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Abstract. In this paper, we have introduced and studied some new notions of R_0 separation axiom in fuzzy soft topological spaces by using quasi-coincident relation for fuzzy soft points. We have observed that all these notions satisfy good extension property. We have shown that these notions are preserved under the one-one, onto and FSP continuous mapping. Finally, we have studied some basic properties of this new concept.

2020 AMS Classification: 03E72, 08A72

Keywords: Fuzzy set, Soft set, Fuzzy soft set, Fuzzy soft topological spaces, Fuzzy soft open set, Quasi-coincidence, Fuzzy soft R_0 separations.

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1. Introduction

In 1999, the Russian researcher Molodtsov [1] introduced the concept of a soft set and pointed out several directions, e.g., game theory, Riemann integration, theory of measurement, smoothness of functions and so on. Maji et al. [2] presented some new definitions on soft sets and discussed in detail the application of soft set theory in decision making problems. Chen et al. [3] studied the parametrization reduction of soft sets. In recent years, after Molodtsov, research on the soft set theory explored soft set theory and obtained interesting results (See [4, 5, 6, 7, 8]). In 2010, Nazmul and Samanta [9] defined soft topological groups, normal soft topological groups and homomorphisms. The concepts of soft topological subspaces such as soft open and soft closed by characterization in soft topological spaces introduced by Şenel and Çağman [10] in 2015. Ahmad and Kharal [11] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. Aktaş and Çağman [12] defined the notion of soft groups and derived some properties. By using the t-norm, the concept of fuzzy soft group was introduced by Aygünoğlu and Aygün [13]. Furthermore, Shabir and Naz [14] introduced the concept of soft topological space and studied neighborhoods and separation axioms. Also, in 2017, Sarma and Tripathy [15] and Debnath and tripathy [16] defined and studied some properties of separation axioms in soft bitopological spaces. Lee et al. [17] introduced the notion of cubic quotient mappings and provided the sufficient conditions for the projection mappings to be cubic open.

In this paper, we introduce some new concepts of fuzzy soft R_0 topological spaces. We discuss some properties of this notions and present their good extension, hereditary. Finally, we show that productive, projective and order preserving properties hold on our concepts fuzzy soft R_0 topological spaces in quasi-coincident sense.

2. Preliminaries

Now we recall some definitions and concepts needed in the next sections.

Definition 2.1 ([1])**.** A pair (F, E) is said to be a *soft set* over an initial universe X, if F is a mapping from E to $P(X)$, where $P(X)$ is the collection of subsets of X.

Definition 2.2 ([18])**.** Let X be an initial universe set and let E be a set of parameters. Let $I^X(I = [0,1])$ denotes the set of all fuzzy sets of X. Let $A \subseteq E$. A pair (F, A) is called a *fuzzy soft set* over X, if $F : A \longrightarrow I^X$ is a mapping such that $F(e) = O_X$ if $e \notin A$ and $O_X \neq F(e) \in I^X$ if $e \in A$, where $O_X = 0$ for all $x \in X$. In this case, F is called an *approximate function* of the fuzzy soft set (F, A) and the value $F(e)$ is a fuzzy set called an *e-element* of the fuzzy soft set (F, A). Thus a fuzzy soft set (F, A) over X can be represented by the set of ordered pairs $(F, A) = \{(e, F(e)) : e \in A, F(e) \in I^X\}$. In other words, the fuzzy soft set is a parameterized family of fuzzy subsets of the set X.

Definition 2.3 ([19])**.** A *fuzzy soft point* x^e_α over X is a fuzzy soft set over X defined as follows: for each $e^{'} \in E$,

$$x^e_\alpha(e^{'}) = \begin{cases} x_\alpha & \text{if } e^{'} = e \\ 0 & \text{if } e^{'} \in E - \{e\}, \end{cases}$$

where x_α is the fuzzy point in X with support x and value α, $\alpha \in (0, 1]$. The set of all fuzzy soft points in $X$is denoted by $FSP(X, E)$. The fuzzy soft point x^e_α is said to *belong to* a fuzzy soft set f_E, denoted by $x^e_\alpha \in f_E$, if $\alpha \leq f(e)(x)$. Every non-null fuzzy soft set f_E can be expressed as the union of all the fuzzy soft points belonging to f_E. The *complemen*t of a fuzzy soft point x^e_α is a fuzzy soft set over X.

Example 2.4. Let $X = \{x, y\}$ and $E = \{e_1, e_2\}$ be a universe set and a parameters set for the universe X respectively. Then the fuzzy soft point $x^{e_1}_{0.5}$ and $y^{e_2}_{0.7}$ is a fuzzy soft set over X given by: for each $e \in E$,

$$x^{e_1}_{0.3}(e) = \begin{cases} x_{0.3} & \text{if } e = e_1 \\ 0 & \text{if } e = e_2, \end{cases}$$

$$y^{e_2}_{0.5}(e) = \begin{cases} 0 & \text{if } e = e_1 \\ y_{0.5} & \text{if } e = e_2. \end{cases}$$

Definition 2.5 ([2])**.** Let (F, A) and (G, B) be two soft sets over a common universe X. The *union* of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cup B$, is defined by: for each $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } \ e \in A - B \\ G(e) & \text{if } \ e \in B - A \\ F(e) \cup G(e) & \text{if } \ e \in B \cap A. \end{cases}$$

It is denoted by $(H, C) = (F, A) \cup (G, B)$.

