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Abstract. The purpose of this paper is to introduce the notion of left and right translational invariant fuzzy subset of a semigroup and the notion of a unit with respect to fuzzy subset of a semigroup and study their properties. We prove that if μ is a translational invariant fuzzy subset of a commutative semigroup with unity then principal ideal generated by an element and fuzzy subset is a prime ideal of a semigroup.

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1. Introduction

Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory and the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal.

Many real-world problems are complicated due to various uncertainties. In addressing them, classical methods may not be the best option. To mention few, artificial intelligence plays an important role in dealing with uncertain information by simulating the peoples needs comprising of uncertain data. Several theories like probability, randomness were introduced. In addressing uncertainty one of the appropriate theory is fuzzy set theory. The fuzzy set theory was developed by Zadeh in 1965 [1]. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory multi agent systems, machine learning, information processing. Rosenfeld introduced the fuzzification of algebraic structure, and he introduced the notion of fuzzy subgroups in 1971 [2]. Fuzzy algebraic structures play a vital role in mathematics with wide applications in many branches such as theoretical physics, computer sciences, control engineering, coding theory etc.

Ray [3] introduced the notion of translational invariant fuzzy subset, and Ray and Ali [4] generalized the results of ring theory by using the notion of translational invariant fuzzy subsets. In this paper, we introduce the notion of units, associates, prime elements with respect to a fuzzy subset, an ideal of a semigroup generated by translational invariant fuzzy subset and an element. We study the properties of image and pre-image of translational invariant fuzzy subset under the semigroup homomorphism.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1 ([5])**.** A *semigroup* is an algebraic system $(M, \cdot)$ consisting of a non-empty set M together with an associative binary operation "$\cdot$". An element $1 \in M$ is said to be *unity*, if $x1 = 1x = x$ for each $x \in M$.

Definition 2.2 ([6])**.** A *subsemigroup* T of a semigroup M is a non-empty subset T of M such that $TT \subseteq T$.

Definition 2.3 ([7])**.** A non-empty subset T of a semigroup M is called a *left* (*right*) *ideal* of M, if $MT \subseteq T(TM \subseteq T)$.

Definition 2.4 ([7])**.** A non-empty subset T of a semigroup M is called an *ideal* of M ,if it is both a left ideal and a right ideal of$M$.

Definition 2.5 ([8])**.** A semigroup M is called a *regular semigroup*, if every element of M is a regular element.

Definition 2.6 ([8])**.** Let M be a semigroup. An ideal P of M is called a *prime ideal* of M, if $ab \in P$ implies $a \in P$ or $b \in P$ for any $a,\ b \in M$.

Definition 2.7 ([8])**.** Let M be a nonempty set. A mapping $f : M \to [0, 1]$ is called a *fuzzy subset* of M.

Definition 2.8 ([3])**.** Let S and T be two sets and $\phi : S \to T$ be any function. A fuzzy subset f of S is said to be *ϕ-invariant*, if $\phi(x) = \phi(y)$ implies $f(x) = f(y)$ for any $x,\ y \in S$.

Definition 2.9 ([3])**.** Let μ be a fuzzy subset of M. μ is said to be *left translational invariant with respect to the binary operation* $*$ on M, if for any $x,\ y \in M$,

$$\mu(x) = \mu(y) \text{ implies } \mu(a * x) = \mu(a * y) \text{ for each } a \in M.$$

Definition 2.10 ([3]). Let μ be a fuzzy subset of M. μ is said to be *right translational invariant with respect to the binary operation* $*$ on M, if for any $x,\ y \in M$,

$$\mu(x) = \mu(y) \text{ implies } \mu(x * a) = \mu(y * a) \text{ for each } a \in M.$$

Definition 2.11 ([3]). A fuzzy subset μ of M is said to be *translational invariant with respect to the binary operation* $*$ on M, if μ is both a left, and a right translational invariant.

