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Abstract. We introduce the concept of ordinary interval-valued fuzzifying topology and obtain some of its basic properties. We show that a neighborhood system in ordinary interval-valued fuzzifying topological spaces has the same properties in a classical neighborhood system. Also, we obtain two characterization of an ordinary interval-valued fuzzifying base and one characterization of an ordinary interval-valued fuzzifying subbase. We define an ordinary interval-valued fuzzifying closure and prove that an ordinary interval-valued fuzzifying topology induced by an ordinary interval-valued fuzzifying closure operator.

2020 AMS Classification: 54A40

Keywords: Ordinary interval-valued fuzzifying (co)topology, Ordinary interval-valued fuzzifying subspace, $\widetilde{a}$-level, Ordinary interval-valued fuzzifying neighborhood system, Ordinary interval-valued fuzzifying base, Ordinary interval-valued fuzzifying subbase, Ordinary interval-valued fuzzifying closure operator.

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1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy sets as the generalization of an ordinary set. In 1986, Chang [2] was the first to introduce the notion of a fuzzy topology by using fuzzy sets. After that, many researchers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] have investigated several properties in fuzzy topological spaces. In particular, Kandil et al [13], Saleh [14, 15], Samanta and Mondal [16] has applied the concept of interval-valued fuzzy set (See [17, 18]) to topology.

However, in their definition of fuzzy topology, fuzziness in the notion of openness of a fuzzy set was absent. In 1992, Samanta et al. [19, 20] introduced the concept of gradation of openness(closedness) of fuzzy sets in X in two different ways, and gave definitions of a fuzzy topology on X. After then, some works have been done by Ramadan [21], Demirci [22], Chattopadhyay and Samanta [23] and Peters [24, 25].

Moreover, Çoker and Demirci [26], and Samanta and Mondal [27, 28] defined intuitionistic gradation of openness (in short IGO) of fuzzy sets in Šostak′s sense [29] by using intuitionistic fuzzy sets introduced by Atanassov [30]. They mainly dealt with intuitionistic gradation of openness of fuzzy sets in the sense of Chang. Lim et al. [31] investigated intuitionistic smooth topological spaces in Lowen′s sense. Kim et al. [32] studied continuities and neighborhood systems in intuitionistic smooth topological spaces. Also Choi et al. [33] studied an interval-valued smooth topology by gradation of openness of interval-valued fuzzy sets introduced by Zadeh [17]. In particular, Ying [34] introduced the concept of the topology (called a fuzzifying topology) considering the degree of openness of an ordinary subset of a set. In 2012, Lim et al. [35] studied some properties in ordinary smooth topological spaces (See [36, 37, 38] for the further topological structures in ordinary smooth topological spaces).

Now we would like to study the topological structures given by the interval number as the degree of openness of an ordinary subset of a set. To do this, we intend to conduct research as follows: We introduce the concepts of ordinary interval-valued fuzzifying topological spaces and subspaces, and study some of their properties. Second, we define an ordinary interval-valued neighborhood system and we show that it has the same properties in a classical neighborhood system. Third, we introduce the notions of ordinary interval-valued fuzzifying bases and subbases, and obtain two characterization of an ordinary interval-valued fuzzifying base and one characterization of an ordinary interval-valued fuzzifying subbase. Finally, we define an ordinary interval-valued fuzzifying closure and prove that an ordinary interval-valued fuzzifying topology induced by an ordinary interval-valued fuzzifying closure operator.

2. Preliminaries

In this section, we list some notations, two definitions and one result needed in the next sections (See [17]). Throughout this paper, I denotes the closed unit interval $[0, 1]$.

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$.

We define an order and $=$ on $[I]$ as follows:

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} \leq \widetilde{b} \Longleftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+),$$
$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow \widetilde{a} \leq \widetilde{b} \text{ and } \widetilde{b} \leq \widetilde{b},\ \text{i.e., } a^- = b^- \text{ and } a^+ = b^+).$$

To say $\widetilde{a} < \widetilde{b}$, we mean $\widetilde{a} \leq \widetilde{b}$ and $\widetilde{a} \neq \widetilde{b}$.

For any $\widetilde{a}$, $\widetilde{b} \in [I]$, their minimum and maximum, denoted by $\widetilde{a} \wedge \widetilde{b}$ and $\widetilde{a} \vee \widetilde{b}$, are defined as follows:

$$\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+],$$
$$\widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+].$$

Let $(\widetilde{a_j})_{j\in J} \subset [I]$. Then its inf and sup, denoted by $\bigwedge_{j\in J}\widetilde{a_j}$ and $\bigvee_{j\in J}\widetilde{a_j}$, are defined as follows:

$$\bigwedge_{j\in J}\widetilde{a_j} = [\bigwedge_{j\in J} a_j^-, \bigwedge_{j\in J} a_j^+],$$
$$\bigvee_{j\in J}\widetilde{a_j} = [\bigvee_{j\in J} a_j^-, \bigvee_{j\in J} a_j^+].$$

For each $\widetilde{a} \in [I]$, its *complement*, denoted by $\widetilde{a}^c$, is defined as follows:

$$\widetilde{a}^c = [1-a^+, 1-a^-].$$

Definition 2.1 ([17])**.** Let X be a nonempty set. Then a mapping $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an IVFS) in X. Let $[I]^X$ denote the set of all IVFSs in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree of membership* of an element x to A, where $A^-, A^+ \in I^X$ are called a *lower fuzzy set* and an *upper fuzzy set* in X respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in X, respectively. We define relations $\subset$ and $=$ on $[I]^X$ as follows:

$$(\forall\ A,\ B \in [I]^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \leq B(x)),$$
$$(\forall\ A,\ B \in [I]^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

Definition 2.2 ([17])**.** Let X be a nonempty set, let $A \in [I]^X$ and let $(A_j)_{j\in J}$ be any subfamily of $[I]^X$. Then the *complement* of A, denoted by A^c, and the *intersection* and the *union* of $(A_j)_{j\in J}$, denoted by $\bigcap_{j\in J} A_j$ and $\bigcup_{j\in J} A_j$, are defined as follows respectively: for each $x \in X$,

$$A^c(x) = [1 - A^+(x), 1 - A^-(x)],$$
$$(\bigcap_{j\in J} A_j)(x) = \bigwedge_{j\in J} A_j(x),$$
$$(\bigcup_{j\in J} A_j)(x) = \bigvee_{j\in J} A_j(x).$$

Definition 2.3 ([16])**.** $A \in [I]^X$ is called an *interval-valued fuzzy point* (briefly, an IVFP) with the support $x \in X$ and the value $\widetilde{a} \in [I]$ with $a^+ > 0$, denoted by $A = x_{\widetilde{a}}$, if for each $y \in X$,

$$x_{\widetilde{a}}(y) = \begin{cases} \widetilde{a} & \text{if } y = x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The set of all IVFPs in X is denoted by $IVF_P(X)$.

For each $x_{\widetilde{a}} \in IVF_P(X)$ and $A \in [I]^X$, we say that $x_{\widetilde{a}}$ *belong to* A, denoted by $x_{\widetilde{a}} \in A$, if $\widetilde{a} \leq A(x)$. It is clear that $A = \bigcup_{x_{\widetilde{a}}\in A} x_{\widetilde{a}}$, for each $A \in [I]^X$.

Result 2.4 (Theorem 1, [16])**.** *Let X be a set, let $A,\ B,\ C \in [I]^X$ and $(A_j)_{j\in J} \subset [I]^X$. Then the followings hold:*

(1) $\widetilde{0} \subset A \subset \widetilde{1}$,

(2) $A \cup B = B \cup A$; $A \cap B = B \cap A$,

(3) $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$,

(4) $A,\ B \subset A \cup B$; $A \cap B \subset A,\ B$,
(5) $A \cap (\bigcup_{j\in J} A_j) = \bigcup_{j\in J}(A \cap A_j)$; $A \cup (\bigcap_{j\in J} A_j) = \bigcap_{j\in J}(A \cup A_j)$,
(6) $(\widetilde{0})^c = \widetilde{1}$; $(\widetilde{1})^c = \widetilde{0}$,
(7) $((A)^c)^c = A$,
(8) $(\bigcup_{j\in J} A_j)^c = \bigcap_{j\in J} A_j^c$; $(\bigcap_{j\in J} A_j)^c = \bigcup_{j\in J} A_j^c$.

We display the interval-valued fuzzy logical and corresponding set-theoretical notations used in this paper.
(1) $[\neg\alpha] := \mathbf{1} - [\alpha]$,
$[\alpha \to \beta] := \mathbf{1} \wedge (\mathbf{1} - [\alpha] + [\beta]) = \mathbf{1} \wedge [1 - \alpha^+ + \beta^-, 1 - \alpha^- + \beta^+]$,
$[\forall x\ \alpha(x)] := \bigwedge_{x\in X}[\alpha(x)]$, $[\exists x\ \alpha(x)] := \bigvee_{x\in X}[\alpha(x)]$,
where X is the universe of discourse.
(2) Let $A,\ B \in [I]^X$ and let $x \in X$. Then
$[x \in A] := A(x)$, $A \subset B := \forall\ x(x \in A \to x \in B)$,
$A \equiv B := A \subset B \wedge B \subset A$.
It can be easily see that $[A \equiv B] = \bigwedge_{x\in X}(\mathbf{1}- \mid A(x) - B(x) \mid)$.

3. Ordinary interval-valued topology

In this section, we define an ordinary interval-valued fuzzifying topological space and obtain some its properties. Throughout this paper, we denote the set of all subsets of a set X as 2^X. For any $A \in 2^X$, we can consider A as the interval-valued fuzzy set in X given by $[\chi_A, \chi_A]$, where χ_A denotes the characteristic function of A (See [39]).

Definition 3.1. Let X be a nonempty set. Then a mapping $\tau = [\tau^-, \tau^+] : 2^X \to [I]$ is called an *ordinary interval-valued fuzzifying topology* (in short, OIVFT) on X, if it satisfies the following axioms: for any $A,\ B \in 2^X$ and each $(A_j)_{j\in J} \subset 2^X$,
(OIVFT1) $\tau(\phi) = \tau(X) = \mathbf{1}$,
(OIVFT2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$,
(OIVFT3) $\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$.
The pair (X, τ) is called an *ordinary interval-valued fuzzy fuzzifying topological space* (in short, OIVFTS).
We will denote the set of all ordinary interval-valued fuzzifying topologies on X as $OIVFT(X)$.

We can easily see that for an OIVFTS (X, τ), (X, τ^-, τ^+) is an ordinary smooth bitopological space such that $\tau^- \subset \tau^+$ (See [35]).

Let $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$. Then we can consider $\mathbf{2}$ as the ordinary two point set $2 = \{0, 1\}$ such that $\mathbf{0} = 0$ and $\mathbf{1} = 1$. Thus $\tau : 2^X \to \mathbf{2}$ satisfy the axioms in Definition 3.1. So $\tau \in T(X)$, where $T(X)$ denotes the set of all classical topologies on X. So we can see that $T(X) \subset OIVFT(X)$.

Example 3.2. (1) Let $X = \{a, b, c\}$. We define the mapping $\tau : 2^X \to [I]$ as follows:
$\tau(\phi) = \tau(X) = \mathbf{1}$,
$\tau(\{a\}) = [0.3, 0.8], \tau(\{b\}) = [0.4, 0.7], \tau(\{c\}) = [0.3, 0.6]$,
$\tau(\{a, b\}) = [0.3, 0.7], \tau(\{b, c\}) = [0.4, 0.6], \tau(\{a, c\}) = [0.3, 0.8]$.
Then we can easily see that $\tau \in OIVFT(X)$.

(2) Let X be a nonempty set. We define the mapping $\tau_\phi : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\tau_\phi(A) = \begin{cases} \mathbf{1} & \text{if either } A = \phi \text{ or } A = X, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then clearly, $\tau_\phi \in OIVT(X)$.

In this case, τ_ϕ [resp. (X, τ_ϕ)] will be called the *ordinary interval-valued fuzzifying indiscrete topology* on X [resp. the *ordinary interval-valued fuzzifying indiscrete space*].

(3) Let X be a nonempty set. We define the mapping $\tau_{_X} : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\tau_{_X}(A) = \mathbf{1}.$$

Then clearly, $\tau_{_X} \in OIVFT(X)$.

In this case, $\tau_{_X}$ [resp. $(X, \tau_{_X})$] will be called the *ordinary interval-valued fuzzifying discrete topology* on X [resp. the *ordinary interval-valued fuzzifying discrete space*].

(4) Let X be an infinite set and let $\widetilde{a} \in [I] \setminus \{\mathbf{0}, \mathbf{1}\}$ be fixed. We define the mapping $\tau_{_{\widetilde{a}}} : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\tau_{_{\widetilde{a}}}(A) = \begin{cases} \mathbf{1} & \text{if either } A = \phi \text{ or } A^c \text{ is finite,} \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau_{_{\widetilde{a}}} \in OIVFT(X)$.

In this case, $\tau_{_{\widetilde{a}}}$ will be called the *$\widetilde{a}$-ordinary interval-valued fuzzifying finite complement topology* on X. $\tau_{_{\widetilde{a}}}$ is of interest only when X is a finite set, because if X is infinite, then $\tau_{_{\widetilde{a}}} = \tau_{_X}$.

(5) Let X be an infinite set and let $\widetilde{a} \in [I] \setminus \{\mathbf{0}, \mathbf{1}\}$ be fixed. We define he mapping $\tau_{_{c,\widetilde{a}}} : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\tau_{_{c,\widetilde{a}}}(A) = \begin{cases} \mathbf{1} & \text{if either } A = \phi \text{ or } A^c \text{ is countable,} \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then clearly, $\tau_{_{c,\widetilde{a}}} \in OIVFT(X)$.

In this case, $\tau_{_{c,\widetilde{a}}}$ will be called the *$\widetilde{a}$-ordinary interval-valued fuzzifying countable complement topology* on X.

(6) Let T be the topology generated by $\mathcal{S} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$ as a subbase, let T_0 be the family of all open sets of $\mathbb{R}$ w.r.t. the usual topology of $\mathbb{R}$ and let $\widetilde{a} \in [I] \setminus \{\mathbf{0}, \mathbf{1}\}$ be fixed. We define the mapping $\tau^{(\mathbb{R},\widetilde{a})} : 2^{\mathbb{R}} \to [I]$ as follows: for each $A \in I^{\mathbb{R}}$,

$$\tau^{(\mathbb{R},\widetilde{a})}(A) = \begin{cases} \mathbf{1} & \text{if } A \in T_0, \\ \widetilde{a} & \text{if } A \in T \setminus T_0, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau^{(\mathbb{R},\widetilde{a})} \in OIVFT(X)$.