Definition 2.6 ([2])**.** Let (F, A) and (G, B) be two soft sets over a common universe X. The *intersection* of two soft sets (F, A) and (G, B) over X is the soft set (H, C), where $C = A \cap B$, is defined by $H(e) = F(e) \cap G(e)$ for each $e \in C$. It is denoted by $(H, C) = (F, A) \cap (G, B)$.

Definition 2.7 ([20])**.** The *complement* of a fuzzy soft set (F, A), denoted by $(F, A)^c$, defined as $(F, A)^c = (F^c, A)$, where $F^c(e) = 1 - F(e)$ for every $e \in A$.

It is obvious that $((F, A)^c)^c = (F, A)$, $(1_E)^c = 0_E$ and $0^c_E = 1_E$.

Definition 2.8 ([21])**.** The fuzzy soft sets (F, E) and (G, E) in (X, E) are said to be *fuzzy soft quasi-coincident*, denoted by $(F, E)q(G, E)$, if there exist $e \in E,\ x \in X$ such that $F(e)(x) + G(e)(x) > 1$. If (F, E) is not fuzzy soft quasi-coincident with (G, E), then we write $(F, E)\bar{q}(G, E)$, i.e., $(F, E)\bar{q}(G, E)$ if and only if $F(e)(x) + G(e)(x) \leq 1$, i.e., $F(e)(x) \leq G^c(e)(x)$ for all $x \in X$ and $e \in E$.

A fuzzy soft point x^e_α is said to be *soft quasi-coincident with fuzzy soft set* (F, E), denoted by $x^e_\alpha q(F, E)$, if there exist $e \in E,\ x \in X$ such that $\alpha + F(e)(x) > 1$ and if $x^e_\alpha \bar{q}(F, E)$, then $\alpha + F(e)(x) \leq 1$.

Definition 2.9 ([18])**.** A *fuzzy soft topology* τ on (X, E) is a family of fuzzy soft sets over (X, E) satisfying the following properties:

(i) $0_E,\ 1_E \in \tau$,

(ii) if $(F, A), (G, B) \in \tau$, then $(F, A) \cap (G, B) \in \tau$,

(iii) if $(F, A)_\alpha \in \tau\ \forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} (F, A)_\alpha \in \tau$.

The triple (X, τ, E) is called a *fuzzy soft topological space*. Each member of τ is called a *fuzzy soft open set* in (X, τ, E). A fuzzy soft set (F, E) over X is called a *fuzzy soft closed set* in X, if $(F, E)^c \in \tau$.

Definition 2.10 ([22])**.** The *Cartesian product* of two fuzzy soft sets (F, A) and (G, B), denoted by $(H, C) = (F, A) \times (G, B)$, is a fuzzy soft set over X defined as follows: for each $(e, e^{'}) \in C$,

$$H(e, e^{'}) = F(e) \times G(e^{'}),$$

where $C = A \times B,\ H : C \to I^X$ and $F(e) \times G(e^{'}) = \{min\{F(e)(x), G(e^{'})(x)\} : x \in X\}$.

The family of all fuzzy soft sets over (X, E) is denoted by $FSS(X, E)$.

Definition 2.11 ([23])**.** Let $F_A \in FSS(X, E)$ and $G_B \in FSS(Y, K)$. Then the *fuzzy soft product* of F_A and G_B, denoted by $F_A \times G_B$, is a fuzzy soft set over $X \times Y$ defined by: for each $(e, k) \in E \times K$ and each $(x, y) \in X \times Y$,

$$(F_A \times G_B)(e, k)(x, y) = (F_A(e) \times G_B(k))(x, y) = min\{F_A(e)(x), G_B(k)(y)\}.$$

Definition 2.12. Let $\{(X_i, E_i), i \in \Lambda\}$ be any family of soft sets and let X and E denote the Cartesian product of these soft sets, i.e., $X = \Pi_{i\in\Lambda} X_i$ and $E = \Pi_{i\in\Lambda} E_i$. The set (X, E) consists of all soft points $P = \langle (x_i)_\alpha^{e_i}, i \in \Lambda \text{ and } \alpha \in (0, 1)\rangle$, where $x_i \in X$ and $e_i \in E_i$. For each $j_0 \in \Lambda$, we define the projection $(P_q)_{j_0}$ from the product soft set (X, E) to the soft co-ordinate space (X_{j_0}, E_{j_0}), i.e., $(P_q)_{j_0}$: $(X, E) \longrightarrow (X_{j_0}, E_{j_0})$ by $(P_q)_{j_0}((x_i)_\alpha^{e_i}) = (x_{j_0})_\alpha^{e_{j_0}}$. These projections are used to define the soft product topology.

