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Abstract. In this paper, the concepts of hybrid quasi-pure hyperideals in an ordered hypersemigroup are introduced. Finally, we characterize quasi-pure hyperideals in ordered hypersemigroups using hybrid quasi-pure hyperideals.

2020 AMS Classification: 06F99, 06F05

Keywords: Ordered hypersemigroup, Hybrid structure, Hybrid quasi-pure hyperideal.

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1. Introduction

The theory of fuzzy sets is the most appropriate theory for dealing with uncertainty and was introduced by Zadeh [1] in 1965. After the introduction of the concept of fuzzy sets by Zadeh, several researchers researched the generalizations of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research and many branches of pure and applied mathematics. For example, Xie et al. [2] applied fuzzy set theory to the switching method.

Molodtsov [3] introduced the concept of the soft set as a new mathematical tool for dealing with uncertainties that are free from the difficulties which have troubled the usual theoretical approaches. The soft sets have many applications in several branches of both pure and applied sciences (See [4, 5, 6]).

As a parallel circuit of fuzzy sets and soft sets, Jun et al. [7] introduced the notion of hybrid structures in a set of parameters over an initial universe set. The hybrid structures can be applied in many areas including mathematics, statistics, computer science, electrical instruments, industrial operations, business, engineering, social decisions, etc. (See [8, 9, 10]).

The algebraic hyperstructure theory was first introduced in 1934 by Marty [11]. Hyperstructures have many applications in several branches of both pure and applied sciences (See [12, 13, 14, 15, 16]). Recently, Heidari and Davvaz applied the hyperstructure theory to ordered semigroups and introduced the concept of ordered semihypergroups (or hypersemigroups)(See [17]), which is a generalization of the concept of ordered semigroups. Furthermore, the ordered semihypergroup theory was enriched by the work of many researchers, for example, [15, 16, 18, 19]. In particular, the hyperideal theory on semihypergroups and ordered hypersemigroups can be seen in [18, 19, 20, 21].

In 1989, Ahsan and Takahashi [22] introduced the notions of pure ideals and purely prime ideals of semigroups. Later, Changphas and Sanborisoot [23] defined the notions of left pure, right pure, left weakly pure, and right weakly pure ideals in ordered semigroups and gave some of their characterizations. In 2020, Changphas and Davvaz [24] studied purity of hyperideals in ordered hyperstructures. They introduced the notions of pure hyperideals and weakly pure hyperideals in an ordered semihypergroup. We now apply hybrid structures to ordered hyperstructures. In this present paper, the concepts of hybrid quasi-pure hyperideals in an ordered hypersemigroup are introduced. Finally, we characterize quasi-pure hyperideals in ordered hypersemigroups in terms of hybrid quasi-pure hyperideals.

2. Preliminaries

In this section, we will recall the basic terms and definitions from the ordered hypersemigroup theory and the hybrid structure theory that we will use later in this paper. Throughout this paper, we will use the concepts of ordered hypersemigroups introduced by Kehayopulu [25] and hybrid structures introduced by Anis [8].

Definition 2.1 ([25])**.** A *hypergroupoid* is a nonempty set H with a hyperoperation

$$\circ : H \times H \to \mathcal{P}^*(H) \mid (a, b) \mapsto a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \to \mathcal{P}^*(H) \mid (A, B) \mapsto A * B$$

on $\mathcal{P}^*(H)$ (induced by the hyperoperation $\circ$) defined by

$$A * B = \bigcup_{a \in A, b \in B} (a \circ b)$$

($\mathcal{P}^*(H)$ is the set of all nonempty subsets of H.)

We have $\{x\} * \{y\} = x \circ y$. Furthermore $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$ for any nonempty subsets $A,\ B$ and C of H.

Definition 2.2 ([25])**.** A hypergroupoid $(H; \circ)$ is called a *hypersemigroup*, if

$$\{x\} * (y \circ z) = (x \circ y) * \{z\}$$

for every $x,\ y,\ z \in H$.

For convenience, the previous equation could be identified as

$$x * (y \circ z) = (x \circ y) * z.$$

Let $(H; \leq)$ be a partial order set. We define a relation $\preceq$ on $\mathcal{P}^*(H)$ as follows: For two nonempty subsets A and B of H,

$$A \preceq B := \{(x, y) \in A \times B \mid x \leq y, \forall x \in A, \exists y \in B\}.$$

Definition 2.3 ([25]). The structure $(H; \circ, \leq)$ is called an *ordered hypersemigroup*, if the following conditions are satisfied:

(i) $(H; \circ)$ is a hypersemigroup,
(ii) $(H; \leq)$ is a partial order set,
(iii) for $a, b, c \in H$, if $a \leq b$, then $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$.

