solution of (3.2). Then there is a $\hat{x} \in \Omega$ such that $Z_i(\hat{x}) \geq Z_i(\overline{x}_\omega)$ for all i and $Z_j(\hat{x}) > Z_j(\overline{x}_\omega)$ for at least one $j \in \{1, 2, 3\}$. Thus we have : for all i, $\Big(\dfrac{P_i(\hat{x})}{Q_i(\hat{x})} \geq \dfrac{P_i(\overline{x}_\omega)}{Q_i(\overline{x}_\omega)} \Longrightarrow$ $P_i(\hat{x}) \geq Q_i(\hat{x}) \dfrac{P_i(\overline{x_\omega})}{Q_i(\overline{x}_\omega)}$ because $Q_i(\hat{x}) > 0$ hence $P_i(\hat{x}) - Z_i^* Q_i(\hat{x}) \geq 0\Big)$ and at least one $j \in \{1,2,3\}$, $\Big(\dfrac{P_j(\hat{x})}{Q_j(\hat{x})} > \dfrac{P_j(\overline{x}_\omega)}{Q_j(\overline{x}_\omega)} \Longrightarrow P_j(\hat{x}) > Q_j(\hat{x})\dfrac{P_j(\overline{x})}{Q_j(\overline{x})}$ because $Q_j(\hat{x}) > 0$ hence $P_j(\hat{x}) - Z_j^* Q_j(\hat{x}) > 0\Big)$

As $\omega_i > 0$, We still have $\omega_i(P_i(\hat{x}) - Z_i^* Q_i(\hat{x})) \geq 0$ for all i and $\omega_j(P_j(\hat{x}) - Z_j^* Q_j(\hat{x})) > 0$ for at least one $j = \{1,2,3\}$. By summing these two relations we obtain $\sum_{i=1}^{3} \omega_i\big(P_i(\hat{x}) - Z_i^* Q_i(\hat{x})\big) > 0$. That allows us to write the following inequality: $\sum_{i=1}^{3} \omega_i\big(P_i(\hat{x}) - Z_i^* Q_i(\hat{x})\big) > 0 = \sum_{i=1}^{3} \omega_i\big(P_i(\overline{x}_\omega) - Z_j^* Q_i(\overline{x}_\omega)\big)$. Hence $\sum_{i=1}^{3} \omega_j\big(P_j(\hat{x}) - Z_j^* Q_j(\hat{x})\big) > \sum_{i=1}^{3} \omega_j\big(P_j(\overline{x}_\omega) - Z_j^* Q_j(\overline{x}_\omega)\big)$. Which is a contradiction with our assumption. Therefore $\overline{x}_\omega$ is also a Pareto optimal of problem (3.2). □

Remark 3.4. The parameter ω is used as the weight values for the objective functions. Then for two different values of ω we get two different resulting problems where their resolution gives two different optimal solutions. So, by varying ω we obtain the whole Pareto optimal set of the initial problem.

Resolution.: This step consists in the resolution of the problem (3.3). This problem is linear single-objective optimization. Therefore, we use the Danzig's simplex method to deduce the optimal solution of the problem (3.3).

Solution initialisation: In this last step, we use the arithmetic operations to transform the obtained deterministic solution to the solution of initial problem which is necessarily fuzzy.

The Algorithm of our method can be write as the following.

Let $\hat{P}$ and $\hat{Q}$ be some fuzzy triangular linear functions and $\frac{\hat{P}(x)}{\hat{Q}(x)}$ the fuzzy fractional linear function. Let P_k and Q_k, $k = \overline{1,3}$ be the deterministic functions from $\frac{\hat{P}(x)}{\hat{Q}(x)}$ by using classical operations on triangular fuzzy numbers. Then