Definition 2.13 ([13]). The soft mappings $(P_q)_i, i \in \{1, 2\}$ is called a *soft projection mapping from* $FSS(X_1, A_1) \times FSS(X_2, A_2)$ *to* $FSS(X_i, A_i)$ and is defined as follows: for each $(F_1, A_1) \in FSS(X_1, A_1)$ and each $(F_2, A_2) \in FSS(X_2, A_2)$,

$$(P_q)_i((F_1, A_1) \times (F_2, A_2)) = P_i(F_1 \times F_2)_{q_i(A_1\times A_2)} = (F_i, A_i),$$

where $P_i : X_1 \times X_2 \longrightarrow X_i$ and $q_i : A_1 \times A_2 \longrightarrow A_i$ are classical projection mappings.

Definition 2.14 ([13]). Let $FSS(X, E)$ and $FSS(Y, K)$ be the collection of all the fuzzy soft sets over X and Y respectively and E, K be the parameters sets for the universe X and Y respectively. Let $u : X \longrightarrow Y$ and $p : E \longrightarrow K$ be two mappings. Let $f_{up} : FSS(X, E) \longrightarrow FSS(Y, K)$ be the fuzzy soft mapping from X to Y and let $(F, A) \in FSS(X, E)$, $(G, B) \in FSS(Y, K)$.

(i) The *image of* (F, A) *under* f_{up}, denoted by $f_{up}(F, A)$, is a fuzzy soft set over Y defined by for each $y \in Y$ and each $k \in K$,

$$f_{up}(F, A)(k)(y) = \begin{cases} sup\{u(x) = y\}\ sup\{p(e) = k\}\ F_A(e)(x) \\ \text{ if } \quad u^{-1}(y) \neq \phi \text{ and } p^{-1}(k) \neq \phi \\ \\ 0\text{otherwise} \end{cases}$$

(ii) The *inverse image of* (G, B) *under* f_{up}, denoted by $f_{up}^{-1}(G, B)$ and is a fuzzy soft set over X defined as: for each $e \in E$ and each $x \in X$,

$$f_{up}^{-1}(G, B)(e)(x) = (G, B)(p(e))(u(x)).$$

Definition 2.15 ([6]). Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces and $f_{up} : (X, \tau_1, E) \longrightarrow (Y, \tau_2, K)$ be a fuzzy soft mapping. Then f_{up} is said to be *fuzzy soft continuous*, if $f_{up}^{-1}(G, E) \in \tau_1$ for each $(G, E) \in \tau_2$.

Definition 2.16. Let X be a non-empty set and T be a soft topology on (X, E), where E be a parameters set. Let $\tau = \omega(T)$ be the set of all fuzzy soft lower semi-continuous mappings from (X, T, E) to I^X. Then $\omega(T) = \{(G, E) \in FSS(X, E) : (G, E)^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in I_1$. It can be shown that $\omega(T)$ is a fuzzy soft topology on (X, E).

Let P be the property of a soft topological space (X, T, E) and FSP be its topological analogue. Then FSP is called a *good extension* of P, if the statement (X, T, E) has P, equivalently, $(X, \omega(T), E)$ has FSP holds good for every soft topological space (X, T, E).

Definition 2.17 ([6]). Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces and $f_{up} : (X, \tau_1, E) \longrightarrow (Y, \tau_2, K)$ be a fuzzy soft mapping. Then f_{up} is said to be fuzzy soft open in X, if $f_{up}(F, E) \in \tau_2$ for each $(F, E) \in \tau_1$.

Definition 2.18 ([6]). Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces and $f_{up} : (X, \tau_1, E) \longrightarrow (Y, \tau_2, K)$ be a fuzzy soft mapping. Then f_{up} is called a *fuzzy soft homeomorphism*, if it is fuzzy soft bijective, fuzzy soft continuous and fuzzy soft open.

Definition 2.19 ([24]). Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces and $f_{up} : (X, \tau_1, E) \longrightarrow (Y, \tau_2, K)$ be a fuzzy soft mapping. Then f_{up} is called a *fuzzy soft one to one mapping*, if $f_{up}(x^e_\alpha) = f_{up}(y^k_\beta)$ implies $x^e_\alpha = y^k_\beta$.

Definition 2.20 ([24]). Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces and $f_{up} : (X, \tau_1, E) \longrightarrow (Y, \tau_2, K)$ be a fuzzy soft mapping. Then f_{up} is called a *fuzzy soft onto mapping*, if $f_{up}(X, E) = (Y, K)$.