(7) Let $T \in T(X)$. We define the mapping $\tau_{_T} : 2^X \to [I]$ as follows : for each $A \in 2^X$,

$$\tau_{_T}(A) = \begin{cases} \mathbf{1} & \text{if } A \in T, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then it is easily seen that $\tau_T \in OIVFT(X)$. Moreover, we can see that if T is the classical indiscrete topology, then $\tau_T = \tau_\phi$ and if T is the classical discrete topology, then $\tau_T = \tau_X$.

Definition 3.3. Let X be a nonempty set. Then a mapping $\mathcal{C} = (\mu_{\mathcal{C}}, \nu_{\mathcal{C}}) : 2^X \to [I]$ is called an *ordinary interval-valued fuzzifying cotopology* (in short, OIVCT) on X, if it is satisfies the following conditions: for any $A, B \in 2^X$ and each $\{A_j\}_{j\in J} \subset 2^X$,

(OIVCT1) $\mathcal{C}(\phi) = \mathcal{C}(X) = \mathbf{1}$,
(OIVCT2) $\mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B)$,
(OIVCT3) $\mathcal{C}(\bigcap_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathcal{C}(A_j)$.

The pair $(X, \mathcal{C})$ is called an *ordinary interval-valued fuzzifying cotopological space* (in short, OIVFCTS). The set of all OIVFCTs in X is denoted by $OIVFCT(X)$.

The following is the immediate result of Definitions 3.1 and 3.3.

Proposition 3.4. *We define two mappings $f : OIVFT(X) \to OIVFCT(X)$ and $g : OIVFCT(X) \to OIVFT(X)$ as follows, respectively:*

$$[f(\tau)](A) = \tau(A^c), \ \ \forall\ \tau \in OIVFT(X), \ \ \forall\ A \in 2^X$$

and

$$[g(\mathcal{C})](A) = \mathcal{C}(A^c), \ \ \forall\ \mathcal{C} \in OIVFCT(X), \ \ \forall\ A \in 2^X.$$

Then f and g are well-defined. Moreover, $g\circ f = id_{OIVFT(X)}$ and $f\circ g = id_{OIVFCT(X)}$.

Remark 3.5. For each $\tau \in OIVFT(X)$ and each $\mathcal{C} \in OIVFCT(X)$, let $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_{\mathcal{C}}$. Then, from Proposition 3.4, we can see that $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_{\mathcal{C}}} = \mathcal{C}$.

Definition 3.6. Let $\tau_1,\ \tau_2 \in OIVFT(X)$ and let $\mathcal{C}_1,\ \mathcal{C}_2 \in OIVFCT(X)$. Then

(i) we say that τ_1 *is finer than* τ_2 or τ_2 *is coarser than* τ_1, denoted by $\tau_2 \leq \tau_1$, if $\tau_2(A) \leq \tau_1(A)$ for each $A \in 2^X$,

(ii) we say that $\mathcal{C}_1$ *is finer than* $\mathcal{C}_2$ or $\mathcal{C}_2$ *is coarser than* $\mathcal{C}_1$, denoted by $\mathcal{C}_2 \leq \mathcal{C}_1$, if $\mathcal{C}_2(A) \leq \mathcal{C}_1(A)$ for each $A \in 2^X$.

We can easily see that τ_1 is finer than τ_2 if and only if $\mathcal{C}_{\tau_1}$ is finer than $\mathcal{C}_{\tau_2}$, and $(OIVFT(X), \leq)$ and $(OIVFCT(X), \leq)$ are posets, respectively.

From Example 3.2 (2) and (3), it is obvious that τ_ϕ is the coarsest ordinary interval-valued topology on X and τ_X is the finest ordinary interval-valued topology on X.

Proposition 3.7. *If $(\tau_j)_{j\in J} \subset OIVFT(X)$, then $\bigcap_{j\in J} \tau_j \in OIVFT(X)$, where $[\bigcap_{j\in J} \tau_j](A) = \bigwedge_{j\in J} \tau_j(A)\ \forall\ A \in 2^X$.*

Proof. From Definitions 2.2 and 3.1, it is obvious. □

From Definition 3.6 and Proposition 3.7, we have the following.

Proposition 3.8. *$(OIVFT(X), \leq)$ is a meet complete lattice with the least element τ_ϕ and the greatest element τ_X.*

Definition 3.9. Let (X, τ) be an $OIVFT$s and let $\widetilde{a} \in [I]$. We define two families $[\tau]_{\widetilde{a}}$ and $[\tau]^*_{\widetilde{a}}$ as follows, respectively:

(i) $[\tau]_{\widetilde{a}} = \{A \in 2^X : \tau(A) \geq \widetilde{a}\}$,
(ii) $[\tau]^*_{\widetilde{a}} = \{A \in 2^X : \tau(A) > \widetilde{a}\}$.

In this case, $[\tau]_{\widetilde{a}}$ [resp. $[\tau]^*_{\widetilde{a}}$] is called the $\widetilde{a}$*-level* [resp. *strong* $\widetilde{a}$*-level*] *set* of τ.

We can easily see that $[\tau]_{\mathbf{0}} = 2^X$ is the classical discrete topology on X and $[\tau]^*_{\mathbf{1}} = \phi$. Moreover, it is obvious that for any $\widetilde{a} \in [I]$, $[\tau]^*_{\widetilde{a}} \subset [\tau]_{\widetilde{a}}$.

Lemma 3.10. *Let* $\tau \in OIVFT(X)$ *and let* $\widetilde{a},\ \widetilde{b} \in [I]$. *Then*

(1) $[\tau]_{\widetilde{a}} \in T(X)$,

(2) *if* $\widetilde{a} \leq \widetilde{b}$, *then* $[\tau]_{\widetilde{b}} \subset [\tau]_{\widetilde{a}}$,

(3) $[\tau]_{\widetilde{a}} = \bigcap_{\widetilde{b}<\widetilde{a}} [\tau]_{\widetilde{b}}$, *where* $\widetilde{a} \in [I] \setminus \{\mathbf{0}\}$,

$(1)'$ $[\tau]^*_{\widetilde{a}} \in T(X)$, *where* $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$,

$(2)'$ *if* $\widetilde{a} \leq \widetilde{b}$, *then* $[\tau]^*_{\widetilde{b}} \subset [\tau]^*_{\widetilde{a}}$,

$(3)'$ $[\tau]^*_{\widetilde{a}} = \bigcup_{\widetilde{b}>\widetilde{a}} [\tau]^*_{\widetilde{b}}$, *where* $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$.

Proof. The proofs of (1), $(1)'$, (2) and $(2)'$ are obvious from Definitions 3.1 and 3.9.

(3) From (2), it is obvious that $([\tau]_{\widetilde{a}})_{\widetilde{a}\in[I]\setminus\{\mathbf{0}\}}$ is a descending family of classical topologies on X. Then clearly, $[\tau]_{\widetilde{a}} \subset \bigcap_{\widetilde{b}<\widetilde{a}} [\tau]_{\widetilde{b}}$ for each $\widetilde{a} \in [I] \setminus \{\mathbf{0}\}$.

Suppose $A \notin [\tau]_{\widetilde{a}}$. Then $\tau^-(A) < a^-$ or $\tau^+(A) < a^+$. Thus

$$\exists b^-,\ b^+ \in I \setminus \{0\} \text{ such that } \tau^-(A) < b^- < a^- \text{ or } \tau^+(A) < b^+ < a^+.$$

So, in either cases, $A \notin [\tau]_{\widetilde{b}}$ for some $\widetilde{b} \in [I] \setminus \{\mathbf{0}\}$ such that $\widetilde{b} < \widetilde{a}$, i.e., $A \notin \bigcap_{\widetilde{b}<\widetilde{a}} [\tau]_{\widetilde{b}}$. Hence $\bigcap_{\widetilde{b}<\widetilde{a}} [\tau]_{\widetilde{b}} \subset [\tau]_{\widetilde{a}}$. Therefore $[\tau]_{\widetilde{a}} = \bigcap_{\widetilde{b}<\widetilde{a}} [\tau]_{\widetilde{b}}$.

$(3)'$ The proof is similar to (3). □

Remark 3.11. From (1) and (2) in Lemma 3.10, we can see that for each $\tau \in OIVT(X)$, $([\tau]_{\widetilde{a}})_{\widetilde{a}\in[I]}$ is a family of descending classical topologies (will be called the $\widetilde{a}$*-level classical topologies* on X w.r.t. τ).

Lemma 3.12. (1) *Let* $(\tau_{\widetilde{a}})_{\widetilde{a}\in[I]}$ *be a descending family of classical topologies on* X *such that* $\tau_{\mathbf{0}}$ *is the classical discrete topology on* X. *We define the mapping* $\tau : 2^X \to [I]$ *as follows: for each* $A \in 2^X$,

$$\tau(A) = \bigvee_{A\in\tau_{\widetilde{a}}} \widetilde{a}.$$

Then $\tau \in OIVFT(X)$.

(2) *If* $\tau_{\widetilde{a}} = \bigcap_{\widetilde{b}<\widetilde{a}} \tau_{\widetilde{b}}$ *for each* $\widetilde{a} \in [I] \setminus \{\mathbf{0}\}$, *then* $[\tau]_{\widetilde{a}} = \tau_{\widetilde{a}}$.

(3) *If* $\tau_{\widetilde{a}} = \bigcup_{\widetilde{b}>\widetilde{a}} \tau_{\widetilde{b}}$ *for each* $\widetilde{a} \in I] \setminus \{\mathbf{1}\}$, *then* $[\tau]^*_{\widetilde{a}} = \tau_{\widetilde{a}}$.

Proof. (1) It is obvious that $\varnothing,\ X \in \tau_{\widetilde{a}}$ for each $\widetilde{a} \in [I]$. Then by the definition of τ, $\tau(\varnothing) = \tau(X) = \mathbf{1}$. Thus the condition (OIVFT1) holds.

Suppose $A,\ B \in 2^X$ such that $\tau(A) = \widetilde{a}$ and $\tau(B) = \widetilde{b}$. If $\widetilde{a} = \mathbf{0}$ or $\widetilde{b} = \mathbf{0}$, then $\tau^-(A \cap B) \geq 0 \geq \tau^-(A) \wedge \tau^-(B)$, $\tau^+(A \cap B) \geq 0 \geq \tau^+(A) \wedge \tau^+(B)$. Thus $\tau(A\cap B) \geq \tau(A)\wedge\tau(A)\wedge\tau(B)$. So without loss of generality, we assume that $\widetilde{a} > \mathbf{0}$,

$\widetilde{b} > \mathbf{0}$, i.e., $\widetilde{a},\ \widetilde{b} \in [I] \setminus \{\mathbf{0}\}$ and let $\varepsilon > 0$. Then by the definition of τ, there are $\widetilde{c_1},\ \widetilde{c_2} \in [I] \setminus \{\mathbf{0}\}$ such that

$$a^- - \varepsilon < c_1^- \leq a^-,\ a^+ - \varepsilon < c_1^+ \leq a^+,\ b^- - \varepsilon < c_2^- \leq b^-,\ b^+ - \varepsilon < c_2^+ \leq b^+$$

and $A \in \tau_{\widetilde{c_1}}$, $B \in \tau_{\widetilde{c_2}}$. Let $c^- = c_1^- \wedge c_2^-$, $c^+ = c_1^+ \wedge c_2^+$ and $d^- = a^- \wedge b^-$, $d^+ = a^+ \wedge b^+$. Then clearly, $\widetilde{c} \in [I] \setminus \{\mathbf{0}\}$ such that $\widetilde{c} \leq \widetilde{a}$ and $\widetilde{c} \leq \widetilde{b}$. Since $(\tau_{\widetilde{a}})_{\widetilde{a} \in [I]}$ is a descending family of classical topologies on X, $\tau_{\widetilde{a}} \subset \tau_{\widetilde{c}}$ and $\tau_{\widetilde{b}} \subset \tau_{\widetilde{c}}$. Since $A \in \tau_{\widetilde{a}}$ and $B \in \tau_{\widetilde{b}}$, $A,\ B \in \tau_{\widetilde{c}}$. Thus $A \cap B \in \tau_{\widetilde{c}}$. So we have

$$\tau^-(A \cap B) \geq c^- > d^- - \varepsilon,\ \tau^+(A \cap B) \geq d^+ > a^+ - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\tau^-(A \cap B) \geq c^- = d^- = a^- \wedge b^-,\ \tau^+(A \cap B) \geq d^+ = a^+ \wedge b^+.$$

Hence $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$. Therefore, in either cases, the condition (OIVFT2) holds.

Finally, let $(A_j)_{j \in J} \subset 2^X$, let $\tau(A_j) = \widetilde{a_j}$ for each $j \in J$ and let $\widetilde{a} = \bigwedge_{j \in J} \widetilde{a_j}$. If $\widetilde{a} = \mathbf{0}$, then $\tau^-(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau^-(A_j)$ and $\tau^+(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau^+(A_j)$. Thus $\tau(\bigcup_{j \in J} A_j) \geq \wedge_{j \in J} \tau(A_j)$. Suppose $\widetilde{a} > \mathbf{0}$ and let $\varepsilon > 0$ such that $a^- > \varepsilon$. Then clearly, $0 < a^- - \varepsilon < a_j^-$ and $0 < a^+ - \varepsilon < a_j^+$ for each $j \in J$. Thus $A_j \in \tau_{[a^- - \varepsilon, a^+ - \varepsilon]}$ for each $j \in J$. Since $\tau_{[a^- - \varepsilon, a^+ - \varepsilon]}$ is a topology on X, $\bigcup_{j \in J} A_j \in \tau_{[a^- - \varepsilon, a^+ - \varepsilon]}$. By the definition of τ, we get

$$\tau^-(\bigcup_{j \in J} A_j) \geq a^- - \varepsilon,\ \tau^+(\bigcup_{j \in J} A_j) \geq a^+ - \varepsilon.$$

Since ε is arbitrary, we have

$$\tau^-(\bigcup_{j \in J} A_j) \geq a^- = \bigwedge_{j \in J} \tau^-(A_j),\ \tau^+(\bigcup_{j \in J} A_j) \geq a^+ = \bigwedge_{j \in J} \tau^+(A_j).$$

So $\tau(\bigcup_{j \in J} A_j) \geq \bigwedge_{j \in J} \tau(A_j)$. Hence, in either cases, the condition (OIVFT3) holds. Therefore $\tau \in OIVFT(X)$.