For simplicity, we denote an ordered hypersemigroup $(H; \circ, \leq)$ by its carrier set as a bold letter $\mathbf{H}$.

Definition 2.4 ([25]). Let $\mathbf{H}$ be an ordered hypersemigroup. A nonempty subset A of H is called a *left* (resp., *right*) *hyperideal* of $\mathbf{H}$, if

(i) $H * A \subseteq A$ (resp., $A * H \subseteq A$).
(ii) for $a \in H,\ b \in A$, if $a \leq b$, then $a \in A$.

A nonempty subset A of H is called a *two-sided hyperideal*, or simply a *hyperideal* of $\mathbf{H}$, if it is both a left and a right hyperideal of $\mathbf{H}$.

Let A be a nonempty subset of H. Define

$$(A] := \{x \in H \mid \ x \leq a \ \text{ for some } a \in A\}.$$

Note that condition (ii) in Definition 2.4 is equivalent to $A = (A]$. If A and B are nonempty subsets of H, then we obtain

(1) $A \subseteq (A]$,
(2) $(A \cup B] = (A] \cup (B]$,
(3) $((A] * (B]] = (A * B]$,
(4) $(A] * (B] \subseteq (A * B]$.

Definition 2.5 ([20]). Let $\mathbf{H}$ be an ordered hypersemigroup. A nonempty subset Q of H is called a *quasi-hyperideal* of $\mathbf{H}$, if it satisfies the following conditions:

(i) $(H * Q) \cap (Q * H) \subseteq Q$,
(ii) For $a \in H, b \in Q$, if $a \leq b$, then $a \in Q$.

Note that condition (ii) in Definition 2.5 is equivalent to $Q = (Q]$.

For an element a of an ordered hypersemigroup $\mathbf{H}$, we denote $Q(a)$ to be the quasi-hyperideal of $\mathbf{H}$ generated by a, and easy to verify that

$$Q(a) = (a \cup (H * a)] \cap (a \cup (a * H)].$$

Definition 2.6. Let $\mathbf{H}$ be an ordered hypersemigroup. A hyperideal A of $\mathbf{H}$ is called a *quasi-pure hyperideal* of $\mathbf{H}$, if $x \in (A * x] \cap (x * A]$ for all $x \in A$.

In what follows, let $I = [0, 1]$ be the unit interval, H a set of parameters and $\mathcal{P}(U)$ the power set of an initial universe set U.

Definition 2.7 ([8])**.** A *hybrid structure* in H over U is defined to be a mapping

$$f := (f^*, f^+) : H \to \mathcal{P}(U) \times I, x \mapsto (f^*(x), f^+(x)),$$

where

$$f^* : H \to \mathcal{P}(U) \text{ and } f^+ : H \to I$$

are mappings.

Let us denote by $Hyb_U(H)$ the set of all hybrid structures in H over U. We define an order $\ll$ on $Hyb_U(H)$ as follows: For all $f, g \in Hyb_U(H)$,

$$f \ll g \Leftrightarrow f^* \sqsubseteq g^*, f^+ \succeq g^+,$$

where $f^* \sqsubseteq g^*$ means that $f^*(x) \subseteq g^*(x)$ and $f^+ \succeq g^+$ means that $f^+(x) \geq g^+(x)$ for all $x \in H$ and $f = g$ if $f \ll g$ and $g \ll f$.

Definition 2.8 ([8])**.** Let f and g be elements in $Hyb_U(H)$. Then the *hybrid intersection* of f and g, denoted by $f \Cap g$, is defined to be a hybrid structure

$$f \Cap g : H \to \mathcal{P}(U) \times I, x \mapsto (f^* \cap g^*)(x), ((f^+ \vee g^+)(x)),$$

where

$$(f^* \cap g^*)(x) := f^*(x) \cap g^*(x) \text{ and } (f^+ \vee g^+)(x) := \max\{f^+(x), g^+(x)\}.$$