Definition 2.21 ([23]). Let $\{(X_i, \tau_i), i \in \Lambda\}$ be a family of fuzzy soft topological spaces relative to the parameters sets E_i respectively, X be a set with parameters set E and for each $i \in \Lambda, (f_{up})_i : X \longrightarrow (X_i, \tau_i)$ be a soft mappings. Then the fuzzy soft topology τ over X is said to be *initial* with respect to the family $\{(f_{up})_i ; i \in \Lambda\}$, if τ has as subbase the set

$$S = \{(f_{up})^{-1}_i (F, Ai) : i \in \Lambda, (F, Ai) \in \tau_i\},$$

i.e., the fuzzy soft topology τ over X is generated by S.

Definition 2.22. Let $\{(X_i, \tau_i), i \in \Lambda\}$ be a family of fuzzy soft topological spaces relative to the parameters sets E_i respectively, X be a non-empty set with parameters set E and for each $i \in \Lambda, (f_{up})_i : (X_i, \tau_i) \longrightarrow X$ be a soft mappings. Then the fuzzy soft topology τ over X is said to be *final* with respect to the family $\{(f_{up})_i ; i \in \Lambda\}$, if τ has as subbase the set

$$S = \{(f_{up})_i (F, Ai) : i \in \Lambda, (F, Ai) \in \tau_i\},$$

i.e., the fuzzy soft topology τ over X is generated by S.

Definition 2.23 ([13]). Let $f_p : FSS(X, A) \longrightarrow FSS(Y, B)$ and $g_q : FSS(Y, B) \longrightarrow FSS(Z, C)$ be two fuzzy soft mappings. Then the *composition* of f_p and g_q, denoted by $f_p o g_q$, is defined by $f_p o g_q = f o g_{poq}$.

3. The main results

In this section, we introduce some notions on fuzzy soft R_0 spaces in quasi-coincident sense and find some of its properties.

Definition 3.1. A fuzzy soft topological space (X, τ, E) is called an:

(i) FSR_0(i)-space, if for any $x^e_r,\ y^e_s$ in (X, E) with $x \neq y$, whenever there exists $(F, E) \in \tau$ with $x^e_r q (F, E)$ and $y^e_s \bar{q} (F, E)$, then there exists $(G, E) \in \tau$ such that $x^e_r \bar{q} (G, E)$ and $y^e_s q (G, E)$.

(ii) FSR_0(ii)-space, if for any $x^e_r,\ y^e_s$ in (X, E) with $x \neq y$, whenever there exists $(F, E) \in \tau$ with $x^e_r \in (F, E)$ and $y^e_s \bar{q} (F, E)$, then there exists $(G, E) \in \tau$ such that $x^e_r \bar{q} (G, E)$ and $y^e_s \in (G, E)$.

(iii) FSR_0(iii)-space, if for any $x^e_r,\ y^e_s$ in (X, E) with $x \neq y$, whenever there exists $(F, E) \in \tau$ with $x^e_r \in (F, E)$ and $y^e_s \cap (F, E) = \phi$, then there exists $(G, E) \in \tau$ such that $x^e_r \cap (G, E) = \phi$ and $y^e_s \in (G, E)$.

(iv) FSR_0(iv)-space, if for any $x_r^e,\ y_s^e$ in (X, E) with $x \neq y$, whenever there exists $(F, E) \in \tau$ with $x_r^e q(F, E)$ and $y_s^e \cap (F, E) = \phi$, then there exists $(G, E) \in \tau$ such that $x_r^e \cap (G, E) = \phi$ and $y_s^e q(G, E)$.

3.1. **Good Extension Properties.** In this subsection, we shall show that our notions satisfy good extension property.

Theorem 3.2. *Let (X, T, E) be a fuzzy soft topological space. Then the followings are equivalent:*

(1) *(X, T, E) is a soft R_0 topological space,*
(2) *$(X, \omega(T), E)$ is an FSR_0(i)-space,*
(3) *$(X, \omega(T), E)$ is an FSR_0(iv)-space.*

Proof. (1) $\Longleftrightarrow$ (2): Let (X, T, E) be soft R_0 and let $x_r^e,\ y_r^e$ be fuzzy soft points in (X, E) with $x \neq y$ and $(F, E) \in \omega(T)$ with $x_r^e q(F, E)$ and $y_r^e \bar{q}(F, E)$. Then we have

$$\begin{aligned} x_r^e q(F, E) &\Rightarrow F(e)(x) + r > 1 \text{ for each } x \in X,\ e \in E \\ &\Rightarrow F(e)(x) > 1 - r \\ &\Rightarrow x \in (F, E)^{-1}(1 - r, 1] \end{aligned}$$

and

$$\begin{aligned} y_r^e \bar{q}(F, E) &\Rightarrow F(e)(y) + r \leq 1 \text{ for each } y \in X,\ e \in E \\ &\Rightarrow F(e)(y) \leq 1 - r \\ &\Rightarrow y \notin (F, E)^{-1}(1 - r, 1]. \end{aligned}$$