(2) Suppose $\tau_{\widetilde{a}} = \bigcap_{\widetilde{b} < \widetilde{a}} \tau_{\widetilde{b}}$ for each $\widetilde{a} \in [I] \setminus \{\mathbf{0}\}$ and let $A \in [\tau]_{\widetilde{a}}$. Then clearly, $\tau(A) \geq \widetilde{a}$. By the definition of τ, $\tau(A) = \bigvee_{A \in \tau_{\widetilde{c}}} \widetilde{c} = \widetilde{d} \geq \widetilde{a}$, where $\widetilde{c} \in [I] \setminus \{\mathbf{0}\}$. Let $\varepsilon > 0$. Then there is $\widetilde{b} \in [I] \setminus \{\mathbf{0}\}$ such that $d^- - \varepsilon < b^-$, $d^+ - \varepsilon < b^+$. Thus we get

$$a^- - \varepsilon \leq d^- - \varepsilon < b^-,\ a^+ - \varepsilon \leq d^+ - \varepsilon < b^+.$$

So $A \in \tau_{[a^- - \varepsilon, a^+ - \varepsilon]}$. Since ε is arbitrary, $A \in \tau_{\widetilde{a}}$. Hence $[\tau]_{\widetilde{a}} \subset \tau_{\widetilde{a}}$. It is clear that $\tau_{\widetilde{a}} \subset [\tau]_{\widetilde{a}}$. Therefore $[\tau]_{\widetilde{a}} = \tau_{\widetilde{a}}$.

(3) The proof is similar to (2). □

From Lemmas 3.10 and 3.12, we have the following result.

Proposition 3.13. *Let $\tau \in OIVFT(X)$ and let $[\tau]_{\widetilde{a}}$ be the $\widetilde{a}$-level classical topology on X w.r.t. τ. We define the mapping $\eta : 2^X \to [I]$ as follows: for each $A \in 2^X$,*

$$\eta(A) = \bigvee_{A \in [\tau]_{\widetilde{a}}} \widetilde{a}.$$

Then $\eta = \tau$.

The fact that an ordinary interval-valued topological space fully determined by it′s decomposition in classical topologies is restated in the following Theorem.

Theorem 3.14. *Let $\tau_1, \tau_2 \in OIVFT(X)$. Then $\tau_1 = \tau_2$ if and only if $[\tau_1]_{\widetilde{a}} = [\tau_2]_{\widetilde{a}}$ for each $\widetilde{a} \in [I]$ or alternatively, if and only if $[\tau_1]^*_{\widetilde{a}} = [\tau_2]^*_{\widetilde{a}}$ for each $\widetilde{a} \in [I]$.*

Remark 3.15. In a similar way, we can construct an ordinary interval-valued fuzzifying cotopology $\mathcal{C}$ on a set X, by using the $\widetilde{a}$-levels,

$$[\mathcal{C}]_{\widetilde{a}} = \{A \in 2^X : \mathcal{C}(A) \geq \widetilde{a}\} \text{ and } [\mathcal{C}]^*_{\widetilde{a}} = \{A \in 2^X : \mathcal{C}(A) > \widetilde{a}\}$$

for each $\widetilde{a} \in [I]$.

Definition 3.16. Let $T \in T(X)$ and let $\tau \in OIVFT(X)$. Then τ is said to be *compatible with* T, if $T = S(\tau)$, where $S(\tau) = \{A \in 2^X : \tau(A) > \mathbf{0}\}$.

Example 3.17. (1) Let T_0 be the classical indiscrete topology on X. Then clearly,

$$S(\tau_\phi) = \{A \in 2^X : \tau_\phi(A) > \mathbf{0}\} = \{\phi, X\} = T_0.$$

Thus τ_ϕ is compatible with T_0.

(2) Let T_1 be the classical discrete topology on X. Then clearly,

$$S(\tau_X) = \{A \in 2^X : \tau_X(A) > \mathbf{0}\} = 2^X = T_1.$$

Thus τ_X is compatible with T_1.

(3) Let X be a nonempty set and let $\widetilde{a} \in [I] \setminus \mathbf{1}$ be fixed. We define the mapping $\tau : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if either } A = \phi \text{ or } A = X, \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then clearly, $\tau \in OIVFT(X)$ and τ is compatible with T_1.

Furthermore, every classical topology can be considered as an ordinary interval-valued topology in the sense of the following result.

Proposition 3.18. *Let (X, τ) be a classical topological space and and let $\widetilde{a} \in [I] \setminus \{\mathbf{0}\}$ be fixed. Then there exists $\tau^{\widetilde{a}} \in OIVFT(X)$ such that $\tau^{\widetilde{a}}$ is compatible with τ. Moreover, $[\tau^{\widetilde{a}}]_{\widetilde{a}} = \tau$.*

In this case, $\tau^{\widetilde{a}}$ is called $\widetilde{a}$*-th ordinary interval-valued fuzzifying topology* on X and $(X, \tau^{\widetilde{a}})$ is called an $\widetilde{a}$*-th ordinary interval-valued fuzzifying topological space.*

Proof. Let $\widetilde{a} \in [I] \setminus \{\mathbf{0}\}$ be fixed and we define the mapping $\tau^{\widetilde{a}} : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\tau^{\widetilde{a}}(A) = \begin{cases} \mathbf{1} & \text{if either } A = \phi \text{ or } A = X, \\ \widetilde{a} & \text{if } A \in \tau \setminus \{\phi, X\}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau^{\widetilde{a}} \in OIVFT(X)$ and $[\tau^{\widetilde{a}}]_{\widetilde{a}} = \tau$. Moreover, by the definition of $\tau^{\widetilde{a}}$,

$$S(\tau^{\widetilde{a}}) = \{A \in 2^X : \tau^{\widetilde{a}}(A) > \mathbf{0}\} = \tau.$$

Thus $\tau^{\widetilde{a}}$ is compatible with τ. □

Proposition 3.19. *Let (X, T) be a classical topological space and let $C(T)$ be the set of all OIVFTs on X compatible with T, let $\widetilde{T} = T \setminus \{\phi, X\}$ and let $[I]_{\mathbf{0}}^{\widetilde{T}}$ be the set of all mappings $f : \widetilde{T} \to [I]$ satisfying the following conditions:*

(i) $f(A) \neq \mathbf{0}$ *for each* $A \in \widetilde{T}$,

(ii) $f(A \cap B) \geq f(A) \wedge f(B)$ *for any* $A,\ B \in \widetilde{T}$,

(iii) $f(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} f(A_j)$ *for any* $(A_j)_{j\in J} \subset \widetilde{T}$.

Then there is a one-to-one correspondence between $C(T)$ and the set $[I]_{\mathbf{0}}^{\widetilde{T}}$.

Proof. We define the mapping $F : [I]_{\mathbf{0}}^{\widetilde{T}} \to C(T)$ as follows: for each $f \in [I]_{\mathbf{0}}^{\widetilde{T}}$,

$$F(f) = \tau_f,$$

where $\tau_f : 2^X \to [I]$ is the mapping defined by: for each $A \in 2^X$,

$$\tau_f(A) = \begin{cases} \mathbf{1} & \text{if either } A = \phi \text{ or } A = X, \\ f(A) & \text{if } A \in \widetilde{T}, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then we easily see that $\tau_f \in C(T)$.

Now we define the mapping $G : C(T) \to [I]_{\mathbf{0}}^{\widetilde{T}}$ as follows: for each $\tau \in C(T)$,

$$G(\tau) = f_\tau,$$

where $f_\tau : \widetilde{T} \to [I]$ is the mapping defined by: for each $A \in \widetilde{T}$,

$$f_\tau(A) = \tau(A).$$

Then clearly, $f_\tau \in [I]_{\mathbf{0}}^{\widetilde{T}}$. Furthermore, we can see that $F \circ G = id_{C(T)}$ and $G \circ F = id_{[I]_{\mathbf{0}}^{\widetilde{T}}}$. Thus $C(T)$ is equipotent to $[I]_{\mathbf{0}}^{\widetilde{T}}$. This completes the proof. □

Proposition 3.20. *Let (X, τ) be an OIVFTS and let $Y \subset X$. We define the mapping $\tau_Y : 2^Y \to [I]$ as follows: for each $A \in 2^Y$,*

$$\tau_Y(A) = \bigvee_{B\in 2^X,\ A=B\cap Y} \tau(B).$$

Then $\tau_Y \in OIVT(Y)$ and $\tau_Y(A) \geq \tau(A)$ for each $A \in 2^Y$.

In this case, (Y, τ_Y) is called an *ordinary interval-valued fuzzifying subspace* of (X, τ) and τ_Y is called the *induced ordinary interval-valued fuzzifying topology on Y by τ.*

Proof. It is obvious that the condition (OIVFT1) holds, i.e., $\tau_Y(\phi) = \tau_Y(Y) = \mathbf{1}$.

Let $A, B \in 2^Y$. Then

$$\begin{aligned} \tau_Y(A) \wedge \tau_Y(B) &= (\textstyle\bigwedge_{C_1\in 2^X,\ A=Y\cap C_1} \tau(C_1)) \wedge (\bigwedge_{C_2\in 2^X,\ B=Y\cap C_2} \tau(C_2)) \\ &= \textstyle\bigwedge_{C_1,\ C_1\in 2^X,\ A\cap B=Y\cap(C_1\cap C_2)} [\tau(C_1) \wedge \tau(C_2)] \\ &\leq \textstyle\bigwedge_{C_1,\ C_1\in 2^X,\ A\cap B=Y\cap(C_1\cap C_2)} \tau(C_1 \cap C_2) \\ &= \tau_Y(A \cap B). \end{aligned}$$

Thus the condition (OIVFT2) holds.

Now let $(A_j)_{j\in J} \subset 2^Y$. Then

$$\begin{aligned} \tau_Y(\textstyle\bigcup_{j\in J} A_j) &= \textstyle\bigwedge_{B_j\in 2^X,\ (\bigcup_{j\in J} B_j)\cap Y=\bigcup_{j\in J} A_j} \tau(\bigcup_{j\in J} B_j) \\ &\geq \textstyle\bigwedge_{B_j\in 2^X,\ (\bigcup_{j\in J} B_j)\cap Y=\bigcup_{j\in J} A_j} [\bigwedge_{j\in J} \tau(B_j)] \end{aligned}$$

$= \bigwedge_{j\in J}[\bigwedge_{B_j\in 2^X,\ (\bigcup_{j\in J} B_j)\cap Y=\bigcup_{j\in J} A_j} \tau(B_j)]$
$= \bigwedge_{j\in J} \tau_Y(A_j)$.

Thus the condition (OIVT3) holds. So $\tau_Y \in OIVFT(Y)$.

Furthermore, we can easily see that for each $A \in 2^Y$, $\tau_Y(A) \geq \tau(A)$. This completes the proof. □

The following is the immediate result of Proposition 3.20.

Corollary 3.21. *Let (Y, τ_Y) be an ordinary interval-valued fuzzifying subspace of (X, τ) and let $A \in 2^Y$.*
(1) $\mathcal{C}_Y(A) = \bigvee_{B\in 2^X, A=B\cap Y} \mathcal{C}(B)$, *where* $\mathcal{C}_Y(A) = \tau_Y(Y - A)$.
(2) *If* $Z \subset Y \subset X$, *then* $\tau_Z = (\tau_Y)_Z$.

4. Ordinary interval-valued fuzzifying neighborhood structures

Definition 4.1. Let (X, τ) be an OIVFTS and let $x \in X$. Then a mapping $\mathcal{N}_x : 2^X \to [I]$ is called the *ordinary interval-valued fuzzifying neighborhood system* of x, if for each $A \in 2^X$,

$$[A \in \mathcal{N}_x] = \mathcal{N}_x(A) = \bigvee_{x\in B\subset A} \tau(B).$$

Example 4.2. Let $X = \{a, b, c\}$ and let (X, τ) be the OIVFTS defined in Example 3.2 (1). Then

$\mathcal{N}_a(\{a\}) = \bigvee_{a\in B\subset\{a\}} \tau(B) = \tau(\{a\}) = [0.2, 0.7]$,
$\mathcal{N}_a(\{a, b\}) = \bigvee_{a\in B\subset\{a,b\}} \tau(B) = \tau(\{a\}) \vee \tau(\{a, b\})$
$= [0.2, 0.7] \vee [0.3, 0.7] = [0.3, 0.7]$,
$\mathcal{N}_a(\{a, c\}) = \bigvee_{a\in B\subset\{a,c\}} \tau(B) = \tau(\{a\}) \vee \tau(\{a, c\})$
$= [0.2, 0.7] \vee [0.3, 0.8] = [0.3, 0.8]$,
$\mathcal{N}_a(X) = \bigvee_{a\in B\subset X} \tau(B) = \tau(\{a\}) \vee \tau(\{a, b\}) \vee \tau(\{a, c\})$
$= [0.2, 0.7] \vee [0.3, 0.7] \vee [0.3, 0.8] = [0.3, 0.8]$.

We have the similar to that of Lemma 3.1 in [34].

Lemma 4.3. *Let (X, τ) be an OIVFTS and let $A \in 2^X$. Then*

$$\bigwedge_{x\in A} \bigvee_{x\in B\subset A} \tau(B) = \tau(A).$$

Proof. It is clear that $\bigwedge_{x\in A} \bigvee_{x\in B\subset A} \tau(B) \geq \tau(A)$. Now let $\mathcal{B}_x = \{B \in 2^X : x \in B \subset A\}$ and let $f \in \Pi_{x\in A}\mathcal{B}_x$. Then clearly, $\bigcup_{x\in A} f(x) = A$. Thus

$$\bigwedge_{x\in A} \tau(f(x)) \leq \tau(\bigcup_{x\in A} f(x)) = \tau(A).$$

So

$$\bigwedge_{x\in A} \bigvee_{x\in B\subset A} \tau(B) = \bigvee_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigwedge_{x\in A} \tau(f(x)) \leq \tau(A).$$

Hence $\bigwedge_{x\in A} \bigvee_{x\in B\subset A} \tau(B) = \tau(A)$. □

Example 4.4. Let $X = \{a, b, c\}$ and let (X, τ) be the OIVFTS defined in Example 3.2 (1). Let $A = \{a, b\}$. Then

$$\begin{aligned}\bigwedge_{x\in A}\bigvee_{x\in B\subset A}\tau(B) &= (\tau(\{a\})\vee\tau(A))\wedge(\tau(\{b\})\vee\tau(A))\\ &= ([0.2, 0.7]\vee[0.3, 0.7])\wedge([0.4, 0.5]\vee[0.2, 0.7])\\ &= [0.3, 0.7]\wedge[0.4, 0.7] = [0.3, 0.7]\\ &= \tau(A).\end{aligned}$$

Thus we can confirm that Lemma 4.3 holds.

We have the similar to Theorem 3.1 in [34].