1. *consider the problem* $\max_{x\in S}\Big\{\dfrac{\hat{P}(x)}{\hat{Q}(x)}\Big\}$,
2. *defuzzify with classical operators on triangular numbers. Hence the form* $\max_{x\in\Omega}\Big\{\dfrac{P_k(x)}{Q_k(x)}, k = \overline{1,3}\Big\}$,
3. *convert to linear form* $\max_{x\in\Omega}\big\{w_k\big(P_k(x) - Z_k^* Q_k(x)\big), k = \overline{1,3}\big\}$ *by using Dinkelbach's theorem,*
4. *put on single-objective form* $\max_{x\in\Omega}\Big\{\sum_{k=1}^{3} w_k\big(P_k(x) - Z_k^* Q_k(x)\big)\Big\}$ *by using weighting sum approach,*
4. *for fixed value of the* $\omega \in]0,1[^3$, *solve* $\max_{x\in\Omega}\Big\{\sum_{k=1}^{3} w_k\big(P_k(x) - Z_k^* Q_k(x)\big)\Big\}$ *by Dantzig's Simplex method.*

3.2. **Didactic examples.** These examples are taken from [7, 16].

Example 1: Let us consider the following fuzzy fractional linear problem:

$$
(3.4)\quad \begin{cases} \max\left(\tilde{Z} = \dfrac{(3,5,7)x_1 + (2,3,4)x_2}{(4,5,6)x_1 + (1,2,3)x_2 + (0,1,2)}\right); \\ \text{subject to} \\ (2,3,4)x_1 + (3,5,7)x_2 \leq (11,15,19); \\ (4,5,6)x_1 + (1,2,3)x_2 \leq (8,10,12); \\ x_1, x_2 \geq 0. \end{cases}
$$

By applying the new method, the fuzzy fractional problem is first transformed into the following deterministic fractional problem:

$$
(3.5)\quad \begin{cases} \max \tilde{Z} = \dfrac{(3x_1 + 2x_2, 5x_1 + 3x_2, 7x_1 + 4x_2)}{(4x_1 + x_2, 5x_1 + 2x_2 + 1, 6x_1 + 3x_2 + 1)}; \\ \text{subject to} \\ 2x_1 + 3x_2 \leq 11; \\ 3x_1 + 5x_2 \leq 15; \\ 4x_1 + 7x_2 \leq 19; \\ 4x_1 + x_2 \leq 8; \\ 5x_1 + 2x_2 \leq 10; \\ 6x_1 + 3x_2 \leq 12; \\ x_1, x_2 \geq 0. \end{cases}
$$

Note that the admissible domain is a convex polytope. The existence of solution is thus guaranteed by the properties of convexity.
Problem (3.5) is equivalent to the following deterministic fractional linear tri-objective problem:

$$
(3.6)\quad \begin{cases} \max Z_1 = \dfrac{3x_1 + 2x_2}{6x_1 + 3x_2 + 1}; \\ \max Z_2 = \dfrac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1}; \\ \max Z_3 = \dfrac{7x_1 + 4x_2}{4x_1 + x_2}; \\ \text{subject to} \\ 2x_1 + 3x_2 \leq 11; \\ 3x_1 + 5x_2 \leq 15; \\ 4x_1 + 7x_2 \leq 19; \\ 4x_1 + x_2 \leq 8; \\ 5x_1 + 2x_2 \leq 10; \\ 6x_1 + 3x_2 \leq 12; \\ x_1, x_2 \geq 0. \end{cases}
$$

We observe that $Z_1, Z_2, Z_3 \geq 0$, for x belonging to the feasible domain. We look for the ideal values: $0.50 \leq Z_1 \leq 0.53$; $1.26 \leq Z_2 \leq 1.27$; $Z_3 = 4$. Let us choose $\omega = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the fractional linear tri-objective optimization problem is equivalent to the following linear optimization problem:

$$(3.7)\quad \begin{cases} \max Z^{\omega} = [\frac{1}{3}((3x_1+2x_2)-0.53(6x_1+3x_2+2))+\frac{1}{3}((5x_1+3x_2) \\ -1.27(5x_1+2x_2+1))+\frac{1}{3}((7x_1+4x_2)-4(4x_1+x_2))]; \\ \text{subject to} \\ 2x_1+3x_2 \leq 11; \\ 3x_1+5x_2 \leq 15; \\ 4x_1+7x_2 \leq 19; \\ 4x_1+x_2 \leq 8; \\ 5x_1+2x_2 \leq 10; \\ 6x_1+3x_2 \leq 12; \\ x_1, x_2 \geq 0. \end{cases}$$