Since (X, T, E) is an R_0 soft topological space, there exists $(V, E) \in T$ such that $y_r^e \in (V, E)$, $x_r^e \notin (V, E)$. From the definition of fuzzy soft lower semi continuous, $1_{(V,E)} \in \omega(T)$ and $1_V(e)(y) = 1$, $1_V(e)(x) = 0$ for each $e \in E$. Thus we get

$$\begin{aligned} 1_V(e)(y) = 1 &\Rightarrow 1_V(e)(y) + r > 1 \\ &\Rightarrow y_r^e q 1_{(V,E)} \end{aligned}$$

and

$$\begin{aligned} 1_V(e)(x) = 0 &\Rightarrow 1_V(e)(x) + r \leq 1 \\ &\Rightarrow x_r^e \bar{q} 1_{(V,E)}. \end{aligned}$$

So there exists $1_{(V,E)} \in \omega(T)$ such that $y_r^e q 1_{(V,E)}$, $x_r^e \bar{q} 1_{(V,E)}$. Hence $(X, \omega(T), E)$ is $FSR_0(i)$.

Conversely, let $(X, \omega(T), E)$ be a fuzzy soft topological space and $(X, \omega(T), E)$ is FSR_0(i). We have to prove that (X, T, E) is SR_0. Let $x,\ y \in X$ with $x \neq y$ and $(U, E) \in T$ with $x_r^e \in (U, E)$ and $y_r^e \notin (U, E)$. From the definition of fuzzy soft lower semi continuous, we have

$$1_{(U,E)} \in \omega(T) \text{ and } 1_U(e)(x) = 1,\ 1_U(e)(y) = 0 \text{ for each } e \in E.$$

Then we get

$$\begin{aligned} 1_U(e)(x) = 1 &\Rightarrow 1_U(e)(x) + r > 1 \\ &\Rightarrow x_r^e q 1_{(U,E)} \end{aligned}$$

and

$$\begin{aligned} 1_U(e)(y) = 0 &\Rightarrow 1_U(e)(y) + r \leq 1 \\ &\Rightarrow y_r^e \bar{q} 1_{(U,E)}. \end{aligned}$$

Since $(X, \omega(T), E)$ is $FSR_0(i)$ topological space, there exists $(G, E) \in \omega(T)$ such that $y_r^e q(G, E)$ and $x_r^e \bar{q}(G, E)$. Thus we have

$$\begin{aligned} y_r^e q(G, E) &\Rightarrow G(e)(y) + r > 1 \text{ for each } y \in X,\ e \in E \\ &\Rightarrow G(e)(y) > 1 - r \end{aligned}$$

$$\Rightarrow y \in (G,E)^{-1}(1-r,1]$$
and $$x^e_r \bar{q}(G,E) \Rightarrow G(e)(x)+r \leq 1 \text{ for each } x \in X,\ e \in E$$
$$\Rightarrow G(e)(x) \leq 1-r$$
$$\Rightarrow x \notin (G,E)^{-1}(1-r,1].$$
So there exists $(G,E)^{-1}(1-r,1] \in T$ such that $y \in (G,E)^{-1}(1-r,1]$, $x \notin (G,E)^{-1}(1-r,1]$. Hence (X,T,E) is a soft R_0 topological space.

(1) $\Longleftrightarrow$ (3): The proof is similar to (1) $\Longleftrightarrow$ (2). □

Remark 3.3. Theorem 3.2 ensures that our axioms are valid and well-defined.

3.2. **Subspaces in fuzzy soft $\mathbf{R}_0$ topological spaces.** In this subsection, we shall show that our notions satisfy hereditary property.

Theorem 3.4. *Let (X,τ,E) be a fuzzy soft topological space, $A \subseteq X$, $t_A = \{(F_A,E) = (F,E) \cap A : (F,E) \in \tau\}$. If (X,τ,E) is FSR$_0$(j), then (A,t_A,E) is FSR$_0$(j) for $j = i,\ ii,\ iii,\ iv$.*