Proposition 4.5. *Let (X, τ) be an OIVFTS, let $A \in 2^X$ and let $x \in X$. Then*

$$\vDash (A \in \tau) \leftrightarrow \forall(x \in A \to \exists B(B \in \mathcal{N}_x \wedge B \subset A)),$$

i.e.,

$$[A \in \tau] = [\forall(x \in A \to \exists B(B \in \mathcal{N}_x \wedge B \subset A)],$$

i.e.,

$$[A \in \tau] = \bigwedge_{x\in A}\bigvee_{B\subset A}\mathcal{N}_x(B).$$

Proof. From Lemma 4.3, it is obvious. □

Definition 4.6. Let $\mathcal{A}$ be an interval-valued fuzzy set in 2^X. Then $\mathcal{A}$ is said to be *normal*, if there is $A_0 \in 2^X$ such that $\mathcal{A}(A_0) = \mathbf{1}$.

We will denote the set of all normal interval-valued fuzzy sets in 2^X as $[I]_N^{2^X}$.

From the following result, we can see that an ordinary interval-valued fuzzy neighborhood system has the same properties in a classical neighborhood system.

Theorem 4.7. *Let (X, τ) be an OIVFTS and let $\mathcal{N} : X \to [I]_N^{2^X}$ be the mapping given by $\mathcal{N}(x) = \mathcal{N}_x$ for each $x \in X$. Then $\mathcal{N}$ has the following properties:*

(1) *for any $x \in X$, $A \in 2^X$, $\vDash A \in \mathcal{N}_x \to x \in A$,*

(2) *for any $x \in X$, $A, B \in 2^X$, $\vDash (A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x) \to A \cap B \in \mathcal{N}_x$,*

(3) *for any $x \in X$, $A, B \in 2^X$, $\vDash (A \subset B) \to (A \in \mathcal{N}_x \to B \in \mathcal{N}_x)$,*

(4) *for any $x \in X$, $\vDash (A \in \mathcal{N}_x) \to \exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))$.*

Conversely, if a mapping $\mathcal{N} : X \to [I]_N^{2^X}$ satisfies the above properties (2) and (3), then there is an ordinary interval-valued fuzzifying topology $\tau : 2^X \to [I]$ on X defined as follows: for each $A \in 2^X$,

$$A \in \tau := \forall x(x \in A \to A \in \mathcal{N}_x),$$

i.e.,

$$[A \in \tau] = \tau(A) = \bigwedge_{x\in A}\mathcal{N}_x(A).$$

In particular, if $\mathcal{N}$ satisfies the above properties (1) and (4) also, then for each $x \in X$, $\mathcal{N}_x$ is an ordinary interval-valued fuzzifying neighborhood system of x with respect to τ.

Proof. (1) Since $A \in 2^X$, we can consider A as a special interval-valued fuzzy set in x represented by $A = [\chi_A, \chi_A]$. Then $[x \in A] = A(x) = \mathbf{1}$. On the other hand,

$$[A \in \mathcal{N}_x] = \bigvee_{x \in C \subset A} \tau(C) \leq \mathbf{1}.$$

Thus $[A \in \mathcal{N}_x] \leq [x \in A]$.

(2) By the definition of $\mathcal{N}_x$, $[A \cap B \in \mathcal{N}_x] = \bigvee_{x \in C \subset A \cap B} \tau(C)$. Then

$$\begin{aligned} \mathcal{N}_x(A \cap B) &= \textstyle\bigvee_{x \in C \subset A \cap B} \tau(C) \\ &= \textstyle\bigvee_{x \in C_1 \subset A,\ x \in C_2 \subset B} \tau(C_1 \cap C_2) \\ &\geq \textstyle\bigvee_{x \in C_1 \subset A,\ x \in C_2 \subset B} [\tau(C_1) \wedge \tau(C_2)] \text{ [By Definition 3.1]} \\ &= \textstyle\bigvee_{x \in C_1 \subset A} \tau(C_1) \wedge \bigvee_{x \in C_2 \subset B} \tau(C_2) \\ &= \mathcal{N}_x(A) \wedge \mathcal{N}_x(B) \\ &= [(A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x)]. \end{aligned}$$

Thus $[A \cap B \in \mathcal{N}_x] \geq [(A \in \mathcal{N}_x) \wedge (B \in \mathcal{N}_x)]$.

(3) From the definition of $\mathcal{N}_x$, we can easily show that $[A \in \mathcal{N}_x] \leq [B \in \mathcal{N}_x]$.

(4) $\quad [\exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))]$

$$\begin{aligned} &= \textstyle\bigvee_{C \subset A} [\mathcal{N}_x(C) \wedge \bigwedge_{y \in C} \mathcal{N}_y(C)] \\ &= \textstyle\bigvee_{C \subset A} [\mathcal{N}_x(C) \wedge \bigwedge_{y \in C} \bigvee_{y \in D \subset C} \tau(D)] \text{ [By Definition 4.1]} \\ &= \textstyle\bigvee_{C \subset A} [\mathcal{N}_x(C) \wedge \tau(C)] \text{ [By Lemma 4.3]} \\ &= \textstyle\bigvee_{C \subset A} \tau(C) \\ &\geq \textstyle\bigvee_{x \in C \subset A} \tau(C) \\ &= [A \in \mathcal{N}_x]. \text{ [By Definition 4.1]} \end{aligned}$$

Then $[\exists C((C \in \mathcal{N}_x) \wedge (C \subset A) \wedge \forall y(y \in C \to C \in \mathcal{N}_y))] \geq [A \in \mathcal{N}_x]$.

Conversely suppose $\mathcal{N}$ satisfies the above properties (2) and (3) and let $\tau : 2^X \to [I]$ be the mapping defined as follows: for each $A \in 2^X$,

$$\tau(A) = \bigwedge_{x \in A} \mathcal{N}_x(A).$$

Then clearly, $\tau(\phi) = \mathbf{1}$. Since $\mathcal{N}_x$ is an interval-valued normal set in 2^X, there is $A_0 \in 2^X$ such that $\mathcal{N}_x(A_0) = \mathbf{1}$. Thus $\mathcal{N}_x(X) = \mathbf{1}$. So $\tau(X) = \bigwedge_{x \in X} \mathcal{N}_x(X) = \mathbf{1}$. Hence τ satisfies the axiom (OIVFT1).

Let $A,\ B \in 2^X$. Then

$$\begin{aligned} \tau(A \cap B) &= \textstyle\bigwedge_{x \in A \cap B} \mathcal{N}_x(A \cap B) \\ &\geq \textstyle\bigvee_{x \in A \cap B} (\mathcal{N}_x(A) \wedge \mathcal{N}_x(B)) \text{ [By (2)]} \\ &= \textstyle\bigwedge_{x \in A \cap B} \mathcal{N}_x(A) \wedge \bigwedge_{x \in A \cap B} \mathcal{N}_x(B) \\ &\geq \textstyle\bigwedge_{x \in A} \mathcal{N}_x(A) \wedge \bigwedge_{x \in B} \mathcal{N}_x(B) \\ &= \tau(A) \wedge \tau(B). \end{aligned}$$

Thus τ satisfies the axiom (OIVFT2).

Now let $(A_j)_{j \in J} \subset 2^X$. Then

$$\begin{aligned} \tau(\textstyle\bigcup_{j \in J} A_j) &= \textstyle\bigwedge_{x \in \bigcup_{j \in J} A_j} \mathcal{N}_x(\bigcup_{j \in J} A_j) \\ &= \textstyle\bigwedge_{j \in J} \bigwedge_{x \in A_j} \mathcal{N}_x(\bigcup_{j \in J} A_j) \\ &\geq \textstyle\bigwedge_{j \in J} \bigwedge_{x \in A_j} \mathcal{N}_x(A_j) \text{ [By (3)]} \\ &= \textstyle\bigwedge_{j \in J} \tau(A_j). \end{aligned}$$

Thus τ satisfies the axiom (OIVFT3). So $\tau \in OIVT(X)$.

Now suppose $\mathcal{N}$ satisfies additionally the above properties (1) and (4). Then from the proof of Theorem 3.2 in [34], we can easily prove that $\mathcal{N}_x$ is the ordinary interval-valued fuzzifying neighborhood system of x with respect to τ for each $x \in X$. This completes the proof. □

5. Ordinary interval-valued fuzzifying bases and subbases

Definition 5.1. Let (X, τ) be an OIVFTS and let $\mathcal{B} : 2^X \to [I]$ be a mapping such that $\mathcal{B} \subset \tau$, i.e., $\mathcal{B}(A) \leq \tau(A)$ for each $A \in 2^X$. Then $\mathcal{B}$ is called an *ordinary interval-valued fuzzifying base* for τ, if for each $A \in 2^X$,

$$\mathcal{B}(A) = \bigvee_{\{B_j\}_{j \in J} \subset 2^X,\ A = \bigcup_{j \in J} B_j} \bigwedge_{j \in J} \mathcal{B}(B_j).$$

Example 5.2. (1) Let X be a set and let $\mathcal{B} : 2^X \to [I]$ be the mapping defined by:

$$\mathcal{B}(\{x\}) = \mathbf{1}\ \forall x \in X.$$

Then $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for τ_X.

(2) Let $X = \{a, b, c\}$, let $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$ be fixed and let $\mathcal{B} : 2^X \to [I]$ be the mapping as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if either } A = \{a, b\} \text{ or } \{b, c\} \text{ or } X, \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then $\mathcal{B}$ is not an ordinary interval-valued fuzzifying base for an OIVFT on X.

Assume that $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for an OIVFT τ on X. Then clearly, $\mathcal{B} \subset \tau$. Thus $\tau(\{a, b\}) = \tau(\{b, c\}) = \mathbf{1}$. So

$$\tau(\{b\}) = \tau(\{a, b\} \cap \tau(\{b, c\}) \geq \tau(\{a, b\} \wedge \tau(\{b, c\} = \mathbf{1}.$$

Hence $\tau(\{b\}) = \mathbf{1}$. On the other hand, by the definition of $\mathcal{B}$,

$$\tau(\{b\}) = \bigvee_{\{A_j\}_{j \in J} \subset 2^X,\ \{b\} = \bigcup_{j \in J} A_j} \bigwedge_{j \in J} \mathcal{B}(A_j) = \widetilde{a}.$$

This is a contradiction. Therefore $\mathcal{B}$ is not an ordinary interval-valued fuzzifying base for an τ on X

Theorem 5.3. *Let (X, τ) be an OIVFTS and let $\mathcal{B} : 2^X \to [I]$ be a mapping such that $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for τ if and only if for each $x \in X$ and each $A \in 2^X$,*

$$\mathcal{N}_x(A) \leq \bigvee_{x \in B \subset A} \mathcal{B}(B).$$

Proof. (⇒): Suppose $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for τ. Let $x \in X$ and let $A \in 2^X$. Then

$\mathcal{N}_x(A) = \bigvee_{x \in B \subset A} \tau(B)$ [By Definition 4.1]
$= \bigvee_{x \in B \subset A} \bigvee_{\{B_j\}_{j \in J} \subset 2^X,\ B = \bigcup_{j \in J} B_j} \bigwedge_{j \in J} \mathcal{B}(B_j)$. [By Definition 5.1]

If $x \in B \subset A$ and $B = \bigcup_{j \in J} B_j$, then there is $j_0 \in J$ such that $x \in B_{j_0}$. Thus

$$\bigwedge_{j \in J} \mathcal{B}(B_j) \leq \mathcal{B}(B_{j_0}) \leq \bigvee_{x \in B \subset A} \mathcal{B}(B).$$

So $\mathcal{N}_x(A) \leq \bigvee_{x\in B\subset A} \mathcal{B}(B)$.

(⇐): suppose the necessary condition holds. Let $A \in 2^X$, where $A = \bigcup_{j\in J} B_j$ and $(B_j)_{j\in J} \subset 2^X$. Then

$$\begin{aligned}\tau(A) &\geq \textstyle\bigwedge_{j\in J} \tau(B_j) \text{ [By the axiom (OIVFT3)]}\\ &\geq \textstyle\bigwedge_{j\in J} \mathcal{B}(B_j). \text{ [Since } \mathcal{B} \subset \tau]\end{aligned}$$

Thus

$$\tau(A) \geq \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ A=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j). \tag{5.3.1}$$

On the other hand,

$$\begin{aligned}\tau(A) &= \textstyle\bigwedge_{x\in A} \bigvee_{x\in B\subset A} \tau(B) \text{ [By Lemma 4.3]}\\ &= \textstyle\bigwedge_{x\in A} \mathcal{N}_x(A) \text{ [By Definition 4.1]}\\ &\leq \textstyle\bigwedge_{x\in A} \bigvee_{x\in B\subset A} \mathcal{B}(B) \text{ [By the hypothesis]}\\ &= \textstyle\bigvee_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigwedge_{x\in A} \mathcal{B}(f(x)),\end{aligned}$$

where $\mathcal{B}_x = \{B \in 2^X : x \in B \subset A\}$. Furthermore, $A = \bigcup_{x\in A} f(x)$ for each $f \in \Pi_{x\in A}\mathcal{B}_x$. So

$$\bigvee_{f\in\Pi_{x\in A}\mathcal{B}_x} \bigwedge_{x\in A} \mathcal{B}(f(x)) = \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ A=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j).$$

Hence

$$\tau(A) \leq \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ A=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j). \tag{5.3.2}$$

By (5.3.1) and (5.3.2), $\tau(A) = \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ A=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j)$. Therefore $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for τ. □

Theorem 5.4. *Let $\mathcal{B} : 2^X \to [I]$ be a mapping. Then $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for some OIVT τ on X if and only if it has the following conditions:*

(1) $\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j) = \mathbf{1}$,

(2) *for any $A_1,\ A_2 \in 2^X$ and each $x \in A_1 \cap A_2$,*

$$\mathcal{B}(A_1) \wedge \mathcal{B}(A_2) \leq \bigvee_{x\in A\subset A_1\cap A_2} \mathcal{B}(A).$$

In fact, $\tau : 2^X \to [I]$ is the mapping defined as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} \mathbf{1} & if\ A = \phi \\ \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ A=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j) & otherwise. \end{cases}$$

In this case, τ is called the *ordinary interval-valued fuzzifying topology on X induced by $\mathcal{B}$.*

Proof. (⇒): Suppose $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for some OIVFT τ on X. Then by Definition 5.1 and the axiom (OIVFT1),

$$\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j) = \tau(X) = \mathbf{1}.$$

Thus the condition (1) holds.

Let $A_1,\ A_2 \in 2^X$ and let $x \in A_1 \cap A_2$. Then

$$\mathcal{B}(A_1) \wedge \mathcal{B}(A_2) \leq \tau(A_1) \wedge \tau(A_2) \leq \tau(A_1 \cap A_2) \leq \mathcal{N}_x(A_1 \cap A_2) \leq \bigvee_{x \in A \subset A_1 \cap A_2} \mathcal{B}(A).$$

Thus $\mathcal{B}(A_1) \wedge \mathcal{B}(A_2) \leq \bigvee_{x \in A \subset A_1 \cap A_2} \mathcal{B}(A)$. So the condition (2) holds.