By solving the problem we find $x_1 = 0$, $x_2 = 2.71$ and the fuzzy optimal solution $\tilde{Z}^{\omega}_{\max} = (0.53, 1.27, 4.00)$.

The comparison with other solutions in the literature is given in the table below.

TABLE 1. Results obtained by the proposed method and other methods

Methods	Fuzzy optimal solution $\tilde{Z}^{\omega}_{\max}$	Ranking value
Our method	**(0.53, 1.27, 4.00)**	**1.76**
Sapan-Edalatpanah-Mandal method	(0.50, 1.00, 2.40)	1.72
Taylor series method	(0.40, 0.75, 2.19)	1.39

We find that our method gives the best solution through the ranking function compared to what is proposed by Sapan[16] and methods using Taylor series [17, 25].

Example 2: Let us consider the following fuzzy fractional linear problem:

$$(3.8)\quad \begin{cases} \max \left(\tilde{Z} = \dfrac{(4,6,8)x_1+(1,2,3)x_2}{(0,1,2)x_1+(0,1,2)x_2+(1,2,3)}\right) \\ \text{subject to} \\ (0,1,2)x_1+(0,1,2)x_2 \leq (3,7,11) \\ (1,2,3)x_1+(2,3,4)x_2 \leq (7,17,27) \\ x_1, x_2 \geq 0 \end{cases}$$

Taking $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$ and after solving we get $x_1 = 5.5$, $x_2 = 0$ and $\tilde{Z}^{\omega}_{\max} = (1.57, 4.40, 44.00)$.
The comparison of this solution and other methods is given in the table below.

TABLE 2. Results obtained by the proposed method and other methods

Methods	Fuzzy optimal solution $\tilde{Z}^{\omega}_{\max}$	Ranking value
Our method	**(1.57, 4.40, 44.00)**	**13.59**
Sapan-Edalatpanah-Mandal method	(1.45, 4.00, 32.00)	10.36
Taylor series method	(1.45, 4.00, 02.32)	03.09

We also find that our method provides the best solution compared to what is proposed by Sapan [16] and methods using Taylor series [17, 25].

3.3. **Result analysis.** Regarding to our obtained solutions and ranking function values on theses two didactic examples we can conclude that our method is better than the two others. Indeed, Sapan, Edalatpanah and Mandal [3, 16, 18] have been evaluated with various existing methods such: Veeramani and Sumathi method [7], Pop et al. [26] method, Stanojevic-Stancu Minasian method [27] and Safaei's method [28] on the same problems. It appears that the Sapan, Edalatpanah and Mandal [16, 18] method provides good solutions.

4. Conclusion

In this paper, we have successfully proposed a hybrid approach to solving fuzzy fractional linear problems. It consisted in combining the algorithm of Veeramani and Sumathi method with Dinkelbach's theorem. We have explained the different stages of this method. Indeed, this method transforms the fuzzy linear fractional optimization problem into a deterministic linear fractional multiobjective optimization problem. Then, this last form is convert to a linear single objective optimization problem. We finally achieve the solution by using Dantzig's simplex method. We have formulated two theorems to prove that the obtained solution by our method is the best of the feasible set. Moreover, two didactic examples have been efficiently solved, whose ranking function value has been calculated. For the two didactic problems, our obtained solutions have been better than for the Sapan-Edaltpanah-Mandal method and the Taylor series method. That allows us to deduce that our new method is the best choice for solving the Fuzzy linear fractional optimization problems.

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