Proof. Let (X,τ,E) be FSR$_0$(j) and let x^e_r, y^e_s be fuzzy soft points in (A,E) with $x \neq y$ and $(F,E) \in t_A$ with $x^e_r q(F,E)$ and $y^e_s \bar{q}(F,E)$. Since $A \subseteq X$, these fuzzy soft points are also fuzzy soft points in (X,E). Since $(F,E) \in t_A$, we can write $(F,E) = (G_A,E) = (G,E) \cap A$, where $(G,E) \in \tau$ with $x^e_r q(G,E)$ and $y^e_s \bar{q}(G,E)$. Also since (X,τ,E) is FSR$_0$(j) fuzzy soft topological space, there exists $(H,E) \in \tau$ such that $x^e_r \bar{q}(H,E)$, $y^e_s q(H,E)$. From the definition of t_A, $(H_A,E) = ((H,E) \cap A) \in t_A$. Then we have
$$y^e_s q(H,E) \Rightarrow H(e)(y)+s > 1 \text{ for each } y \in X,\ e \in E$$
$$\Rightarrow H(e)(y) \cap A(y) + s > 1 \text{ for each } y \in A \subseteq X$$
$$\Rightarrow ((H,E) \cap A)(e)(y) + s > 1$$
$$\Rightarrow (H_A,E)(e)(y) + s > 1$$
$$\Rightarrow y^e_s q(H_A,E)$$
and
$$x^e_r \bar{q}(H,E) \Rightarrow H(e)(x)+r \leq 1 \text{ for each } x \in X,\ e \in E$$
$$\Rightarrow H(e)(x) \cap A(x) + r \leq 1 \text{ for each } x \in A \subseteq X$$
$$\Rightarrow ((H,E) \cap A)(e)(x) + r \leq 1$$
$$\Rightarrow (H_A,E)(e)(x) + r \leq 1$$
$$\Rightarrow x^e_r \bar{q}(H_A,E).$$
Thus there exists $(H_A,E) \in t_A$ such that $x^e_r \bar{q}(H_A,E)$, $y^e_s q(H_A,E)$. So (A,tA,E) is FSR$_0$(j). □

3.3. **Productivity and projectivity in fuzzy soft $\mathbf{R}_0$ topological spaces.** In this subsection, we shall show that our notions satisfy productive and projective properties.

Theorem 3.5. *Let (X_i,τ_i,E_i), $i \in \Lambda$ be a fuzzy soft topological spaces and $X = \Pi_{i\in\Lambda} X_i$, $E = \Pi_{i\in\Lambda} E_i$ and τ be the fuzzy soft topology on (X,E). Then for all $i \in \Lambda$, (X_i,τ_i,E_i) is FSR$_0$(j) if and only if (X,τ,E) is FSR$_0$(j) for $j = i,\ ii,\ iii,\ iv$.*

Proof. Suppose (X_i,τ_i,E_i) is FSR$_0$(j) space for all $i \in \Lambda$ and let x^e_r, y^e_s be fuzzy soft points in (X,E) with $x \neq y$ and $(F,E) \in \tau$ such that $x^e_r q(F,E)$, $y^e_s \bar{q}(F,E)$. Then $(x_i)^{e_i}_r$, $(y_i)^{e_i}_s$ are fuzzy soft points with $(x_i) \neq (y_i)$ for some $i \in \Lambda$ and $(F_i,E_i) \in \tau_i$

such that $(x_i)_r^{e_i} q(F_i, E_i)$, $(y_i)_s^{e_i} \bar{q}(F_i, E_i)$. Since (X_i, τ_i, E_i) is FSR_0(j), there exists $(G_i, E_i) \in \tau_i$ such that $(x_i)_r^{e_i} \bar{q}(G_i, E_i)$ and $(y_i)_s^{e_i} q(G_i, E_i)$. But we have

$$PX_i(x) = x_i,\ PX_i(y) = y_i,\ qE_i(e) = e_i.$$

On the other hand, we get

$$\begin{aligned}(x_i)_r^{e_i} \bar{q}(G_i, E_i) &\Rightarrow G_i(e_i)(x_i) + r \leq 1 \text{ for each } x_i \in X_i,\ e_i \in E_i\\ &\Rightarrow G_i(qE_i(e))(PX_i(x)) + r \leq 1 \text{ for each } x \in X,\ e \in E\\ &\Rightarrow (G_i oqE_i)(e)(PX_i(x)) + r \leq 1\\ &\Rightarrow (G_i oqE_i oPX_i)(e)(x) + r \leq 1\\ &\Rightarrow x_r^e \bar{q}(G_i oqE_i oPX_i, E)\end{aligned}$$

and

$$\begin{aligned}(y_i)_s^{e_i} q(G_i, E_i) &\Rightarrow G_i(e_i)(y_i) + s > 1 \text{ for each } y_i \in X_i \text{ and each } e_i \in E_i\\ &\Rightarrow G_i(qE_i(e))(PX_i(y)) + s > 1 \text{ for each } y \in X,\ e \in E\\ &\Rightarrow (G_i oqE_i)(e)(PX_i(y)) + s > 1\\ &\Rightarrow (G_i oqE_i oPX_i)(e)(y) + s > 1\\ &\Rightarrow y_s^e q(G_i oqE_i oPX_i, E).\end{aligned}$$

Then there exists $(G_i oqE_i oPX_i, E) \in \tau_i$ such that

$$x_r^e \bar{q}(G_i oqE_i oPX_i, E),\ y_s^e q(G_i oqE_i oPX_i, E).$$

Thus (X, τ, E) is FSR_0(j).