Definition 2.12 ([4]). Let M be a ring and μ be a fuzzy subset of M. Then μ is said to be *left* (*right*) *translational invariant*, if it satisfies the following conditions: for any $a,\ x,\ y \in M$,

$$\mu(x) = \mu(y) \text{ implies } \mu(x + a) = \mu(y + a) \text{ and } \mu(xa) = \mu(ya)(\mu(ax) = \mu(ay)).$$

Definition 2.13 ([4]). Let M be a ring, $a, b \in M$ and $\mu(a) \neq \mu(0)$. Then a is said to be μ-*divisor* of b, if there exists $c \in M$ such that $\mu(b) = \mu(ac)$. It is denoted by $(a/_b)_\mu$.

Definition 2.14 ([4]). Let M be a ring, $a, b \in M$ and $\mu(a) \neq \mu(0)$. Then a and b are said to be μ- *associates*, if $(a/_b)_\mu$ and $(b/_a)_\mu$.

Definition 2.15 ([4]). Let $f : M \to S$ be a homomorphism of rings and μ be a fuzzy subset of M. We define a fuzzy subset $f(\mu)$ of S by

$$f(\mu)(x) = \begin{cases} \sup\limits_{y \in f^{-1}(x)} \mu(y) & \text{if } f^{-1}(x) \neq \phi \\ 0 & \text{otherwise .} \end{cases}$$

Definition 2.16. Let $\phi : M \to M'$ be a homomorphism of semigroups and μ be a fuzzy subset of M'. We define a fuzzy subset $\phi^{-1}(\mu)$ of M by $\phi^{-1}(\mu)(x) = \mu(\phi(x))$ for all $x \in M$. We call $\phi^{-1}(\mu)$ is the *preimage* of μ.

3. Translational invariant fuzzy subset of M

In this section, we introduce the notion of left and right translational invariant fuzzy subset of a semigroup M, the notion of a unit with respect to a fuzzy subset and study their properties. Throughout this paper, M is a semigroup with the unity 1.

Definition 3.1. Let M be a semigroup, $a \in M$ and μ be a fuzzy subset of M. The set $\{x \in M : \mu(x) = \mu(ra),\ r \in M\}$, is denoted by $L(a, \mu)$ and the set $\{x \in M : \mu(x) = \mu(ar),\ r \in M\}$, is denoted by $R(a, \mu)$.

Example 3.2. Let $M = \{0, a, b, c\}$. Define the binary operation $'\cdot'$ on M by the following table:

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	a	b	c

Then $(M, \cdot)$ is a semigroup with unity and $B = \{0, b\}$ is a right ideal.

We define $\mu : M \to [0,1]$ as follows:

$$\mu(0) = 1,\ \mu(a) = 0.5,\ \mu(b) = 0.3,\ \mu(c) = 0$$

$$\mu(b0) = \mu(0) = 1,\ \mu(ba) = \mu(b) = 0.3,\ \mu(bb) = \mu(b) = 0.3,\ \mu(bc) = \mu(b) = 0.3.$$

Then $R(b,\mu) = \{0, b\}$.

Theorem 3.3. *Let μ be a left translational invariant fuzzy subset of a semigroup M. Then for any $a \in M$, $L(a,\mu)$ is a left ideal of M for each $a \in M$.*

Proof. Let μ be a left translational invariant fuzzy subset of M. Since $a = 1a$, $\mu(a) = \mu(1a)$. Since μ is left translational invariant, $\mu(a) = \mu(1a)$. Since $1 \in M$, $a \in L(a,\mu)$. Then $L(a,\mu) \neq \varnothing$. Suppose $s \in L(a,\mu)$ and $r \in M$. Then $\mu(s) = \mu(ya)$ for some $y \in M$. Thus $\mu(rs) = \mu\big(r(ya)\big)$. So $\mu(rs) = \mu\big((ry)a\big)$. Hence $rs \in L(a,\mu)$. Therefore $L(a,\mu)$ is a left ideal of the semigroup M. □