We denote $\widetilde{H} := (H^*, H^+)$ the hybrid structure in H over U and is defined as follows:

$$\widetilde{H} : H \longrightarrow \mathcal{P}(U) \times I : x \mapsto \left(H^*(x), H^+(x)\right),$$

where

$$H^*(x) := U \text{ and } H^+(x) := 0.$$

Let a be an element in H. Then we set

$$\mathbf{H}_a := \{(x, y) \in H \times H \mid a \preceq x \circ y\}.$$

Definition 2.9. Let f and g be elements in $Hyb_U(H)$. Then the *hybrid products* of f and g, denoted by $f \otimes g$ is defined to be a hybrid structure

$$f \otimes g : H \to \mathcal{P}(U) \times I, x \mapsto ((f^* \odot g^*)(x), (f^+ \oplus g^+)(x)),$$

where

$$(f^* \odot g^*)(x) := \begin{cases} \bigcup\limits_{(a,b) \in \mathbf{H}_x} (f^*(a) \cap g^*(b)) & \text{if } \mathbf{H}_x \neq \varnothing \\ \varnothing & \text{otherwise,} \end{cases}$$

and

$$(f^+ \oplus g^+)(x) := \begin{cases} \bigwedge\limits_{(a,b) \in \mathbf{H}_x} \{\max\{f^+(a), g^+(b)\}\} & \text{if } \mathbf{H}_x \neq \varnothing \\ 1 & \text{otherwise.} \end{cases}$$

Let A be a nonempty subset of H. We denote by $\chi_A := (\chi_A^*, \chi_A^+)$ the characteristic hybrid structure of A in H over U and is defined to be a hybrid structure

$$\chi_A : H \to \mathcal{P}(U) \times I, x \mapsto (\chi_A^*(x), \chi_A^+(x)),$$

where

$$\chi_A^*(x) := \begin{cases} U & \text{if } x \in A \\ \varnothing & \text{otherwise,} \end{cases}$$

and

$$\chi_A^+(x) := \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise.} \end{cases}$$

We set $\chi_H := \widetilde{H}$.

3. Main results

In this main section, we introduce the concepts of hybrid quasi-pure hyperideals in ordered hypersemigroups. Finally, we characterize quasi-pure hyperideals in ordered hypersemigroups in terms of hybrid quasi-pure hyperideals.

Definition 3.1. Let $\mathbf{H}$ be an ordered hypersemigroup. A hybrid structure $f = (f^*, f^+)$ in H over U is called a *hybrid left* (resp. *right*) *hyperideal* in $\mathbf{H}$ over U, if for every $x, y \in H$,

(i) $\bigcap_{a \in x \circ y} f^*(a) \supseteq f^*(y)$ (resp. $\bigcap_{a \in x \circ y} f^*(a) \supseteq f^*(x)$),
(ii) $\bigvee_{a \in x \circ y} f^+(a) \leq f^+(y)$ (resp. $\bigvee_{a \in x \circ y} f^+(a) \leq f^+(x)$),
(iii) $x \leq y$ implies $f^*(x) \supseteq f^*(y)$ and $f^+(x) \leq f^+(y)$.

A hybrid structure f is called a *hybrid hyperideal* in $\mathbf{H}$ over U, if it is both a hybrid left and a hybrid right hyperideal in $\mathbf{H}$ over U.

Example 3.2. Let $H = \{a, b, c\}$. We define a binary operation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$
c	H	H	H

and $\leq := \{(a, c), (b, c)\} \cup \Delta_H$, where $\Delta_H := \{(x, x) : x \in H\}$. Then $\mathbf{H} := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \mathbb{N}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	a	b	c
$f^*(x)$	U	$2U$	$4U$
$f^+(x)$	0.2	0.7	0.8

Then $f = (f^*, f^+)$ is a hybrid right hyperideal in H over U.

Example 3.3. Let $H = \{a, b, c\}$. We define a binary operation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c
a	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a, b\}$	$\{c\}$

and $\leq := \{(a, b)\} \cup \Delta_H$, where $\Delta_H := \{(x, x) : x \in H\}$. Then $\mathbf{H} := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \mathbb{N}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	a	b	c
$f^*(x)$	$2U$	U	$4U$
$f^+(x)$	0.7	0.2	0.8

Then $f = (f^*, f^+)$ is a hybrid left hyperideal in H over U.