($\Leftarrow$): Suppose the necessary conditions (1) and (2) are satisfied. From the definition of τ and the condition (1), it is obvious that $\tau(X) = \tau(\phi) = \mathbf{1}$. Then τ satisfies the axiom (OIVFT1).

Let $(A_j)_{j \in J} \subset 2^X$ and let $\mathcal{B}_j = \{\{B_{\delta_j} : \delta_j \in J_j\} : \bigcup_{\delta_j \in J_j} B_{\delta_j} = A_j\}$. Let $f \in \Pi_{\alpha \in \Gamma} \mathcal{B}_\alpha$. Then clearly, $\bigcup_{j \in J} \bigcup_{B_{\delta_j} \in f(j)} B_{\delta_j} = \bigcup_{j \in J} A_j$. Thus

$$\begin{aligned}\tau(\textstyle\bigcup_{j \in J} A_j) &= \textstyle\bigvee_{\bigcup_{\delta \in J} B_\delta = \bigcup_{j \in J} A_j} \bigwedge_{\delta \in J} \mathcal{B}(B_\delta) \\ &\geq \textstyle\bigvee_{f \in \Pi_{j \in J} \mathcal{B}_j} \bigwedge_{j \in J} \bigwedge_{B_{\delta_j} \in f(j)} \mathcal{B}(B_{\delta_j}) \\ &= \textstyle\bigwedge_{j \in J} \bigvee_{\{B_{\delta_j} : \delta_j \in J_j\} \in \mathcal{B}_j} \bigwedge_{\delta_j \in J_j} \mathcal{B}(B_{\delta_j}) \\ &= \textstyle\bigwedge_{j \in J} \tau(A_j).\end{aligned}$$

So τ satisfies the axiom (OIVFT3).

Now let $A,\ B \in 2^X$ and suppose $\tau(A) > \widetilde{a}$ and $\tau(B) > \widetilde{a}$, for $\widetilde{a} \in [I]$. Then there are $\{A_{j_1} : j_1 \in J_1\}$ and $\{B_{j_2} : j_2 \in J_2\}$ such that $\bigcup_{j_1 \in J_1} A_{j_1} = A$, $\bigcup_{j_2 \in J_2} B_{j_2} = B$, and $\mathcal{B}(A_{j_1}) > \widetilde{a}$ for each $j_1 \in J_1$ and $\mathcal{B}(B_{j_2}) > \widetilde{a}$ for each $j_2 \in J_2$. Let $x \in A \cap B$. Then there are $j_{1x} \in J_1$ and $j_{2x} \in J_2$ such that $x \in A_{j_{1x}} \cap B_{j_{2x}}$. Thus from the assumption,

$$\widetilde{a} < \mathcal{B}(A_{j_{1x}}) \wedge \mathcal{B}(B_{j_{2x}}) \leq \bigvee_{x \in C \subset A_{j_{1x}} \cap B_{j_{2x}}} \mathcal{B}(C).$$

Moreover, there is C_x such that $x \in C_x \subset A_{j_{1x}} \cap B_{j_{2x}} \subset A \cap B$ and $\mathcal{B}(C_x) > \widetilde{a}$. Since $\bigcup_{x \in A \cap B} C_x = A \cap B$, we obtain

$$\widetilde{a} \leq \bigwedge_{x \in A \cap B} \mathcal{B}(C_x) \leq \bigvee_{\bigcup_{j \in J} B_j = A \cap B} \bigwedge_{j \in J} \mathcal{B}(B_j) = \tau(A \cap B).$$

Now let $\widetilde{b} = \tau(A) \wedge \tau(B)$ and let n be any natural number, where $\widetilde{b} \in [I]$. Then $\tau(A) > \widetilde{b} - \frac{1}{n}$ and $\tau(B) > \widetilde{b} - \frac{1}{n}$, where $\widetilde{b} - \frac{1}{n} = [b^- - \frac{1}{n}, b^+ + \frac{1}{n}]$. Thus $\tau(A \cap B) \geq \widetilde{b} - \frac{1}{n}$. So $\tau(A \cap B) \geq \widetilde{b} = \tau(A) \wedge \tau(B)$. Hence τ satisfies the axiom (OIVFT2). This completes the proof. □

Example 5.5. (1) Let $X = \{a, b, c\}$ and let $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$ be fixed. We define the mapping $\mathcal{B} : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\mathcal{B}(A) = \begin{cases} \mathbf{1} & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\} \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then we can easily see that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5.4. Thus $\mathcal{B}$ is an ordinary interval-valued base for an OIVFT τ on X. In fact, $\tau : 2^X \to [I]$ is defined as follows: for each $A \in 2^X$,

$$\tau(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\} \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

(2) Let $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$ be fixed. We define the mapping $\mathcal{B} : 2^{\mathbb{R}} \to [I]$ as follows: for each $A \in 2^{\mathbb{R}}$,

$$\mathcal{B}(A) = \begin{cases} \mathbf{1} & \text{if } A = (a, b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then it can be easily seen that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5.4. Thus $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for an OIVT $\tau_{\widetilde{a}}$ on $\mathbb{R}$.

In this case, $\tau_{\widetilde{a}}$ will be called the *$\widetilde{a}$-ordinary interval-valued fuzzifying usual topology* on $\mathbb{R}$ and we will write $\tau_{\widetilde{a}} = \mathcal{U}_{\widetilde{a}}$.

(3) Let $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$ be fixed. We define the mapping $\mathcal{B} : 2^{\mathbb{R}} \to [I]$ as follows: for each $A \in 2^{\mathbb{R}}$,

$$\mathcal{B}(A) = \begin{cases} \mathbf{1} & \text{if } A = [a, b) \text{ for } a,\ b \in \mathbb{R} \text{ with } a \leq b \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then we can easily see that $\mathcal{B}$ satisfies the conditions (1) and (2) in Theorem 5.4. Thus $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for an OIVFT τ_l on $\mathbb{R}$.

In this case, τ_l will be called the *$\widetilde{a}$-ordinary interval-valued fuzzifying lower-limit topology* on $\mathbb{R}$.

Definition 5.6. Let $\tau_1,\ \tau_2 \in OIVFT(X)$, and let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary interval-valued fuzzifying bases for τ_1 and τ_2 respectively. Then we say that $\mathcal{B}_1$ and $\mathcal{B}_1$ are *equivalent*, if $\tau_1 = \tau_2$.

Theorem 5.7. *Let $\tau_1,\ \tau_2 \in OIVFT(X)$, and let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary interval-valued fuzzifying bases for τ_1 and τ_2 respectively. Then τ_1 is coarser than τ_2 if and only if for each $x \in X$ and each $A \in 2^X$, if $x \in A$, then $\mathcal{B}_1(A) \leq \bigvee_{x \in B \subset A} \mathcal{B}_2(B)$.*

Proof. ($\Rightarrow$): Suppose τ_1 is coarser than τ_2. For each $x \in X$, let $x \in A \in 2^X$. Then

$\mathcal{B}_1(A) \leq \tau_1(A)$ [Since $\mathcal{B}_1$ is an ordinary interval-valued fuzzifying base for τ_1]
$\leq \tau_2(A)$ [By the hypothesis]
$= \bigvee_{\{A_j\}_{j \in J} \subset 2^X,\ A = \bigcup_{j \in J} A_j} \bigwedge_{j \in J} \mathcal{B}_2(A_j)$.
[Since $\mathcal{B}_2$ is an ordinary interval-valued fuzzifying base for τ_2]

Since $x \in A$ and $A = \bigcup_{j \in J} A_j$, there is $j_0 \in J$ such that $x \in A_{j_0}$. Thus

$$\bigvee_{\{A_j\}_{j \in J} \subset 2^X,\ A = \bigcup_{j \in J} A_j} \bigwedge_{j \in J} \mathcal{B}_2(A_j) \preceq \mathcal{B}_2(A_{j_0}) \leq \bigvee_{x \in B \subset A} \mathcal{B}_2(B).$$

So $\mathcal{B}_1(A) \leq \bigvee_{x \in B \subset A} \mathcal{B}_2(B)$.

($\Leftarrow$): Suppose the necessary conditions hold. Let $A \in 2^X$. Then

$\tau_1(A) = \bigwedge_{x \in A} \bigvee_{x \in B \subset A} \mathcal{B}_1(B)$ [By Lemma 4.3]
$\leq \bigwedge_{x \in A} \bigvee_{x \in B \subset A} \bigvee_{x \in C \subset B} \mathcal{B}_2(C)$ [By the hypothesis]
$= \bigvee_{x \in C \subset A} \bigwedge_{x \in A} \mathcal{B}_2(C)$
$= \bigvee_{\{C_x\}_{x \in A} \subset 2^X,\ A = \bigcup_{x \in A} C_x} \bigwedge_{x \in A} \mathcal{B}_2(C_x)$
$= \tau_2(A)$.

Thus $\tau_1 \leq \tau_2$. So τ_1 is coarser than τ_2. This completes the proof. □

The following is the immediate result of Definition 5.6 and Theorem 5.7.

Corollary 5.8. *Let $\mathcal{B}_1$ and $\mathcal{B}_1$ be ordinary interval-valued fuzzifying bases for two ordinary interval-valued fuzzy topologies on a set X respectively. Then $\mathcal{B}_1$ and $\mathcal{B}_1$ are equivalent if and only if*

(1) *for each $B_1 \in 2^X$ and each $x \in B_1$, $\mathcal{B}_1(B_1) \leq \bigvee_{x\in B_2\subset B_1} \mathcal{B}_2(B_2)$,*
(2) *for each $B_2 \in 2^X$ and each $x \in B_2$, $\mathcal{B}_2(B_2) \leq \bigvee_{x\in B_1\subset B_2} \mathcal{B}_1(B_1)$.*

It is obvious that every ordinary interval-valued fuzzifying topology itself forms an ordinary interval-valued fuzzifying base. Then the following provides a sufficient condition for one to see if a mapping $\mathcal{B} : 2^X \to [I]$ such that $\mathcal{B} \subset \tau$ is an ordinary interval-valued base for τ, where $\tau \in OIVFT(X)$.

Proposition 5.9. *Let (X, τ) be an OIVTS and let $\mathcal{B} : 2^X \to [I]$ be a mapping such that $\mathcal{B} \subset \tau$. Then for each $x \in X$ and each $A \in 2^X$ such that $x \in A$ and $\tau(A) \leq \bigvee_{x\in B\subset A} \mathcal{B}(B)$, $\mathcal{B}$ is an ordinary interval-valued fuzzifying base for τ.*

Proof.
$$\begin{aligned}
&\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j)\\
&\leq \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \tau(B_j) \text{ [Since } \mathcal{B} \subset \tau]\\
&\leq \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \tau(\textstyle\bigcup_{j\in J} B_j) \text{ [By the axiom (OIVFT3)]}\\
&= \tau(X)\\
&= \bigwedge_{x\in X} \bigvee_{x\in B\subset X} \tau(B) \text{ [By Lemma 4.3]}\\
&\leq \bigwedge_{x\in X} \bigvee_{x\in B\subset X} \bigvee_{x\in C\subset B} \mathcal{B}(C) \text{ [By the hypothesis]}\\
&= \bigvee_{x\in C\subset X} \bigwedge_{x\in X} \mathcal{B}(C)\\
&= \bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j).
\end{aligned}$$
Since $\tau \in OIVFT(X)$, $\tau(X) = \mathbf{1}$. Thus $\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \mathcal{B}(B_j) = \mathbf{1}$. So the condition (1) of Theorem 5.4 holds.

Now let $A_1,\ A_2 \in 2^X$ and let $x \in A_1 \cap A_2$. Then
$$\begin{aligned}
\mathcal{B}(A_1) \wedge \mathcal{B}(A_2) &\leq \tau(A_1) \wedge \tau(A_2) \text{ [Since } \mathcal{B} \subset \tau]\\
&\leq \tau(A_1 \cap A_2) \text{ [By the axiom (OIVFT2)]}\\
&\leq \bigvee_{x\in A\subset A_1\cap A_2} \mathcal{B}(A). \text{ [By the hypothesis]}
\end{aligned}$$
Thus the condition (2) of Theorem 5.4 holds. So, by Theorem 5.4, $\mathcal{B}$ is an ordinary interval-valued base for τ. This completes the proof. □

Definition 5.10. Let (X, τ) be an OIVFTS and let $\varphi : 2^X \to [I]$ be a mapping. Then φ is called an *ordinary interval-valued fuzzifying subbase* for τ, if $\varphi^{\sqcap}$ is an ordinary interval-valued fuzzifying base for τ, where $\varphi^{\sqcap} : 2^X \to [I]$ is the mapping defined as follows: for each $A \in 2^X$,

$$\varphi^{\sqcap}(A) = \bigvee_{\{B_j\}\sqsubset 2^X,\ A=\bigcap_{j\in J} B_j} \bigwedge_{j\in J} \varphi(B_j),$$

where $\{B_j\} \sqsubset 2^X$ means that $\{B_j\}$ is a finite subset of 2^X.

Example 5.11. Let $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$ be fixed. We define the mapping $\varphi : 2^{\mathbb{R}} \to [I]$ as follows: for each $A \in 2^{\mathbb{R}}$,

$$\varphi(A) = \begin{cases} \mathbf{1} & \text{if } A = (a, \infty) \text{ or } (\infty, b) \text{ or } (a, b)\\ \widetilde{a} & \text{otherwise,} \end{cases}$$

where $a,\ b \in \mathbb{R}$ such that $a \leq b$. Then we can easily see that φ is an ordinary interval-valued fuzzifying subbase for the $\widetilde{a}$-ordinary interval-valued fuzzifying usual topology $\mathcal{U}_{\widetilde{a}}$ on $\mathbb{R}$.

Theorem 5.12. *Let $\varphi : 2^X \to [I]$ be a mapping. Then φ is an ordinary interval-valued fuzzifying subbase for some OIVFT if and only if*

$$\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_J} \bigwedge_{j\in J} \varphi(B_j) = \mathbf{1}.$$

Proof. ($\Rightarrow$): Suppose φ is an ordinary interval-valued fuzzifying subbase for some OIVFT. Then by Definition 5.10, it is clear that the necessary condition holds.