The proof of the following theorem is similar as that of Theorem 3.4

Theorem 3.4. *Let M be a semigroup and μ be a right translational invariant fuzzy subset of M and $a \in M$. Then the set $R(a,\mu)$ is a right ideal of a semigroup M.*

Corollary 3.5. *Let M be a commutative semigroup and μ be a translational invariant fuzzy subset of M. Then for any $a \in M$, $L(a,\mu)$ is an ideal of a semigroup M.*

Theorem 3.6. *Let μ be a translational invariant fuzzy subset of a semigroup.*
(1) *If $a \in L(b,\mu)$, then $L(a,\mu) \subseteq L(b,\mu)$.*
(2) *If $a \in R(b,\mu)$, then $R(a,\mu) \subseteq R(b,\mu)$.*

Proof. (1) Suppose $a \in L(b,\mu)$ and let $t \in L(a,\mu)$. Then there exist $x,\ y \in M$ such that

$$\mu(a) = \mu(xb) \text{ and } \mu(t) = \mu(ya).$$

Thus $\mu(t) = \mu(ya) = \mu(yxb)$. So $t \in L(b,\mu)$. Hence $L(a,\mu) \subseteq L(b,\mu)$.
Similarly, we can prove (2). □

Theorem 3.7. *Let μ be a translational invariant fuzzy subset of a semigroup. and $a,\ b \in M$. If $\mu(a) = \mu(b)$ then $L(a,\mu = L(b,\mu)$ and $R(a,\mu) = R(b,\mu)$.*

Proof. Suppose $\mu(a) = \mu(b)$ and let $x \in L(a,\mu)$. Then $\mu(x) = \mu(ra)$ for some $r \in M$. Since $\mu(a) = \mu(b)$, $\mu(ra) = \mu(rb)$. Thus $\mu(x) = \mu(rb)$. So $x \in L(b,\mu)$. Hence $L(a,\mu) \subseteq L(b,\mu)$.

Now let $y \in L(b,\mu)$. Then $\mu(y) = \mu(sb)$ for some $s \in M$. Thus $\mu(y) = \mu(sb) = \mu(sa)$, i.e., $y \in L(a,\mu)$. So $L(b,\mu) \subseteq L(a,\mu)$. Hence $L(a,\mu) = L(b,\mu)$. Similarly, we can prove $R(a,\mu) = R(b,\mu)$. □

Theorem 3.8. *Let M be a semigroup and μ be a translational invariant fuzzy subset of M. For any $a \in M$, the left ideal $Ma = \{ra : r \in M\}$ of M is contained in left ideal $L(a,\mu)$ and the right ideal $aM = \{ar : r \in M\}$ is contained in right ideal $R(a,\mu)$.*

Proof. Suppose $x \in Ma$. Then $x = ra$ for some $r \in M$. Thus $\mu(x) = \mu(ra)$. So $x \in L(a,\mu)$. Hence $Ma \subseteq L(a,\mu)$. Similarly, we can prove $aM \subseteq R(a,\mu)$. □

Definition 3.9. Let M be a commutative semigroup and $a \in M$. If $L(a,\mu) = R(a,\mu)$, then the ideal $L(a,\mu)$ is denoted by $I(a,\mu)$ and $I(a,\mu)$ is called the μ-*principal ideal* of M generated by a and μ.

Theorem 3.10. *If M is a commutative semigroup, μ is a translational invariant fuzzy subset of M and $a \in M$, then the principal ideal $\langle a \rangle$ is a subset of $I(a,\mu)$.*

Proof. From the hypothesis, it is clear that $L(a,\mu) = R(a,\mu) = I(a,\mu)$. Then $\langle a \rangle \subseteq I(a,\mu)$. □

Definition 3.11. Let M be a semigroup, μ be a translational invariant fuzzy subset of a semigroup M and $\mu(0) \neq \mu(1)$. An element $a \in M$ with $\mu(a) \neq \mu(0)$ is called a μ-*unit* of M, if there exists an element $u \in M$ such that $\mu(u) \neq \mu(0)$ and $\mu(au) = \mu(ua) = \mu(1)$.