Example 3.4. Let $H = \{a, b, c, d\}$. We define a binary operation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$

and $\leq := \{(a, b)\} \cup \Delta_H$, where $\Delta_H := \{(x, x) : x \in H\}$. Then $\mathbf{H} := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \{1, 2, 3, 4, 5\}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	a	b	c	d
$f^*(x)$	$\{1, 2, 4, 5\}$	$\{1, 2, 5\}$	$\{1, 5\}$	$\{5\}$
$f^+(x)$	0.1	0.2	0.5	0.8

Then $f = (f^*, f^+)$ is a hybrid hyperideal in H over U.

Definition 3.5. Let $\mathbf{H}$ be an ordered hypersemigroup. A hybrid structure $f = (f^*, f^+)$ in H over U is called a *hybrid quasi-hyperideal* in $\mathbf{H}$ over U, if it satisfies the following conditions:

(i) $(f \otimes \widetilde{H}) \Cap (\widetilde{H} \otimes f) \ll f$,
(ii) for $x,\ y \in H$, if $x \leq y$, then $f^*(x) \supseteq f^*(y)$ and $f^+(x) \leq f^+(y)$.

Example 3.6. Let $H = \{a, b, c, d, e\}$. We define a binary operation $\circ$ and a binary relation $\leq$ on H as follows:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{e\}$
b	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a\}$	$\{e\}$
c	$\{a\}$	$\{a, c\}$	$\{a\}$	$\{a\}$	$\{e\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{e\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

and $\leq := \{(a, b), (a, c)\} \cup \Delta_H$, where $\Delta_H := \{(x, x) : x \in H\}$. Then $\mathbf{H} := (H; \circ, \leq)$ is an ordered hypersemigroup. Let $U = \{1, 2, 3\}$. Define a hybrid structure $f := (f^*, f^+)$ in H over U as follows:

H	a	b	c	d	e
$f^*(x)$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\varnothing$	$\{1, 2, 3\}$
$f^+(x)$	0	0	0	1	0

Then $f = (f^*, f^+)$ is a hybrid hyperideal in H over U and it is easy to verify that $f = (f^*, f^+)$ also a hybrid quasi-pure hyperideal in H over U.

The following remarks are useful tools in calculating the purity of hybrid hyperideals.

Remark 3.7. Let $f = (f^*, f^+)$ be a hybrid quasi-hyperideal in $\mathbf{H}$ over U. Then we have

(1) $(f^* \odot H^*) \cap (H^* \odot f^*) \sqsubseteq f^*$,
(2) $(f^+ \oplus H^+) \vee (H^+ \oplus f^+) \succeq f^+$.

Definition 3.8. Let $\mathbf{H}$ be an ordered hypersemigroup. A hybrid hyperideal $f = (f^*, f^+)$ in $\mathbf{H}$ over U is said to be *quasi-pure*, if $f \Cap g = (f \otimes g) \Cap (g \otimes f)$ for every hybrid quasi-hyperideal $g = (g^*, g^+)$ in $\mathbf{H}$ over U.

The following lemmas are important in illustrating our first theorem.

Lemma 3.9. *Let* $\mathbf{H}$ *be an ordered hypersemigroup and* A, B *nonempty subsets of* H. *Then the following conditions hold:*

(1) $A \subseteq B$ *if and only if* $\chi_A \ll \chi_B$,
(2) $\chi_A \Cap \chi_B = \chi_{A \cap B}$,
(3) $\chi_A \otimes \chi_B = \chi_{(A*B]}$.

Proof. We will give a proof of (3) only. Let $x \in (A * B]$. Then there exists $c \in a \circ b$ for some $a \in A$ and $b \in B$ such that $x \leq c$, which means that $x \preceq a \circ b$. Thus $\mathbf{H}_x \neq \varnothing$ and we obtain

$$\begin{aligned} U &\supseteq (\chi_A^* \odot \chi_B^*)(x) \\ &= \bigcup_{(y,z)\in \mathbf{H_x}} [\chi_A^*(y) \cap \chi_B^*(z)] \\ &\supseteq \chi_A^*(a) \cap \chi_B^*(b) \\ &= U. \end{aligned}$$

This implies that $(\chi_A^* \odot \chi_B^*)(x) = U = \chi_{(A*B]}^*(x)$ and

$$\begin{aligned} 0 &\leq (\chi_A^+ \oplus \chi_B^+)(x) \\ &= \bigwedge_{(y,z)\in \mathbf{H_x}} \{\max\{\chi_A^+(y), \chi_B^+(z)\}\} \\ &\leq \max\{\chi_A^+(a), \chi_B^+(b)\} \\ &= 0. \end{aligned}$$