Conversely, suppose (X, τ, E) is FSR_0(j) and let a_i be a fixed element in X_i. Let $A_i = \{x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j \text{for some } i \neq j\}$. Then A_i is a subset of X and thus (A_i, τ_{A_i}, E_i) is a subspace of (X, τ, E). Since (X, τ, E) is FSR_0(j), (A_i, τ_{A_i}, E_i) is FSR_0(j). Note that (A_i, E_i) is homeomorphic image of (X_i, E_i). So (X_i, τ_i, E_i) is FSR_0(j) space for all $i \in \Lambda$. □

3.4. Mappings in fuzzy soft $\mathbf{R}_0$ topological spaces.

Theorem 3.6. *Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces. Let $u : X \longrightarrow Y$, $p : E \longrightarrow K$ be one-one, onto and fuzzy soft open maps and thus a fuzzy soft mapping $f_{up} : FSS(X, E) \longrightarrow FSS(Y, K)$ be a one-one, onto and fuzzy soft open map. If (X, τ_1, E) is FSR_0(j), then (Y, τ_2, K) is FSR_0(j) for $j = i,\ ii,\ iii,\ iv$.*

Proof. Suppose (X, τ_1, E) is FSR_0(j) and let x_r^k, y_s^k be fuzzy soft points in (Y, K) with $x \neq y$ and let $(F, K) \in \tau_2$ with $x_r^k q(F, K)$, $y_s^k \bar{q}(F, K)$. Since u and p are onto, there exist $\acute{x}, \acute{y} \in X$ with $u(\acute{x}) = x, u(\acute{y}) = y$ and $p(e) = k; \forall e \in E, k \in K$ and also $\acute{x}_r^e, \acute{y}_s^e$ are fuzzy soft points in (X, E) with $\acute{x} \neq \acute{y}$ as u is one-one. Again, since f_{up} is soft continuous and $(F, K) \in \tau_2, f_{up}^{-1}(F, K) \in \tau_1$. On the other hand, we have

$$\begin{aligned}x_r^k q(F, K) &\Rightarrow F(k)(x) + r > 1 \text{ for each } x \in Y,\ k \in K\\ &\Rightarrow (F, K)(p(e))(u(x^{'})) + r > 1 \text{ for each } x^{'} \in X,\ e \in E\\ &\Rightarrow f_{up}^{-1}(F, K)(e)(\acute{x}) + r > 1\\ &\Rightarrow \acute{x}_r^e q f_{up}^{-1}(F, K)\end{aligned}$$

and

$$\begin{aligned}y_s^k \bar{q}(F, K) &\Rightarrow F(k)(y) + s \leq 1 \text{ for each } y \in Y,\ k \in K\\ &\Rightarrow (F, K)(p(e))(u(y^{'})) + s \leq 1 \text{ for each } y^{'} \in X,\ e \in E\\ &\Rightarrow f_{up}^{-1}(F, K)(e)(\acute{y}) + s \leq 1\end{aligned}$$

$\Rightarrow \acute{y}_s^e \bar{q} f_{up}^{-1}(F, K)$.

Since (X, τ_1, E) is FSR_0(i) space, there exists $(G, E) \in \tau_1$ such that

$$(y^{'})_s^e q(G, E),\ (x^{'})_r^e \bar{q}(G, E).$$

By the definition, we get: for some $x^{'}$ and $y^{'}$,

$$f_{up}(F, E)(k)(x) = \sup\{u(x^{'}) = x\} \sup\{p(e) = k\} F(e)(x^{'}) = F(e)(x^{'})$$

and

$$f_{up}(G, E)(k)(y) = \sup\{u(y^{'}) = y\} \sup\{p(e) = k\} G(e)(y^{'}) = G(e)(y^{'}).$$

Furthermore, we have

$$\begin{aligned}(y^{'})_s^e q(G, E) &\Rightarrow G(e)(y^{'}) + s > 1 \text{ for each } y^{'} \in X,\ e \in E\\ &\Rightarrow \sup\{u(y^{'}) = y\} \sup\{p(e) = k\} G(e)(y^{'}) + s > 1\\ &\quad \text{for each } y \in Y,\ k \in K\\ &\Rightarrow f_{up}(G, E)(k)(y) + s > 1\\ &\Rightarrow y_s^k q f_{up}(G, E)\end{aligned}$$

and

$$\begin{aligned}((x^{'})_r^e \bar{q}(G, E) &\Rightarrow G(e)(x^{'}) + r \leq 1 \text{ for each } x^{'} \in X,\ e \in E\\ &\Rightarrow \sup\{u(\acute{x}) = x\}\ \sup\{p(e) = k\} G(e)(x^{'}) + r \leq 1\\ &\quad \text{for each } x \in Y,\ k \in K\\ &\Rightarrow f_{up}(G, E)(k)(x) + r \leq 1\\ &\Rightarrow x_r^k \bar{q} f_{up}(G, E).\end{aligned}$$

Since f_{up} is fuzzy soft open mapping, $f_{up}(G, E) \in \tau_2$. Then there exists $f_{up}(G, E) \in \tau_2$ such that $y_s^k q f_{up}(G, E),\ x_r^k \bar{q} f_{up}(G, E)$. Thus (Y, τ_2, K) is FSR_0(j) space. □