Example 3.12. Let M be the semigroup of non-negative integers with respect to usual multiplication of integers as binary operation. Then we have

$$< 6 >= \{0, 6, 12 \cdots\} = \text{ all } \text{ even } integers$$

and

$$I(6,\mu) = \{\mu(x) = \mu(r6) = \mu(6r)\} \text{ for } \text{ some } r \in M.$$

Thus $< 6 > \subseteq I(6,\mu) \subseteq M$.

Theorem 3.13. *Let M be a semigroup. If a be μ-unit of M, then $L(a,\mu) = R(a,\mu) = M$.*

Proof. Suppose a is a μ-unit of M. Then there exists $u \in M$ such that $\mu(u) \neq \mu(0)$, $\mu(au) = \mu(ua) = \mu(1)$. Let $x \in M$. Then $\mu(1) = \mu(au)$. Thus $\mu(1x) = \mu(aux)$. Since $1x = x$, $\mu(x) = \mu(aux)$. So $x \in R(a,\mu)$. Hence $M \subseteq R(a,\mu)$. Similarly, we can prove $M \subseteq L(a,\mu)$. Therefore $L(a,\mu) = R(a,\mu) = M$. □

Theorem 3.14. *Let M be a semigroup, $a \in M$ and μ be a right translational invariant fuzzy subset of M. If $1 \in R(a,\mu)$, then $R(a,\mu) = M$.*

Proof. Suppose $1 \in R(a,\mu)$. Let $x \in M$. Then $\mu(1) = \mu(ay)$ for some $y \in M$. Thus $\mu(x) = \mu(1x) = \mu(ayx)$. So $x \in R(a,\mu)$, i.e., $M \subseteq R(a,\mu)$. Hence $R(a,\mu) = M$. □

Corollary 3.15. *Let M be a commutative semigroup, $a \in M$ and μ be a translational invariant fuzzy subset of M. If $1 \in I(a,\mu)$, then $I(a,\mu) = M$.*

Theorem 3.16. *Let M be a semigroup, $a \in M$ and μ be a right translational invariant fuzzy subset of M. If $x \in R(a,\mu)$ is invertible, then $R(a,\mu) = M$.*

Proof. Suppose $x \in R(a,\mu)$ is invertible. Then there exist $y \in M$ such that $xy = 1$. Thus by Theorem 3.5, $R(a,\mu)$ is a right ideal of M. So $1 = xy \in R(a,\mu)$. Hence by Theorem 3.14, $R(a,\mu) = M$. □

Corollary 3.17. *Let M be a commutative semigroup, $a \in M$ and μ be a translational invariant fuzzy subset of M. If $x \in I(a,\mu)$ is invertible then $I(a,\mu) = M$.*

4. Prime ideal $I(a, \mu)$

In this section, we introduce the notion of associates, prime elements with respect to a fuzzy subset, an ideal of a semigroup generated by translational fuzzy subset and an element. We study the properties of image and pre-image of a translational invariant fuzzy subset under the semigroup homomorphism. Throughout in this section M is a commutative semigroup.

Theorem 4.1. *Let M be a semigroup, $a, b \in M$ and $\mu(a) \neq \mu(0)$. If $(a/_b)_\mu$, then $I(b, \mu) \subseteq I(a, \mu)$.*

Proof. Suppose $(a/_b)_\mu$. Then $\mu(b) = \mu(ac)$ for some $c \in M$. Thus $b \in I(a, \mu)$. So by Definition 3.9, $I(b, \mu) \subseteq I(a, \mu)$. □

Theorem 4.2. *Let f be a homomorphism from semigroup M onto semigroup S, μ be a f-invariant fuzzy subset of M and $x \in M$. If $x = f(a)$, then $f(\mu)(x) = \mu(a)$.*