This implies that $(\chi_A^+ \oplus \chi_B^+)(x) = 0 = \chi_{(A*B]}^+(x)$. So $\chi_A \otimes \chi_B = \chi_{(A*B]}$. □

Lemma 3.10. *Let* $\mathbf{H}$ *be an ordered hypersemigroup and* A *a nonempty subset of* H. *Then the following conditions are equivalent:*

(1) A *is a right (resp., left) hyperideal of* $\mathbf{H}$,
(2) $\chi_A = (\chi_A^*, \chi_A^+)$ *is a hybrid right (resp. left) hyperideal in* $\mathbf{H}$ *over* U.

Proof. (1)⇒(2). Let A be a right hyperideal of an ordered hypersemigroup $\mathbf{H}$. Firstly, let $x,\ y \in H$. If $x \in A$, then $x \circ y \subseteq A$ and we obtain

$$\bigcap_{a \in x \circ y} \chi^*_A(a) = U \supseteq \chi^*_A(x) \quad \text{and} \quad \bigvee_{a \in x \circ y} \chi^+_A(a) = 0 \leq \chi^+_A(x).$$

If $x \notin A$, we obtain

$$\bigcap_{a \in x \circ y} \chi^*_A(a) \supseteq \varnothing = \chi^*_A(x) \quad \text{and} \quad \bigvee_{a \in x \circ y} \chi^+_A(a) \leq 1 = \chi^+_A(x).$$

Secondly, let $x,\ y \in H$ be such that $x \leq y$. If $y \in A$, then $x \in A$ and then

$$\chi^*_A(x) = U \supseteq \chi^*_A(y) \quad \text{and} \quad \chi^+_A(x) = 0 \leq \chi^*_A(y).$$

If $y \notin A$, we obtain

$$\chi^*_A(x) \supseteq \varnothing = \chi^*_A(y), \quad \text{and} \quad \chi^+_A(x) \leq 1 = \chi^+_A(y).$$

Altogether, it is complete to prove that χ_A is a hybrid right hyperideal in $\mathbf{H}$ over U.

(2)⇒(1). Let χ_A be a hybrid right hyperideal in $\mathbf{H}$ over U. Firstly, let $x \in A$ and $y \in H$. We obtain $U \supseteq \bigcap_{a \in x \circ y} \chi^*_A(a) \supseteq \chi^*_A(x) = U$, which implies that $\bigcap_{a \in x \circ y} \chi^*_A(a) = U$ and since $U = \bigcap_{a \in x \circ y} \chi^*_A(x) \subseteq \chi^*_A(a) \subseteq U$, we obtain $\chi^*_A(a) = U$. Similarly, since $0 \leq \bigvee_{a \in x \circ y} \chi^+_A(a) \leq \chi^+_A(x) = 0$, which implies that $\bigvee_{a \in x \circ y} \chi^+_A(a) = 0$, and since $0 = \bigvee_{a \in x \circ y} \chi^+(a) \geq \chi^+_A(a) \geq 0$, we obtain that $\chi^+_A(a) = 0$. Altogether, we have $a \in A$ true.

Secondly, let $x, y \in H$ be such that $x \leq y$ and $y \in A$. We obtain $U \supseteq \chi^*_A(x) \supseteq \chi^*_A(y) = U$, which implies that $\chi^*_A(x) = U$. It means that $x \in A$. For χ^+_A need not to consider. Therefore A is a right hyperideal of $\mathbf{H}$. Similarly, we can show that A is a left hyperideal if and only if χ_A is a hybrid left hyperideal. □

As a consequence of the above lemma, we have that A is a hyperideal of $\mathbf{H}$ if and only if χ_A is a hybrid hyperideal in $\mathbf{H}$ over U.