($\Leftarrow$): Suppose the necessary condition holds. We only show that $\varphi^{\sqcap}$ satisfies the condition (2) in Theorem 5.4. Let $A,\ B \in 2^X$ and $x \in A \cap B$ for each $x \in X$. Then

$$\begin{aligned}
&\varphi^{\sqcap}(A) \wedge \varphi^{\sqcap}(B)\\
&= (\textstyle\bigvee_{\bigcap_{j_1\in J_1} B_{j_1}=A} \bigwedge_{j_1\in J_1} \varphi(B_{j_1})) \wedge (\bigvee_{\bigcap_{j_2\in J_2} B_{\alpha_2}=B} \bigwedge_{j_2\in J_2} \varphi(B_{j_2}))\\
&= \textstyle\bigvee_{\bigcap_{j_1\in J_1} B_{j_1}=A} \bigvee_{\bigcap_{j_2\in J_2} B_{j_2}=B} (\bigwedge_{j_1\in J_1} \varphi(B_{j_1}) \wedge \bigwedge_{j_2\in J_2} \varphi(B_{j_2}))\\
&\leq \textstyle\bigvee_{\bigcap_{j\in J} B_j=A\cap B} \bigwedge_{j\in J} \varphi(B_j)\\
&= \varphi^{\sqcap}(A \cap B).
\end{aligned}$$

Since $x \in A \cap B$, $\varphi^{\sqcap}(A) \wedge \varphi^{\sqcap}(B) \leq \varphi^{\sqcap}(A \cap B) \leq \bigvee_{x\in C\subset A\cap B} \varphi^{\sqcap}(C)$. Thus $\varphi^{\sqcap}$ satisfies the condition (2) in Theorem 5.4. This completes the proof. □

Example 5.13. Let $X = \{a, b, c, d, e\}$ and let $\widetilde{a} \in [I] \setminus \{\mathbf{1}\}$ be fixed. We define the mapping $\varphi : 2^X \to [I]$ as follows: for each $A \in 2^X$,

$$\varphi(A) = \begin{cases} \mathbf{1} & \text{if } A \in \{\{a\}, \{a, b, c\}, \{b, c, d\}, \{c, e\}\} \\ \widetilde{a} & \text{otherwise.} \end{cases}$$

Then $X = \{a\} \cup \{b, c, d\} \cup \{c, e\}$ and $\varphi^{\sqcap}(\{a\}) = \varphi^{\sqcap}(\{b, c, d\}) = \varphi^{\sqcap}(\{c, e\}) = \mathbf{1}$. Thus

$$\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \varphi(B_j) = \mathbf{1}.$$

So by Theorem 5.12, φ is an ordinary interval-valued fuzzifying subbase for some OIVFT.

The following is an immediate result of Corollary 5.8 and Theorem 5.12.

Proposition 5.14. $\varphi_1,\ \varphi_2 : 2^X \to [I]$ *be two mappings such that*

$$\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j \bigwedge_{j\in J} \varphi_1} (B_j) = \mathbf{1}$$

and

$$\bigvee_{\{B_j\}_{j\in J}\subset 2^X,\ X=\bigcup_{j\in J} B_j} \bigwedge_{j\in J} \varphi_2(B_j) = 1.$$

Suppose the two conditions hold:

(1) *for each $S_1 \in 2^X$ and each $x \in S_1$, $\varphi_1(S_1) \leq \bigvee_{x\in S_2\subset S_1} \varphi_2(S_2)$,*

(2) *for each $S_2 \in 2^X$ and each $x \in S_2$, $\varphi_2(S_2) \leq \bigvee_{x\in S_1\subset S_2} \varphi_1(S_1)$.*

Then φ_1 and φ_2 are ordinary interval-valued fuzzifying subbases for the same ordinary interval-valued fuzzifying topology on X.

6. Ordinary interval-valued fuzzifying derived sets and closures

Definition 6.1. Let (X, τ) be an OIVTS and let $A \in 2^X$. Then $A^{'}$ is called the *ordinary interval-valued fuzzifying derived set* of A, denoted by $A^{'}$, is an interval-valued fuzzy set in X defined as follows: for each $x \in X$,

$$x \in A^{'} := \forall B(B \in \mathcal{N}_x \to B \cap (A - \{x\} \neq \phi), \text{ i.e.,}$$

$$A^{'}(x) = \bigwedge_{B\cap(A-\{x\}),\ B\in 2^X} (\mathbf{1} - \mathcal{N}_x(B)) = \bigwedge_{B\cap(A-\{x\}),\ B\in 2^X} [1 - \mathcal{N}_x^{+}(B), 1 - \mathcal{N}_x^{-}(B)].$$

Example 6.2. (1) Let $X = \{a, b, c\}$, let $A = \{a, b\}$ and let (X, τ) be the OIVFTS defined in Example 3.2 (1). Then

$$\begin{aligned} A^{'}(a) &= \textstyle\bigwedge_{B\cap(A-\{a\}),\ B\in 2^X} [1 - \mathcal{N}_a^{+}(B), 1 - \mathcal{N}_a^{-}(B)] \\ &= [1 - \mathcal{N}_a^{+}(\{a\}), 1 - \mathcal{N}_a^{-}(\{a\})] \wedge [1 - \mathcal{N}_a^{+}(\{c\}), 1 - \mathcal{N}_a^{-}(\{c\})] \\ &\quad \wedge [1 - \mathcal{N}_a^{+}(\{a, c\}), 1 - \mathcal{N}_a^{-}(\{a, c\})] \end{aligned}$$

and from Example 4.2,

$$\mathcal{N}_a(\{a\}) = [0.2, 0.7],\ \mathcal{N}_a(\{a, c\})] = [0.3, 0.8].$$

On the other hand, we have $\mathcal{N}_a(\{c\}) = \bigvee_{a\in B\subset\{c\}} \tau(B) = \tau(\phi) = [1, 1]$. Thus

$$A^{'}(a) = [1 - 0.7, 1 - 0.2] \wedge [1 - 1, 1 - 1] \wedge [1 - 0.8, 1 - 0.3] = [0.0].$$

Similarly, we have $A^{'}(a) = [0, 0] = A^{'}(b)$ and $A^{'}(c) = [0.4, 0.7]$. So

$$A^{'} = \{(a, [0, 0]), (b, [0, 0]), (c, [0.4, 0.7])\}.$$

(2) Let (X, τ_ϕ) be the interval-valued fuzzifying indiscrete space (See Example 4.2 (2)). Suppose X has at least two points. Let $x \in X$ and let $A \in 2^X$. Then there is $\phi \neq B \in 2^X$ such that $B \cap A - \{x\} = \phi$. Since $B \neq X$, $\tau_\phi(B) = [0, 0]$. Thus by the definition of $A^{'}$, $A^{'}(x) = [1, 1]$. So $A^{'} = \mathbf{1}$.

Suppose X is a singleton set $\{x\}$. Then clearly, we have $B = \phi$ such that $B \cap A - \{x\} = \phi$. Thus $\tau_\phi(B) = \tau_\phi(\phi) = [1, 1]$. So $A^{'}(x) = [0, 0]$. Hence $A^{'} = \mathbf{0}$.

(3) Let (X, τ_X) be the interval-valued fuzzifying discrete space (See Example 4.2 (3)). Let $A \in 2^X$ and let $x \in X$. Consider $B \in 2^X$ such that $B \cap A - \{x\} = \phi$. Then clearly, by the definition of τ_X, $\tau_X(B) = [1, 1]$. Thus $A^{'}(x) = [0, 0]$. So $A^{'} = \mathbf{0}$.

Lemma 6.3. *Let (X, τ) be an OIVFTS and let $A \in 2^X$. Then for each $x \in X$,*

$$A^{'}(x) = \mathbf{1} - \mathcal{N}_x(A^c \cup \{x\}).$$

Proof. From Definition 6.1, it is clear. □

Theorem 6.4. *For each $A \in 2^X$, $\models A \in \mathcal{C} \leftrightarrow A^{'} \subset A$, i.e., $\mathcal{C}(A) = [A^{'} \subset A]$, where for each $B \in [I]^X$ and each $A \in 2^X$, $[B \subset A] = \bigwedge_{x\in A^c}(\mathbf{1} - B(x))$.*

Proof. Let $x \in X$. Then

$$\begin{aligned} [A^{'} \subset A] &= \textstyle\bigwedge_{x\in A^c}(\mathbf{1} - A^{'}(x)) \\ &= \textstyle\bigwedge_{x\in A^c} \mathcal{N}_x(A^c \cup \{x\}) \text{ [By Lemma 6.3]} \\ &= \textstyle\bigwedge_{x\in A^c} \mathcal{N}_x(A^c) \text{ [Since } x \in A^c] \\ &= \textstyle\bigwedge_{x\in A^c} \bigvee_{x\in C\subset A^c} \tau(C) \text{ [By Definition 4.1]} \\ &= \tau(A^c) \end{aligned}$$

$= \mathcal{C}(A)$ [By Proposition 3.4]
$= [A \in \mathcal{C}]$. □

Definition 6.5. Let (X, τ) be an OIVTS and let $A \in 2^X$. Then the *ordinary interval-valued fuzzifying closure* of A, denoted by $\bar{A}$, is an interval-valued fuzzy set in X defined as follows: for each $x \in X$,

$$x \in \bar{A} := \forall B(B \supset A) \wedge (B \in \mathcal{C}) \rightarrow x \in B), \text{ i.e.,}$$

$$\bar{A}(x) = \bigwedge_{x \notin B \supset A,\ B \in 2^X} (\mathbf{1} - \mathcal{C}(B)).$$

In fact, we can think that the ordinary interval-valued fuzzifying closure "$-$" is a mapping $^- : 2^X \rightarrow [I]^X$.

Lemma 6.6. *Let (X, τ) be an OIVFTS and let $A \in 2^X$. Then for each $x \in X$,*

$$\bar{A}(x) = \mathbf{1} - \mathcal{N}_x(A^c).$$

Proof. It follows directly from Proposition 3.4. □

Theorem 6.7. *Let (X, τ) be an OIVTS, let $x \in X$ and let $A \in 2^X$. Then*
(1) $\models \bar{A} \equiv A \cup A'$,
(2) $\models x \in \bar{A} \leftrightarrow \forall B(B \in \mathcal{N}_x \rightarrow A \cap B \neq \phi)$,
(3) $\models A \equiv \bar{A} \leftrightarrow A \in \mathcal{C}$.

Proof. (1) The proof is straightforward form Lemma 6.6.
(2) $[\forall B(B \in \mathcal{N}_x \rightarrow A \cap B \neq \phi)] = \bigwedge_{A \cap B = \phi}(\mathbf{1} - \mathcal{N}_x(B))$
$= \mathbf{1} - \bigvee_{A \cap B = \phi} \mathcal{N}_x(B)$
$= \mathbf{1} - \bigvee_{x \in C \subset B \subset A^c} \tau(C)$
$= \mathbf{1} - \mathcal{N}_x(A^c)$
$= \bar{A}(x)$. [By Lemma 6.6]
(3) It follows from Theorem 6.4 and (1). □

In order to distinguish an interval-valued fuzzy set from an ordinary set, we will denote interval-valued fuzzy sets as $\widetilde{A}$, $\widetilde{B}$, $\cdots$, etc. For each $\widetilde{A} \in [I]^X$ and $\widetilde{a} \in [I]$, $\widetilde{a}\widetilde{A}$ is the interval-valued fuzzy set in X defined as follows: for each $x \in X$,

$$(\widetilde{a}\widetilde{A})(x) = \widetilde{a} \wedge \widetilde{A}(x) = [a^- \wedge \widetilde{A}^-(x), a^+ \wedge \widetilde{A}^+(x)].$$

In fact, for each $A \in 2^X$ and each $\widetilde{a} \in [I]$, we can easily see that $\widetilde{a}A$ is the interval-valued fuzzy set in X given by:

$$\widetilde{a}A = \widetilde{a}[\chi_A, \chi_A].$$

Then for each $x \in X$,

$$(\widetilde{a}A) = \begin{cases} \widetilde{a} & \text{if } x \in A \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Definition 6.8. Let $\widetilde{A} \in [I]^X$ and let $\widetilde{a} \in [I]$. Then $[\widetilde{A}]_{\widetilde{a}}$ and $[\widetilde{A}]^*_{\widetilde{a}}$ are subsets of X defined as follows:
(i) $[\widetilde{A}]_{\widetilde{a}} = \{x \in X : \widetilde{A}(x) \geq \widetilde{a}\}$ is called the $\widetilde{a}$-*level subset* of X [32],
(ii) $[\widetilde{A}]^*_{\widetilde{a}} = \{x \in X : \widetilde{A}(x) > \widetilde{a}\}$ is called the $\widetilde{a}$-*strong level subset* of X.

It is obvious that for any $\widetilde{a},\ \widetilde{b} \in [I]$ such that $\widetilde{a} \preceq \widetilde{b}$,

$$[\widetilde{A}]_{\widetilde{a}} \supset [\widetilde{A}]_{\widetilde{b}},\ [\widetilde{A}]^*_{\widetilde{a}} \supset [\widetilde{A}]^*_{\widetilde{b}}.$$

Definition 6.9. Let (X, τ) be an OIVFTS and let $^- : 2^X \to [I]^X$ be the ordinary interval-valued fuzzifying closure mapping. Let $cl : [I]^X \to [I]^X$ be the mapping (will be called the *extension* of $-$) defined as follows: for each $\widetilde{A} \in [I]^X$,

$$cl(\widetilde{A}) = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}}}.$$

Suppose cl satisfies the following Kuratovski closure axioms: for any $\widetilde{A},\ \widetilde{B} \in [I]^X$,
(i) $cl(\mathbf{0}) = \mathbf{0}$,
(ii) $\widetilde{A} \subset cl(\widetilde{A})$,
(iii) $cl(\widetilde{A} \cup \widetilde{B}) = cl(\widetilde{A}) \cup cl(\widetilde{B})$,
(iv) $cl(cl(\widetilde{A})) \subset cl(\widetilde{A})$.
Then $^- : 2^X \to [I]^X$ is called an *ordinary interval-valued fuzzifying closure operator*.

Lemma 6.10. *Let (X, τ) be an OIVFTS and let $\widetilde{A} \in [I]^X$. Then*

$$cl(\widetilde{A}) = \bigcup_{x \in X} \widetilde{A}(x)\overline{[\widetilde{A}]_{\widetilde{A}(x)}}.$$

Proof. It follows directly from Definition 6.1 □

Proposition 6.11. *Let (X, τ) be an OIVFTS. Then $^-$ satisfies the following Kuratovski closure axioms: for any $A,\ B \in 2^X$,*
(1) $\overline{\phi} = \phi$,
(2) $A \subset \overline{A}$,
(3) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$,
(4) $\overline{\overline{A}} \subset \overline{A}$.