Proof. Suppose $x = f(a)$. Then $f^{-1}(x) = a$. Let $t \in f^{-1}(x)$. Then $x = f(t)$. Thus $f(a) = x = f(t)$. Since μ is a f-invariant fuzzy subset of M, $\mu(a) = \mu(t)$. So $f(\mu)(x) = \sup\limits_{t \in f^{-1}(x)} \{\mu(t)\} = \mu(a)$. □

Theorem 4.3. *Let f be a homomorphism from semigroup M onto semigroup S. If μ is a translational invariant and $f-$invariant fuzzy subset of M, then $f(\mu)$ is a translational invariant fuzzy subset of S.*

Proof. Suppose μ is a translational invariant and f-invariant fuzzy subset of M, and let $x,\ y \in S$. Then $f(\mu)(x) = f(\mu)(y)$. Since f is onto, there exist $a,\ b \in M$ such that $f(a) = x$ and $f(b) = y$. By Theorem 4.2, $f(\mu)(x) = \mu(a)$ and $f(\mu)(y) = \mu(b)$. Thus $\mu(a) = \mu(b)$. Now let $z \in S$. Then there exists $c \in M$ such that $f(c) = z$. Thus we have

$$xz = f(a)f(c) = f(ac),\ yz = f(b)f(c) = f(bc).$$

So we get

$$f(\mu)(xz) = \mu(ac),\ f(\mu)(yz) = mu(bc). \tag{4.1}$$

Since μ is translational invariant, $\mu(a) = \mu(b)$. Hence $\mu(ac) = \mu(bc)$. By 4.1), $f(\mu)(xz) = f(\mu)(yz)$. Therefore $f(\mu)$ is a translational invariant fuzzy subset of S. □

Theorem 4.4. *Let f be a homomorphism from semigroup M onto semigroup S. If μ is an f-invariant, translational invariant fuzzy subset of M, then*

$$f(I(a, \mu)) = I(f(a), f(\mu)) \text{ for all } a \in M.$$

Proof. Suppose μ is an $f-$invariant, translational invariant fuzzy subset of M, and let $y \in I(f(a), f(\mu))$. Then we have

$$\begin{aligned}
& f(\mu)(y) = f(\mu)(sf(a)) \text{ for some } s \in S \\
\Leftrightarrow\ & f(\mu)f(x) = f(\mu)\big(f(r)f(a)\big) \\
& \quad \text{[Since f is onto, there exist } x,\ r \in M \text{ such that } f(x) = y,\ f(r) = s.] \\
\Leftrightarrow\ & f(\mu)f(x) = f(\mu)\big(f(ra)\big) \\
\Leftrightarrow\ & \mu(x) = \mu(ra) \\
\Leftrightarrow\ & x \in I(a, \mu)
\end{aligned}$$

$\Leftrightarrow y = f(x) \in f\big(I(a,\mu)\big)$.
Thus $I\big(f(a), f(\mu)\big) = f\big(I(a,\mu)\big)$. □

Theorem 4.5. *Let f be a homomorphism from semigroup M into semigroup S. If μ is a translational invariant fuzzy subset of S, then $f^{-1}(\mu)$ is a translational invariant fuzzy subset of a semigroup M.*

Proof. Suppose μ is a translational invariant fuzzy subset of S and let $a,\ b \in M$ such that $f^{-1}(\mu)(a) = f^{-1}(\mu)(b)$. Then $\mu(f(a)) = \mu(f(b))$. Let $x \in M$. Then $f(x) = y \in S$. Since μ is translational invariant fuzzy subset of S, we have

$$\begin{aligned} &\mu(f(a)y) = \mu(f(b)y) \\ \Rightarrow &\mu(f(a)f(x)) = \mu(f(b)f(x)) \\ \Rightarrow &\mu(f(ax)) = \mu(f(bx)) \\ \Rightarrow &f^{-1}(\mu)(ax) = f^{-1}(\mu)(bx). \end{aligned}$$