Lemma 3.11. *Let $\mathbf{H}$ be an ordered hypersemigroup and Q a nonempty subset of H. Then the following conditions are equivalent:*

(1) *Q is a quasi-hyperideal of $\mathbf{H}$,*
(2) *$\chi_Q = (\chi^*_Q, \chi^+_Q)$ is a hybrid quasi-hyperideal in $\mathbf{H}$ over U.*

Proof. (1)⇒(2). Let Q be a quasi-hyperideal of an ordered hypersemigroup $\mathbf{H}$. By Lemma 3.9, we obtain

$$\begin{aligned}(\chi_Q \otimes \widetilde{H}) \Cap (\widetilde{H} \otimes \chi_Q) &= (\chi_Q \otimes \chi_H) \Cap (\chi_H \otimes \chi_Q)\\ &= \chi_{(Q*H]} \Cap \chi_{(H*Q]}\\ &= \chi_{(Q*H] \cap (H*Q)}\\ &\ll \chi_Q.\end{aligned}$$

Let $x, y \in H$ be such that $x \leq y$. If $y \in Q$, then since Q is a quasi-hyperideal of H we have $x \in Q$, and then

$$\chi^*_Q(x) = U = \chi^*_Q(y) \quad \text{and} \quad \chi^+_Q(x) = 0 = \chi^+_Q(y).$$

If $y \notin Q$, then we have

$$\chi_Q^*(x) \supseteq \varnothing = f_Q^*(y) \text{ and } \chi_Q^+(x) \leq 1 = \chi_Q^+(y).$$

Thus $\chi_Q = (\chi_Q^*, \chi_Q^+)$ is a hybrid quasi-hyperideal in $\mathbf{H}$ over U.

(2)⇒(1). Let $\chi_Q = (\chi_Q^*, \chi_Q^+)$ is a hybrid quasi-hyperideal in $\mathbf{H}$ over U. By Lemma 3.9, we get

$$\begin{aligned}
\chi_{(Q*H]\cap(H*Q]} &= \chi_{(Q*H]} \Cap \chi_{(H*Q]} \\
&= (\chi_Q \otimes \chi_H) \Cap (\chi_H \otimes \chi_Q) \\
&= (\chi_Q \otimes \widetilde{H}) \Cap (\widetilde{H} \otimes \chi_Q) \\
&\subseteq \chi_Q.
\end{aligned}$$

Then

$$U \supseteq \chi_Q^*(x) \supseteq \chi_Q^*(y) = U.$$

This implies that $\chi_Q^*(x) = U$. Thus $x \in Q$ and

$$0 \leq \chi_Q^+(x) \leq \chi_Q^+(y) = 0.$$

This implies that $\chi_Q^+(x) = 0$. So $x \in Q$ is true. Hence Q is a quasi-hyperideal of $\mathbf{H}$. □

Lemma 3.12. *Let $\mathbf{H}$ be an ordered hypersemigroup and A a hyperideal of $\mathbf{H}$. Then the following conditions are equivalent:*

(1) *A is a quasi-pure hyperideal of $\mathbf{H}$,*
(2) *$A \cap Q = (Q * A] \cap (A * Q]$ for every quasi-hyperideal Q of $\mathbf{H}$.*

Proof. (1)⇒(2). Suppose A is a quasi-pure hyperideal of $\mathbf{H}$ and let Q be a quasi-hyperideal of $\mathbf{H}$. If $x \in A \cap Q$, then $x \in (x * A] \cap (A * x] \subseteq (Q * A] \cap (A * Q]$. Thus $A \cap Q \subseteq (Q * A] \cap (A * Q]$. It is is clear that $(Q * A] \cap (A * Q] \subseteq (A] \cap (Q] = A \cap Q$. So $A \cap Q = (Q * A] \cap (A * Q]$.

(2)⇒(1). Suppose (2) holds andlLet $x \in A$. Then by the hypothesis, we have

$$\begin{aligned}
x \in A \cap Q(x) &= (Q(x) * A] \cap (A * Q(x)] \\
&= ((x \cup (x * H)] \cap (x \cup (H * x)]) * A] \cap \\
&\quad\ (A * ((x \cup (x * H)] \cap (x \cup (H * x)]] \\
&\subseteq ((x \cup (x * H)] * A] \cap (A * (x \cup (H * x)]]] \\
&\subseteq ((x * A) \cup ((x * H) * A)] \cap ((A * x) \cup (A * (H * x))] \\
&= ((x * A) \cup (x * (H * A))] \cap (A * x) \cup ((A * H) * x)] \\
&\subseteq ((x * A) \cup (x * A)] \cap ((A * x) \cup (A * x)] \\
&= (x * A] \cap (A * x].
\end{aligned}$$

Thus A is a quasi-pure hyperideal of $\mathbf{H}$. □

The following main theorem provides a characterization of quasi-pure hyperideals in ordered hypersemigroups using hybrid quasi-pure hyperideals.