Proof. (1) $\overline{\phi}(x) = \mathbf{1} - \mathcal{N}_x(\phi^c)$ [By Lemma 6.6]
$= \mathbf{1} - \mathcal{N}_x(X)$
$= \mathbf{1} - \bigvee_{x \in B \subset X} \tau(B)$[By Definition 4.1]
$= \mathbf{1} - \tau(X) = \mathbf{1} - \mathbf{1}$
$= [0, 0] = [\chi_\phi, \chi_\phi](x)$.
Thus $\overline{\phi} = \phi$.
(2) The proof is straightforward from Theorem 6.7 (1).
(3) $\overline{(A \cup B)}(x) = \mathbf{1} - \mathcal{N}_x((A \cup B)^c)$
$= \mathbf{1} - \mathcal{N}_x(A^c \cap B^c)$
$= \mathbf{1} - \bigvee_{x \in C \subset A^c \cap B^c} \tau(C)$
$= \mathbf{1} - \bigvee_{x \in C_1 \subset A^c,\ x \in C_2 \subset B^c} \tau(C_1 \cap C_2)$
$\leq \mathbf{1} - \bigvee_{x \in C_1 \subset A^c,\ x \in C_2 \subset B^c} [(\tau(C_1) \wedge (\tau(C_2))]$
$= (\mathbf{1} - \bigvee_{x \in C_1 \subset A^c} \tau(C_1) \vee (\mathbf{1} - \bigvee_{x \in C_2 \subset B^c} \tau(C_2)$
$= (\mathbf{1} - \mathcal{N}_x(A^c)) \vee (\mathbf{1} - \mathcal{N}_x(B^c))$
$= \overline{A}(x) \vee \overline{B}(x)$
$= (\overline{A} \cup \overline{B})(x)$.
Then $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

Suppose $A,\ B \in 2^X$ such that $A \subset B$ and let $\widetilde{a} \in [I]$. Then

$$[A]_{\widetilde{a}} = \{x \in X : A(x) = [\chi_A(x), \chi_A(x)] \geq \widetilde{a}\} \subset [B]_{\widetilde{a}},$$

where $[A]_{\widetilde{a}} = \widetilde{a}$, if $x \in A$ and $[A]_{\widetilde{a}} = \phi$, if $x \notin A$. Thus by Definition 6.9,

$$\overline{A} = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[A]_{\widetilde{a}}} \subset \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[B]_{\widetilde{a}}} = \overline{B}.$$

Since $A \subset A \cup B$ and $B \subset A \cup B$, $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$. So

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B}.$$

Hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(4) Since $\overline{A} \in [I]^X$, by Lemma 6.10, $\overline{\overline{A}} = \bigcup_{x \in X} \overline{A}(x)\overline{[\overline{A}]_{\overline{A}(x)}}$. Then for each $y \in X$,

$$\overline{\overline{A}}(y) = \bigvee_{x \in X} [\overline{A}(x) \wedge \overline{[\overline{A}]_{\overline{A}(x)}}(y)].$$

For any $x \in X$, let $K_x = [\overline{A}]_{\overline{A}(x)}$. Then

$$\begin{aligned} \overline{A}(x) \wedge \overline{K_x}(y) &\leq \textstyle\bigwedge_{z \in K_x} \overline{A}(z) \wedge \overline{K_x}(y) \\ &= \textstyle\bigwedge_{z \in K_x} [(\mathbf{1} - \mathcal{N}_z(A^c)) \wedge (\mathbf{1} - \mathcal{N}_y(K_x^c))] \\ &= \mathbf{1} - \textstyle\bigvee_{z \in K_x} [\mathcal{N}_z(A^c) \vee \mathcal{N}_y(K_x^c)]. \end{aligned}$$

By the procedure of proof of Theorem 5.3 in [34],

$$\{D : \{z, y\} \subset D \subset A^c\} \subset \{B : y \in B \subset A^c\} \text{ for each } z \in K_x$$

and

$$\{D : y \in D \subset A^c,\ z \notin D,\ \text{for each } z \in K_x\} \subset \{C : y \in C \subset K_x^c\}.$$

Thus

$$\begin{aligned} &\textstyle\bigvee_{z \in K_x} [\mathcal{N}_z(A^c) \vee \mathcal{N}_y(K_x^c)] \\ &= \textstyle\bigvee_{z \in K_x} [\bigvee_{z \in B \subset A^c} \tau(B) \vee \bigvee_{y \in C \subset K_x^c} \tau(C)] \\ &\geq \textstyle\bigvee_{z \in K_x} [\bigvee_{\{y,z\} \subset D \subset A^c} \tau(D) \vee \bigvee_{y \in D \subset A^c,\ z \notin D \text{ for each } z \in K_x} \tau(D) \\ &= \textstyle\bigvee_{y \in D \subset A^c} \tau(D) \\ &= \mathcal{N}_y(A^c). \end{aligned}$$

Furthermore, $\overline{A}(x) \wedge \overline{K_x}(y) \leq \overline{A}(y)$ for each $x \in X$. So

$$\overline{\overline{A}}(y) = \bigvee_{x \in X} [\overline{A}(x) \wedge \overline{[\overline{A}]_{\overline{A}(x)}}(y) \leq \overline{A}(y).$$

Hence $\overline{\overline{A}} \subset \overline{A}$. □

Lemma 6.12. *Let (X, τ) be an OIVFTS and let $\widetilde{A} \in [I]^X$. Then $cl(\widetilde{A}) = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A}]^*_{\widetilde{a}}}$.*

Proof. $[\widetilde{A}]_{\widetilde{a}} \supset [\widetilde{A}]^*_{\widetilde{a}}$ for each $\widetilde{a} \in [I]$. Then $cl(\widetilde{A}) \supset \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A}]^*_{\widetilde{a}}}$. For each $\widetilde{a} \in [I]$, let $\widetilde{a_n} \in [0, a_n^-) \times [0, a_n^+)$ $(n = 1,\ 2,\ \cdots)$ such that $\widetilde{a_n} \uparrow \widetilde{a}$, i.e., $a_n^- \uparrow a^-$ and $a_n^+ \uparrow a^+$. Then clearly, $[\widetilde{A}]^*_{\widetilde{a_n}} \supset [\widetilde{A}]_{\widetilde{a}}$. Thus $\overline{[\widetilde{A}]^*_{\widetilde{a_n}}} \supset \overline{[\widetilde{A}]_{\widetilde{a}}}$. So $\bigcup_{n=1}^{\infty} \widetilde{a_n}\overline{[\widetilde{A}]^*_{\widetilde{a_n}}} \supset \overline{[\widetilde{A}]_{\widetilde{a}}}$. Hence

$$\bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A}]^*_{\widetilde{a}}} \supset \bigcup_{\widetilde{a} \in [I]} (\bigcup_{n=1}^{\infty} \widetilde{a_n}\overline{[\widetilde{A}]^*_{\widetilde{a_n}}}) \supset \bigcup_{\widetilde{a} \in [I]} \overline{[\widetilde{A}]_{\widetilde{a}}} = cl(\widetilde{A}).$$

Therefore $cl(\widetilde{A}) = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A}]^*_{\widetilde{a}}}$. □

Lemma 6.13. *Let (X, τ) be an OIVFS, let $\widetilde{A} \in [I]^X$ and let $\widetilde{a} \in [I]$. Then*

$$\widetilde{a}[cl(\widetilde{A})]^*_{\widetilde{a}} \subset \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}}}.$$

Proof. Let $x \in [cl(\widetilde{A})]^*_{\widetilde{a}}$. Then

$$[cl(\widetilde{A})](x) = [\bigcup_{\widetilde{b}\in[I]} \widetilde{b}\overline{[\widetilde{A}]_{\widetilde{b}}}](x) = \bigvee_{\widetilde{b}\in[I]} [b \wedge \overline{[\widetilde{A}]_{\widetilde{b}}}(x)] > \widetilde{a}.$$

Thus there is $\widetilde{b_0} \in [I]$ such that $b_0 \wedge \overline{[\widetilde{A}]_{\widetilde{b_0}}}(x) > \widetilde{a}$, i.e., $b_0 > \widetilde{a}$ and $\overline{[\widetilde{A}]_{\widetilde{b_0}}}(x) > \widetilde{a}$. So $\overline{[\widetilde{A}]_{\widetilde{a}}}(x) \geq \overline{[\widetilde{A}]_{\widetilde{b_0}}}(x) > \widetilde{a}$. Hence $(\widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}}})(x) = \widetilde{a} \wedge \overline{[\widetilde{A}]_{\widetilde{a}}}(x) = \widetilde{a} = (\widetilde{a}[cl(\widetilde{A})]^*_{\widetilde{a}})(x)$. Therefore $\widetilde{a}[cl(\widetilde{A})]^*_{\widetilde{a}} \subset \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}}}$. □

Lemma 6.14. *Let (X, τ) be an OIVFTS, let $\widetilde{A} \in [I]^X$ and let $\widetilde{a} \in [I]$. Then*

$$\widetilde{a}cl(\widetilde{A}) = cl(\widetilde{a}\widetilde{A}).$$

Proof. Let $\widetilde{b} \in [I]$ such that $b^- \in [0, a^-)$ and $b^+ \in [0, a^+)$. Then

$$\begin{aligned}[\widetilde{a}\widetilde{A}]_{\widetilde{b}} &= \{x \in X : [\widetilde{a}\widetilde{A}](x) \geq \widetilde{b}\} \\ &= \{x \in X : \widetilde{a} \wedge \widetilde{A}(x) \geq \widetilde{b}\} \\ &= \{x \in X : \widetilde{A}(x) \geq \widetilde{b}\} \text{ [Since } \widetilde{b} < \widetilde{a}] \\ &= [\widetilde{A}]_{\widetilde{b}}.\end{aligned}$$

Thus

(6.14.1) $\quad [\widetilde{a}\widetilde{A}]_{\widetilde{b}} = [\widetilde{A}]_{\widetilde{b}}$, for each $\widetilde{b} \in [I]$ such that $b^- \in [0, a^-)$ and $b^+ \in [0, a^+)$.

Now let $\widetilde{b} \in [I]$ such that $b^- \in (a^-, 1]$ and $b^+ \in (a^+, 1]$. Then

$$\begin{aligned}[\widetilde{a}\widetilde{A}]_{\widetilde{b}} &= \{x \in X : [\widetilde{a}\widetilde{A}](x) \geq \widetilde{b}\} \\ &= \{x \in X : \widetilde{a} \wedge \widetilde{A}(x) \geq \widetilde{b}\} \\ &= \{x \in X : \widetilde{a} \geq \widetilde{b}, \widetilde{A}(x) \geq \widetilde{b}\} \\ &= \phi. \text{ [Since } \widetilde{a} < \widetilde{b}]\end{aligned}$$

Thus

(6.14.2) $\quad [\widetilde{a}\widetilde{A}]_{\widetilde{b}} = \phi$, for each $\widetilde{b} \in [I]$ such that $b^- \in (a^-, 1]$ and $b^+ \in (a^+, 1]$.

Let $\widetilde{b} \in [I]$ such that $\widetilde{b} \succ \widetilde{a}$. Then clearly, $[\widetilde{A}]_{\widetilde{b}} \subset [\widetilde{A}]_{\widetilde{a}}$. Thus $\overline{[\widetilde{A}]_{\widetilde{b}}} \subset \overline{[\widetilde{A}]_{\widetilde{a}}}$. So by (6.14.2), $\bigcup_{\widetilde{b}\in[I],\ b^-\in(a^-,1],\ b^+\in(a^+,1]} \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{b}}} \subset \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}}}$. Hence

$$\begin{aligned}\widetilde{a}cl(\widetilde{A}) &= \widetilde{a}\bigcup_{\widetilde{b}\in[I]} \widetilde{b}\overline{[\widetilde{A}]_{\widetilde{b}}} \text{ [By Definition 6.9]} \\ &= \bigcup_{\widetilde{b}\in[I]} (\widetilde{a} \wedge \widetilde{b})\overline{[\widetilde{A}]_{\widetilde{b}}} \\ &= (\bigcup_{\widetilde{b}\in[I],\ b^-\in[0,a^-],\ b^+\in[0,a^+]} \widetilde{b}\overline{[\widetilde{A}]_{\widetilde{b}}}) \cup (\bigcup_{\widetilde{b}\in[I],\ b^-\in(a^-,1],\ b^+\in(a^+,1]} \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{b}}}) \\ &= \bigcup_{\widetilde{b}\in[I],\ b^-\in[0,a^-],\ b^+\in[0,a^+]} \widetilde{b}\overline{[\widetilde{A}]_{\widetilde{b}}} \text{ [By (6.14.1) and (6.14.2)]} \\ &= \bigcup_{\widetilde{b}\in[I]} \widetilde{b}\overline{[\widetilde{a}\widetilde{A}]_{\widetilde{b}}} \text{ [By (6.14.2)]} \\ &= cl(\widetilde{a}\widetilde{A}).\end{aligned}$$

□

Lemma 6.15. *Let $\widetilde{A}^{(\widetilde{a})}, \widetilde{B} \in [I]^X$ $(\widetilde{a} \in [I])$. If $\widetilde{A}^{(\widetilde{a})} \supset [\widetilde{B}]_{\widetilde{a}}$ $(\widetilde{a} \in [I])$ and $\bigcup_{\widetilde{a}\in[I]} \widetilde{a}\widetilde{A}^{(\widetilde{a})} = \bigcup_{\widetilde{a}\in[I]} \widetilde{a}[\widetilde{B}]_{\widetilde{a}}$, then $[\widetilde{A}^{(\widetilde{a})} \equiv [\widetilde{B}]_{\widetilde{a}}] \geq \mathbf{1} - \widetilde{a} = [1 - a^+, 1 - a^-]$.*

Proof. $\widetilde{A}^{(\widetilde{a})} \equiv [\widetilde{B}]_{\widetilde{a}} := (\widetilde{A}^{(\widetilde{a})} \subset [\widetilde{B}]_{\widetilde{a}}) \wedge ([\widetilde{B}]_{\widetilde{a}} \subset \widetilde{A}^{(\widetilde{a})})$
$:= [\forall x(x \in \widetilde{A}^{(\widetilde{a})} \to x \in [\widetilde{B}]_{\widetilde{a}})] \wedge [\forall x(x \in [\widetilde{B}]_{\widetilde{a}} \to x \in \widetilde{A}^{(\widetilde{a})})]$.
Then $[\widetilde{A}^{(\widetilde{a})} \equiv [\widetilde{B}]_{\widetilde{a}}] = (\bigwedge_{x \notin [\widetilde{B}]_{\widetilde{a}}} (\mathbf{1} - \widetilde{A}^{(\widetilde{a})}(x))) \wedge (\bigwedge_{x \in [\widetilde{B}]_{\widetilde{a}}} \widetilde{A}^{(\widetilde{a})}(x) \geq [\widetilde{B}]_{\widetilde{a}}(x))$
$= \bigwedge_{x \notin [\widetilde{B}]_{\widetilde{a}}} (\mathbf{1} - \widetilde{A}^{(\widetilde{a})}(x))$. [Since $(\bigwedge_{x \in [\widetilde{B}]_{\widetilde{a}}} \widetilde{A}^{(\widetilde{a})}(x) = \mathbf{1}$.]