Thus $f^{-1}(\mu)$ is a translational invariant fuzzy subset of M. □

Theorem 4.6. *Let M be a semigroup and $a,\ b \in M,\ \mu(a),\ \mu(b) \neq \mu(0)$. If a and b are μ-assoicates, then $I(a,\mu) = I(b,\mu)$.*

Proof. Suppose a and b are μ-assoicates. Then by Definition 2.14, $(a/b)_\mu$ and $(b/a)_\mu$. Thus by Theorem 4.1, $I(b,\mu) \subseteq I(a,\mu),\ I(a,\mu) \subseteq I(b,\mu)$. So $I(b,\mu) = I(a,\mu)$. □

Theorem 4.7. *Let M be a semigroup, $a,\ b \in M$ and $\mu(a),\ \mu(b) \neq \mu(0)$. If $\mu(a) = \mu(bu)$ for some μ-unit $u \in M$, then a and b are μ-associates.*

Proof. Suppose $\mu(a) = \mu(bu)$ for some μ-unit $u \in M$. Then $(b/_a)_\mu$. Since u is a μ -unit, there exists $v \in M$ and $\mu(v) \neq \mu(0)$ such that $\mu(uv) = \mu(e)$. Since $\mu(a) = \mu(bu)$, we get

$$\mu(av) = \mu(buv) = \mu(be) = \mu(b).$$

Thus $(a/_b)_\mu$. So a and b are μ-associates. □

Definition 4.8. Let M be a semigroup and μ be a translational invariant fuzzy subset of M. Suppose element a is not a unit and $\mu(a) \neq \mu(0)$. Then a is said to be *μ-prime element*, if $\left(a/_{bc}\right)_\mu$ implies $\left(a/_b\right)_\mu$ or $\left(a/_c\right)_\mu$ for all $b,\ c \in M$.

Theorem 4.9. *Let M be a semigroup, μ be a translational invariant fuzzy subset of M, $a \in M, \mu(a) \neq \mu(0)$ and $I(a,\mu) \neq M$. Then a is μ-prime element if and only if the ideal $I(a,\mu)$ is a prime ideal of M.*

Proof. Suppose a is a μ-prime element. Then by Corollary 3.6, $I(a,\mu)$ is an ideal of M. Let $x,\ y \in M$ and $xy \in I(a,\mu)$. Then $\mu(xy) = \mu(ar)$ for some $r \in M$. Thus $\left(a/_{xy}\right)_\mu$. Since a is a μ-prime, $\left(a/_x\right)_\mu$ or $\left(a/_y\right)_\mu$. If $\left(a/_x\right)_\mu$, then $\mu(x) = \mu(ac)$ for some $c \in M$. Thus $x \in I(a,\mu)$. If $\left(a/_y\right)_\mu$, then $y \in I(a,\mu)$. So $I(a,\mu)$ is a prime ideal of M.

Conversely, suppose that $I(a,\mu)$ is a prime ideal of M and let $x,\ y \in M$, and $\left(a/_{(xy)}\right)_\mu$. Then $\mu(xy) = \mu(ac)$ for some $c \in M$. Thus $xy \in I(a,\mu)$. Since $I(a,\mu)$ is a prime ideal, $x \in I(a,\mu)$ or $y \in I(a,\mu)$. If $x \in I(a,\mu)$, then $\mu(x) = \mu(ya)$ for some $y \in M$. Thus $\left(a/_x\right)_\mu$. Similarly, we can show that if $y \in I(a,\mu)$, then $(a/_y)_\mu$. So a is a μ-prime element of M. □

Theorem 4.10. *Let f be a homomorphism from semigroup M onto semigroup S and μ be a translational invariant and f-invariant fuzzy subset of M. If a is a μ-prime element of M, then $f(a)$ is a $f(\mu)$-prime element of S.*