Theorem 3.7. *Let (X, τ_1, E) and (Y, τ_2, K) be two fuzzy soft topological spaces. Let $u : X \longrightarrow Y$, $p : E \longrightarrow K$ be one-one and soft continuous maps and thus a fuzzy soft mapping $f_{up} : FSS(X, E) \longrightarrow FSS(Y, K)$ be a one-one and fuzzy soft continuous map. If (Y, τ_2, K) is FSR_0(j), then (X, τ_1, E) is FSR_0(j)for $j = i,\ ii\ , iii,\ iv$.*

Proof. Suppose (Y, τ_2, K) is FSR_0(j) and let $x_r^e,\ y_s^e$ be fuzzy soft points in (X, E) with $x \neq y$ and let $(F, E) \in \tau_1$ such that $x_r^e q(F, E),\ y_s^e \bar{q}(F, E)$. Then there exist fuzzy soft points $(x^{'})_r^k,\ (y^{'})_s^k$ in (Y, K) with $u(x) = x^{'},\ u(y) = y^{'}$ and $x^{'} \neq y^{'}$ as u is one-one. Since p is one-one, $p(e) = k$ for each $e \in E,\ k \in K$. Since f_{up} is soft open mapping and $(F, E) \in \tau_1,\ f_{up}(F, E) \in \tau_2$. By the definition, we have : for some x and y,

$$f_{up}(F, E)(k)(\acute{x}) = \sup\{u(x) = \acute{x}\} \sup\{p(e) = k\} F(e)(x) \Rightarrow f_{up}(F, E)(k)(\acute{x}) = F(e)(x)$$

and

$$f_{up}(F, E)(k)(\acute{y}) = \sup\{u(y) = \acute{y}\} \sup\{p(e) = k\} F(e)(y) \Rightarrow f_{up}(F, E)(k)(\acute{y}) = F(e)(y).$$

On the other hand we get

$$\begin{aligned}x_r^e q(F, E) &\Rightarrow F(e)(x) + r > 1 \text{ for each } x \in X,\ e \in E\\ &\Rightarrow f_{up}(F, E)(k)(x^{'}) + r > 1 \text{ for each } x^{'} \in Y,\ k \in K\\ &\Rightarrow (x^{'})_r^k q f_{up}(F, E)\end{aligned}$$

and

$$\begin{aligned}y_s^e \bar{q}(F, E) &\Rightarrow F(e)(y) + s \leq 1 \text{ for each } y \in X,\ e \in E\\ &\Rightarrow f_{up}(F, E)(k)(y^{'}) + s \leq 1\ y^{'} \in Y,\ k \in K\end{aligned}$$

$$\Rightarrow \acute{y}^k_s \bar{q} f_{up}(F,E).$$

Since (Y,τ_2,K) is FSR_0(i) space , there exists $(G,K)\in\tau_2$ such that $(y^{'})^k_s q(G,K)$, $(x^{'})^k_r \bar{q}(G,K)$. Also since f_{up} is continuous, $f^{-1}_{up}(G,K)\in\tau_1$. Furthermore, we have

$$\begin{aligned}(y^{'})^k_s q(G,K) &\Rightarrow G(k)(y^{'})+s>1 \text{ for each } y^{'}\in Y,\ k\in K\\ &\Rightarrow (G,K)(p(e))(u(y))+s>1 \text{ for each } y\in X,\ e\in E\\ &\Rightarrow f^{-1}_{up}(G,K)(e)(y)+s>1\\ &\Rightarrow y^e_s q f^{-1}_{up}(G,K)\end{aligned}$$

and

$$\begin{aligned}(x^{'})^k_r \bar{q}(G,K) &\Rightarrow G(k)(\acute{x})+r\le 1 \text{ each } x^{'}\in Y,\ k\in K\\ &\Rightarrow (G,K)(p(e))(u(x))+r\le 1 \text{ for each } x\in X,\ e\in E\\ &\Rightarrow f^{-1}_{up}(G,K)(e)(x)+r\le 1\\ &\Rightarrow x^e_r \bar{q} f^{-1}_{up}(G,K).\end{aligned}$$

Thus there exists $f^{-1}_{up}(G,K)\in\tau_1$ such that $y^e_s q f^{-1}_{up}(G,K), x^e_r \bar{q} f^{-1}_{up}(G,K)$. So (X,τ_1,E) is FSR_0(j) space. The proof is complete. □

4. Conclusions

The main result of this paper is introducing some new concepts of fuzzy soft R_0 topological spaces. We discuss some features of this concepts and present their good extension, hereditary. We hope that these results will be useful for the future study on fuzzy soft topology to carry out general framework for the practical applications and to solve the complicated problems containing uncertainties in engineering, medical, environment and in general man-machine systems of various types.

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