Theorem 3.13. *Let A be a hyperideal of an ordered hypersemigroup $\mathbf{H}$. Then the following conditions are equivalent:*

(1) *A is a quasi-pure hyperideal of* $\mathbf{H}$,
(2) $\chi_A = (\chi^*_A, \chi^+_A)$ *is a hybrid quasi-pure hyperideal in* $\mathbf{H}$ *over* U.

Proof. (1)⇒(2). Suppose A is a quasi-pure hyperideal of $\mathbf{H}$. Since A is a hyperideal of $\mathbf{H}$, by Lemma 3.10, $\chi_A = (\chi^*_A, \chi^+_A)$ is a hybrid hyperideal in $\mathbf{H}$ over U. Let f be a hybrid quasi-hyperideal in $\mathbf{H}$ over U and $a \in H$. Suppose that $a \notin A$, we consider two cases as follows: If $\mathbf{H}_a = \varnothing$, then we have

$$\begin{aligned}
[(f^* \odot \chi^*_A) \cap (\chi^* \odot f^*)](a) &= (f^* \odot \chi^*_A)(a) \cap (\chi^* \odot f^*)(a) \\
&= \varnothing \\
&= \chi^*_A(a) \\
&= f^*(a) \cap \chi^*_A(a) \\
&= (f^* \cap \chi^*_A)(a)
\end{aligned}$$

and

$$\begin{aligned}
[(f^+ \oplus \chi^+_A) \vee (f^+ \oplus \chi^+_A)](a) &= \max\{(f^+ \oplus \chi^+_A)(a), (f^+ \oplus \chi^+_A)(a)\} \\
&= 1 \\
&= \chi^+_A(a) \\
&= \max\{f^+(a), \chi^+_A(a)\} \\
&= (f^+ \vee \chi^+_A)(a).
\end{aligned}$$

If $\mathbf{H}_a \neq \varnothing$, then, by a quasi-purity of A, $u,\ v \notin A$ for all $(u, v) \in \mathbf{H}_a$. Thus

$$\begin{aligned}
(f^* \odot \chi^*_A)(a) &= \bigcup_{(x,y)\in\mathbf{H}_a} [f^*(x) \cap \chi^*_A(y)] \\
&= \varnothing \\
&= f^*(a) \cap \chi^*_A(a) \\
&= (f^* \cap \chi^*_A)(a).
\end{aligned}$$

Similarly, we have $(\chi^*_A \odot f^*)(a) = (f^* \cap \chi^*_A)(a)$. It follows that

$$\begin{aligned}
[(f^* \odot \chi^*_A) \cap (\chi^*_A \odot f^*)](a) &= (f^* \odot \chi^*_A)(a) \cap (\chi^*_A \odot f^*)(a) \\
&= (f^* \cap \chi^*_A)(a) \cap (f^* \cap \chi^*_A)(a) \\
&= (f^* \cap \chi^*_A)(a).
\end{aligned}$$

So we get

$$\begin{aligned}
(f^+ \oplus \chi^+_A)(a) &= \bigwedge_{(x,y)\in\mathbf{H}_a} \{\max\{f^+(x), \chi^+_A(y)\}\} \\
&= 1 \\
&= \max\{f^+(a), \chi^+_A(a)\} \\
&= (f^+ \vee \chi^+_A)(a).
\end{aligned}$$

Similarly, we get $(\chi^+_A \oplus f^+)(a) = (f^+ \vee \chi^+_A)(a)$. It follows that

$$\begin{aligned}
[(f^+ \oplus \chi^+_A) \vee (\chi^+_A \oplus f^+](a) &= \max\{(f^+ \oplus \chi^+_A)(a), (\chi^+_A \oplus f^+)(a)\} \\
&= \max\{(f^+ \vee \chi^+_A)(a), (f^+ \vee \chi^+_A)(a)\} \\
&= (f^+ \vee \chi^+_A)(a).
\end{aligned}$$

Now, we assume that $a \in A$. By the quasi-purity of A, we have $\mathbf{H}_a \neq \varnothing$. More precisely, there exists $(a, x) \in \mathbf{H}_a$ such that $x \in A$. Then, by the hybrid quasi-hyperideality of f and χ_A, we have

$$\begin{aligned}
(f^* \cap \chi_A^*)(a) &= f^*(a) \cap \chi_A^*(a) \\
&= f^*(a) \\
&= f^*(a) \cap \chi_A^*(x) \\
&\subseteq \bigcup_{(u,v)\in \mathbf{H_a}} [f^*(u) \cap \chi_A^*(v)] \\
&\subseteq \bigcup_{(u,v)\in \mathbf{H}_a} [f^*(uv) \cap \chi_A^*(uv)] \\
&\subseteq \bigcup_{(u,v)\in \mathbf{H}_a} [f^*(a) \cap \chi_A^*(a)] \\
&= f^*(a) \cap \chi_A^*(a) \\
&= (f^* \cap \chi_A^*)(a).
\end{aligned}$$