Proof. Let μ be a translational invariant and f-invariant fuzzy subset of M and f be a homomorphism from semigroup M onto semigroup S. By Theorem 4.3, $f(\mu)$ is a translational invariant fuzzy subset of S. Suppose a is a μ-prime element of M and $\left(f(a)/_{yz}\right)_{f(\mu)}$ for $y,\ z \in S$. Since f is an onto homomorphism, there exist $b,\ c \in M$ such that $f(b) = y,\ f(c) = z$. Then $f(bc) = f(b)f(c) = yz$. Since $\left(f(a)/_{yz}\right)_{f(\mu)}$, there exists $d \in M$ such that $f(\mu)\big(f(a)f(d)\big) = f(\mu)f(bc)$. Thus $\mu(ad) = \mu(bc)$. So $\left(a/_{bc}\right)_\mu$. Since a is μ -prime element, $\left(a/_b\right)_\mu$ or $\left(a/_c\right)_\mu$. Hence we get

$$\text{there exists } d \in M \text{ such that } f(\mu)\big(f(a)f(d)\big) = f(\mu)f(bc)$$
$$\Rightarrow \left(a/_{bc}\right)_\mu.$$

Since a is a μ-prime element, we have

$$\left(a/_b\right)_\mu \text{ or } \left(a/_c\right)_\mu$$
$$\Rightarrow \mu(as) = \mu(b) \text{ or } \mu(ar) = \mu(c) \text{ for some } s,\ r \in M$$
$$\Rightarrow f(\mu)\Big(f(as)\Big) = f(\mu)\Big(f(b)\Big) \text{ or } f(\mu)\Big(f(ar)\Big) = f(\mu)\Big(f(c)\Big)$$
$$\Rightarrow f(\mu)\Big(f(a)f(s)\Big) = f(\mu)\Big(f(b)\Big) \text{ or } f(\mu)\Big(f(a)f(r)\Big) = f(\mu)\Big(f(c)\Big)$$
$$\Rightarrow \left(f(a)/_{f(b)}\right)_{f(\mu)} \text{ or } \left(f(a)/_{f(c)}\right)_{f(\mu)}$$
$$\Rightarrow \left(f(a)/_y\right)_{f(\mu)} \text{ or } \left(f(a)/_z\right)_{f(\mu)}.$$

Therefore $f(a)$ is a $f(\mu)$-prime element of S. □

Theorem 4.11. *Let f be a homomorphism from semigroup M onto semigroup S and μ be a translational invariant and f-invarianat fuzzy subset of M. If a is a μ-prime element of M, then the homomorphic image of $I(a,\mu)$ is a prime ideal of S.*

Proof. Let f be a homomorphism from semigroup M onto semigroup S and μ be a translational invariant and f-invarianat fuzzy subset of M. Then by Theorem 4.3, $f(\mu)$ is a fuzzy translational invariant fuzzy subset of S. Thus by Theorem 4.9, the μ-principal ideal $I(a,\mu)$ is a prime ideal of M, where a is a μ-prime element and μ is a f-invariant, translational invariant fuzzy subset of M. So by Theorem 4.10, $f(I(a,\mu)) = I(f(a), f(\mu))$. Hence $f(a)$ is $f(\mu)$-prime element of S. Therefore by Theorem 4.4, $I(f(a), f(\mu))$ is a prime ideal of S. □

5. Conclusion

In this paper, we introduced the notion of left and right translational invariant fuzzy subset of a semigroup the notion of a unit with respect to fuzzy subset and studied their properties, the notion of associates, prime elements with respect to a fuzzy subset, an ideal of a semigroup generated by translational fuzzy subset and an element. The properties of image and pre-image of translational invariant fuzzy subset under the semigroup homomorphism, and every homomorphic image of an ideal of a semigroup generated by μ-prime element and fuzzy translational invariant fuzzy subset μ is a prime ideal of semigroup were studied. Our future work on this topic, we will extend these results to other algebraic structures and ordered semigroups.

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