Assume that $[\widetilde{A}^{(\widetilde{a})} \equiv [\widetilde{B}]_{\widetilde{a}}] < \mathbf{1} - \widetilde{a}$. Then there is $x_0 \notin [\widetilde{B}]_{\widetilde{a}}$ such that $\widetilde{A}^{(\widetilde{a})}(x_0) > \widetilde{a}$. Thus $(\bigcup_{\widetilde{b} \in [I]} \widetilde{b}\widetilde{A}^{(\widetilde{b})})(x_0) \geq \widetilde{a}$. On the other hand,

$$(\bigcup_{\widetilde{b} \in [I]} \widetilde{b}[\widetilde{B}]_{\widetilde{b}})(x_0) = \bigvee_{\widetilde{b} \in [I],\ b^- \in [0,a^-],\ b^+ \in [0,a^+]} (\widetilde{b} \wedge [\widetilde{B}]_{\widetilde{b}}(x_0)) < \widetilde{a}.$$

Suppose $\bigvee_{\widetilde{b} \in [I],\ b^- \in [0,a^-),\ b^+ \in [0,a^+)} [\widetilde{B}]_{\widetilde{b}}(x_0) \geq \widetilde{a}$. Then for any $\widetilde{b} \in [I]$ such that $b^- \in [0, a^-)$ and $b^+ \in [0, a^+)$, $x_0 \in [\widetilde{B}]_{\widetilde{b}}$. Thus

$$x_0 \in \bigcap_{\widetilde{b} \in [I],\ b^- \in [0,a^-),\ b^+ \in [0,a^+)} [\widetilde{B}]_{\widetilde{b}} = [\widetilde{B}]_{\widetilde{a}}.$$

This is a contradiction. This completes the proof. □

The following theorem shows that an ordinary interval-valued fuzzifying closure operator completely determines an OIVFT τ and that in τ, the operator is the closure.

Theorem 6.16. *Let X be an OIVFTS. Then $^-: 2^X \to [I]^X$ is an ordinary interval-valued fuzzifying closure operator.*

Conversely, let $-^*: 2^X \to [I]^X$ be an ordinary interval-valued fuzzifying closure operator on X and let $\tau: 2^X \to [I]^X$ be the mapping defined as follows: for each $A \in [I]^X$,

$$A \in \tau := \overline{A^c}^* \equiv A^c, \text{ i.e., } \tau(A) = [\overline{A^c}^* \equiv A^c].$$

Then τ is an oivt. Moreover, for each $A \in [I]^X$, $\overline{A}^* = \overline{A}$, where $\overline{A}$ denotes the ordinary interval-valued fuzzifying closure with respect to τ.

Proof. (⇒): Let $cl : [I]^X \to [I]^X$ be the extension of $^-$. Then we will prove that cl satisfies the Kuratovski closure axioms.

(i) $cl(\mathbf{0}) = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\mathbf{0}]_{\widetilde{a}}}$
$= \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{\phi}$
$= \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\mathbf{0}$ [By Proposition 6.11 (1)]
$= \mathbf{0}$.

(ii) Let $\widetilde{A} \in [I]^X$. Then by Definition 6.9 and Proposition 6.11 (2),

$$cl(\widetilde{A}) = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}}} \supset \bigcup_{\widetilde{a} \in [I]} \widetilde{a}[\widetilde{A}]_{\widetilde{a}} = \widetilde{A}.$$

(iii) Let $\widetilde{A},\ \widetilde{B} \in [I]^X$. Then clearly, by the procedure of the proof of Proposition 6.11 (3),

$$cl(\widetilde{A} \cup \widetilde{B}) \supset cl(\widetilde{A}) \cup cl(\widetilde{B}).$$

On the other hand,

$cl(\widetilde{A} \cup \widetilde{B}) = \bigcup_{\widetilde{a} \in [I]} \widetilde{a}\overline{[\widetilde{A} \cup \widetilde{B}]_{\widetilde{a}}}$

$= \bigcup_{\widetilde{a}\in[I]} \widetilde{a}\overline{[\widetilde{A}]_{\widetilde{a}} \cup [\widetilde{B}]_{\widetilde{a}}}$ [Since $[\widetilde{A} \cup \widetilde{B}]_{\widetilde{a}} = [\widetilde{A}]_{\widetilde{a}} \cup [\widetilde{B}]_{\widetilde{a}}$]
$\subset \bigcup_{\widetilde{a}\in[I]} (\widetilde{A}]_{\widetilde{a}} \cup [\widetilde{B}]_{\widetilde{a}})$ [By Proposition 6.11 (3)]
$= (\bigcup_{\widetilde{a}\in[I]} \widetilde{A}]_{\widetilde{a}}) \cup (\bigcup_{\widetilde{a}\in[I]} \widetilde{B}]_{\widetilde{a}})$
$= cl(\widetilde{A}) \cup cl(\widetilde{B})$.

Thus $cl(\widetilde{A} \cup \widetilde{B}) = cl(\widetilde{A}) \cup cl(\widetilde{B})$.

(iv) Let $\widetilde{A} \in [I]^X$. Then

$cl(cl(\widetilde{A})) = \bigcup_{\widetilde{a}\in[I]} \widetilde{a}\overline{[cl(\widetilde{A})]^*_{\widetilde{a}}}$ [By Lemma 6.12]
$= \bigcup_{\widetilde{a}\in[I]} \widetilde{a}\overline{[\bigcup_{\widetilde{b}\in[I]} \widetilde{b}\overline{[A]^*_{\widetilde{b}}}]^*_{\widetilde{a}}}$
$= \bigcup_{\widetilde{a}\in[I]} \bigcup_{\widetilde{b}\in[I]} \widetilde{a}\widetilde{b}\overline{[\overline{[A]^*_{\widetilde{b}}}]^*_{\widetilde{a}}}$ [By Proposition 6.11 (3)]
$= \bigcup_{\widetilde{a}\in[I]} \widetilde{a}\overline{\overline{[A]^*_{\widetilde{a}}}}$ [Since either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{b} \leq \widetilde{a}$, say $\widetilde{a} \leq \widetilde{b}$]
$= \bigcup_{\widetilde{a}\in[I]} \overline{\widetilde{a}\overline{[A]^*_{\widetilde{a}}}}$ [By Lemma 6.14]
$\subset \bigcup_{\widetilde{a}\in[I]} \overline{\widetilde{a}\overline{[A]_{\widetilde{a}}}}$ [By Lemma 6.13]
$= \bigcup_{\widetilde{a}\in[I]} \widetilde{a}\overline{\overline{[A]_{\widetilde{a}}}}$
$\subset \bigcup_{\widetilde{a}\in[I]} \widetilde{a}\overline{[A]_{\widetilde{a}}}$ [By Proposition 6.11 (4)]
$= cl(\widetilde{A})$.

Thus $cl(cl(\widetilde{A})) \subset cl(\widetilde{A})$. So $-$ is an ordinary interval-valued fuzzifying closure operator.

($\Leftarrow$): Let $\mathcal{C} : 2^X \to [I]^X$ be the mapping defined as follows: for each $A \in 2^X$,

$$A \in \mathcal{C} := \overline{A}^* \equiv A, \text{ i.e., } \mathcal{C}(A) = [\overline{A}^* \equiv A].$$

(OIVCFT1) By Definition 6.9 (i), $\overline{\phi}^* = \mathbf{0} \equiv \phi$. Then $[\overline{\phi}^* \equiv \phi] = \mathbf{1}$. Thus $\mathcal{C}(\phi) = \mathbf{1}$. Moreover, by Theorem 6.7 (1), $\overline{X}^* \equiv X$, i.e., $[\overline{X}^* \equiv X] = \mathbf{1}$. So $\mathcal{C}(X) = \mathbf{1}$. Hence $\mathcal{C}$ satisfies the axiom (OIVCT1).

(OIVCFT2) Let $A,\ B \in 2^X$. Then

$\mathcal{C}(A \cup B) = [\overline{A \cup B}^* \equiv A \cup B]$
$= [\overline{A}^* \cup \overline{B}^* \equiv A \cup B]$ [By Proposition 6.11 (3)]
$\geq [\overline{A}^* \equiv A] \wedge [\overline{B}^* \equiv B]$
$= \mathcal{C}(A) \wedge \mathcal{C}(B)$.

Thus $\mathcal{C}$ satisfies the axiom (OIVCFT2).

(OIVCFT3) The proof is similar to (c) of Theorem 5.3 in [34].

Finally, we show that $\overline{A}^* = \overline{A}$ for each $A \in 2^X$, where $-$ denotes the ordinary interval-valued fuzzifying closure with respect to τ. Let $A \in 2^X$ and let $x \in X$. Then

$$\overline{A}(x) = \mathbf{1} - \mathcal{N}_x(A^c) = \mathbf{1} - \bigvee_{x \in B \subset A^c} \tau(B) = \mathbf{1} - \bigvee_{x \in B \subset A^c} [\overline{B^c}^* \equiv B^c].$$

Thus $\overline{A}(x) = \bigwedge_{y \in B \subset A^c} (\mathbf{1} - [\overline{B^c}^* \equiv B^c])$. Since $B^c \subset \overline{B^c}^*$,

$$[\overline{B^c}^* \equiv B^c] = \bigwedge_{y \in B} [\mathbf{1} - \overline{B^c}^*](y) = \mathbf{1} - \bigvee_{y \in B} \overline{B^c}^*(y).$$

So

$$\overline{A}(x) = \bigwedge_{x\in B\subset A^c} \bigvee_{y\in B} \overline{B^c}^*(y). \tag{6.16.1}$$

Let $\mathcal{B} = \{B : x \in B \subset A^c\}$ and let $f_0 : \mathcal{B} \to \bigcup \mathcal{B}$ be the mapping defined by:

$$f_0(B) = x \text{ for each } B \in \mathcal{B}.$$

Then

$$\begin{aligned}\overline{A}(x) &= \textstyle\bigwedge_{B\in\mathcal{B}} \bigvee_{f\in\Pi_{B\in\mathcal{B}}B} \overline{B^c}^*(f(B)) \text{ [By (6.16.1)]}\\ &= \textstyle\bigvee_{f\in\Pi_{B\in\mathcal{B}}B} \bigwedge_{B\in\mathcal{B}} \overline{B^c}^*(f(B))\\ &\geq \textstyle\bigwedge_{B\in\mathcal{B}} \overline{B^c}^*(f_0(B))\\ &= \textstyle\bigwedge_{B\in\mathcal{B}} \overline{B^c}^*(x)\\ &= \textstyle\bigwedge_{x\notin A\subset B^c} \overline{B^c}^*(x) \text{ [By the definition of } \mathcal{B}]\\ &\geq \overline{A}^*(x).\end{aligned}$$

Thus $\overline{A} \supset \overline{A}^*$.

Now let $A \in 2^X$, $\widetilde{B} \in [I]^X$ such that $\widetilde{B} \supset A$ and $\overline{\widetilde{B}}^* = \widetilde{B}$. Let $x \in X$ and for any positive integer n, let $\widetilde{b} = \mathbf{1} - \widetilde{B}(x) - [\frac{1}{n}, \frac{1}{n}] \geq \mathbf{0}$. Then clearly, $x \notin [\widetilde{B}]_{\mathbf{1}-\widetilde{b}}$. Since $A = [\chi_A, \chi_A] \subset \widetilde{B}$, $A \subset [\widetilde{B}]_{\mathbf{1}-\widetilde{b}}$. Thus

$$\bigvee_{x\notin D\supset A} [\overline{D}^* \equiv D] \geq [\overline{[\widetilde{B}]_{\mathbf{1}-\widetilde{b}}}^* \equiv [\widetilde{B}]_{\mathbf{1}-\widetilde{b}}].$$

On the other hand,

$$\bigvee_{\widetilde{a}\in[I]} \overline{\widetilde{a}[\widetilde{B}]_{\widetilde{a}}}^* = \overline{\widetilde{B}}^* = \widetilde{B} = \bigvee_{\widetilde{a}\in[I]} \widetilde{a}[\widetilde{B}]_{\widetilde{a}} \text{ and } \overline{[\widetilde{B}]_{\widetilde{a}}}^* \supset [\widetilde{B}]_{\widetilde{a}}.$$

By Lemma 6.15, $[\overline{[\widetilde{B}]_{\mathbf{1}-\widetilde{b}}}^* \equiv [\widetilde{B}]_{\mathbf{1}-\widetilde{b}}] \geq \widetilde{b}$. So

$$\bigvee_{x\notin D\supset A} [\overline{D}^* \equiv D] \geq \mathbf{1} - \widetilde{B}(x) - [\frac{1}{n}, \frac{1}{n}].$$

Let $n \to \infty$. Then clearly, $\bigvee_{x\notin D\supset A}[\overline{D}^* \equiv D] \geq \mathbf{1} - \widetilde{B}(x)$. Moreover,

$$\overline{A}(x) = \mathbf{1} - \bigvee_{x\notin D\supset A} [\overline{D}^* \equiv D] \leq \widetilde{B}(x).$$

So $\overline{A} \subset \overline{A}^*$. Hence $\overline{A} = \overline{A}^*$. □

7. Conclusions

We defined an ordinary interval-valued fuzzifying topology and level set of an OIVFT, and obtain some their basic properties and gave some examples. Second, we introduced the concept of ordinary interval-valued fuzzifying neighborhood systemS and and we proved that an ordinary interval-valued fuzzifying neighborhood system has the same properties in a classical neighborhood system (See Theorem 4.7). Third, we defined an ordinary interval-valued fuzzifying base and an ordinary interval-valued fuzzifying subbase, and obtain two characterization of an ordinary interval-valued fuzzifying base (See Theorems 5.3 and 5.4) and one characterization of an ordinary interval-valued fuzzifying subbase (See Theorem 5.12), and gave some their examples. Finally, we proved that an ordinary interval-valued fuzzifying topology induced by an ordinary interval-valued fuzzifying closure operator (See Theorem 6.16).

In the future, by defining the mapping (will be called an *interval-valued fuzzifying toplogy* on X) $\tau : [I]^X \to [I]$ satisfying the following axioms: for any $A,\ B \in [I]^X$ and any $(A_j)_{j\in J} \subset [I]^X$,

(i) $\tau(\widetilde{0}) = \tau(\widetilde{1}) = [1,1]$,

(ii) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$,

(iii) $\tau(\bigcup_{j\in J} A_j) \geq \bigwedge_{j\in J} \tau(A_j)$,

we will try to obtain various its properties and find some relations among ordinary interval-valued fuzzifying topologies and interval-valued fuzzifying topologies.

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