This implies that

$$(f^* \cap \chi_A^*)(a) = \bigcup_{(u,v)\in \mathbf{H_a}} [f^*(u) \cap \chi_A^*(v)] = (f^* \odot \chi_A^*)(a).$$

Similarly, we have

$$(\chi_A^* \cap f^*)(a) = \bigcup_{(u,v)\in \mathbf{H_a}} [\chi_A^*(v) \cap f^*(u)] = (\chi_A^* \odot f^*)(a).$$

Thus we get

$$(\chi_A^* \cap f^*)(a) = (\chi_A^* \odot f^*)(a) \cap (f^* \odot \chi_A^*)(a) = [(\chi_A^* \odot f^*) \cap (f^* \odot \chi_A^*)](a)$$

and

$$\begin{aligned}
(f^+ \vee \chi_A^+)(a) &= \max\{f^+(a), \chi_A^+(a)\} \\
&= f^+(a) \\
&= \max\{f^+(a), \chi_A^+(x)\} \\
&\geq \bigwedge_{(u,v)\in \mathbf{H}_a} \{\max\{f^+(u), \chi_A^+(v)\}\} \\
&\geq \bigwedge_{(u,v)\in \mathbf{H}_a} \{\max\{f^+(uv), \chi_A^+(uv)\}\} \\
&\geq \bigwedge_{(u,v)\in \mathbf{H}_a} \{\max\{f^+(a), \chi_A^+(a)\}\} \\
&= \max\{f^+(a), \chi_A^+(a)\} \\
&= (f^+ \vee \chi_A^+)(a).
\end{aligned}$$

This implies that

$$(f^+ \vee \chi_A^+)(a) = \bigwedge_{(u,v)\in \mathbf{H}_a} \{\max\{f^+(u), \chi_A^+(v)\}\} = (f^+ \oplus \chi_A^+)(a).$$

Similarly, we get

$$(\chi_A^+ \vee f^+)(a) = \bigwedge_{(u,v)\in \mathbf{H}_a} \{\max\{\chi_A^+(u), f^+(v)\}\} = (\chi_A^+ \oplus f^+)(a).$$

So

$$(\chi_A^+ \vee f^+)(a) = \max\{(\chi_A^+ \oplus f^+)(a), (f^+ \oplus \chi_A^+)(a)\} = [(\chi_A^+ \oplus f^+) \vee (f^+ \oplus \chi_A^+)](a).$$

Altogether, we have $\chi_A \Cap f = (\chi_A \otimes f) \Cap (f \otimes \chi_A)$. Hence χ_A is a hybrid quasi-pure hyperideal in $\mathbf{H}$ over U.

(2)⇒(1). Suppose $\chi_A = (\chi_A^*, \chi_A^+)$ is a hybrid quasi-pure hyperideal in $\mathbf{H}$ over U. Let B be a quasi-hyperideal of $\mathbf{H}$. By Lemma 3.11, χ_B is a hybrid quasi-hyperideal in $\mathbf{H}$ over U. By the hypothesis, we obtain

$$\begin{aligned} \chi_{B\cap A} &= \chi_B \Cap \chi_A \\ &= (\chi_B \otimes \chi_A) \Cap (\chi_A \otimes \chi_B) \\ &= \chi_{(B*A]} \Cap \chi_{(A*B]} \\ &= \chi_{(B*A]\cap(A*B]}. \end{aligned}$$

By Lemma 3.9, we have $B \cap A = (B * A] \cap (A * B]$ and by Lemma 3.12, we obtain A is a quasi-pure hyperideal in $\mathbf{H}$ over U. □

4. Conclusion

In this paper, the concepts of hybrid quasi-pure hyperideals in ordered hypersemigroups were introduced. We characterized quasi-pure hyperideals in an ordered hypersemigroup in terms of hybrid quasi-pure hyperideals. In our future work, we will apply the notions of hybrid quasi-pure hyperideals to the theory of hypersemirings, hypergroups, etc.

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