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Abstract. This paper examines the logical steps to demonstrate how the category **GTOP**, consisting of all crisp generalized topological spaces and all generalized continuous mappings is embedded in the category **ESGTOP**, consisting of all soft generalized topological spaces and all soft generalized continuous mappings.

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1. Introduction

Crisp data are not always required to solve problems in researches. In order to deal with uncertainty mathematically, various theories have been developed, including such as fuzzy set theory [1, 2], intuitionistic fuzzy sets [3], and rough sets [4]. All of these hypotheses have issues, comparable to those mentioned by Molodtsov in [5]. To deal with ambiguity and uncertainties, Molodtsov introduced the idea of soft set theory as a new theory and mathematical instrument. The soft set theory was successfully implemented by Molodtsov in a number of ways. Many studies that are defined over initial universal sets with a specified set of parameters have presented and examined the idea of soft topological spaces (See [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]). For some important concepts of generalized topology and generalized continuity, see [23]. Thomas and John [19] introduced soft generalized topology, soft generalized neighborhood systems and some properties and Öztürk [24] introduced and studied some soft mappings on soft generalized topological spaces. The contents of this paper is organized as follows: In section two, I recall some fundamental concepts and related definitions, moreover, some important results are outlined. In section three, by some logical steps, it is obtained the main result that"the category **GTOP** of the crisp generalized topological spaces is embedded in the category **ESGTOP**".

2. Preliminaries and some important results

Definition 2.1 ([25, 26, 27, 28])**.** A *category* is a quadruple $\mathbb{C} = (\partial, hom, id, \circ)$ consisting of:

(i) a class ∂, whose members are called $\mathbb{C}$-*objects*,
(ii) for each pair (A, B) of $\mathbb{C}$-objects, a set $hom(A, B)$, whose members are called $\mathbb{C}$-*morphisms from* A *to* B,
(iii) for each $\mathbb{C}$-object A, a morphism $idA : A \to A$, called the $\mathbb{C}$-*identity on* A,
(iv) a composition law associating with each $\mathbb{C}$-morphism $f : A \to B$ and each $\mathbb{C}$-morphism $g : B \to C$ an $\mathbb{C}$-morphism $g \circ f : A \to C$, called the *composite* of f and g, subject to the following conditions:
   a. composition is associative,
   b. $\mathbb{C}$-identities act as identities with respect to composition,
   c. the sets $hom(A, B)$ are pairwise disjoint.

For more information about the notions of category theory, see[25] and [26].

Definition 2.2 ([5])**.** Let E be a set of parameters. A pair (F, A) is called a *soft set* over the initial universe set U, if $F : A \longrightarrow P(U)$ is a mapping from $A \subset E$ to the power set $P(U)$ of U.

Each soft set (F, A) will be reformulated to be a soft set of the form (K, E), where

(1) $K(e) = F(e)$ for all $e \in A$,
(2) $K(e) = \varnothing$ for all $e \in E - A$.

Definition 2.3 ([29])**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a *soft subset* of (G, B), denoted by $(F, A) \widetilde{\subset} (G, B)$, if

(i) $A \subseteq B$,
(ii) for all $e \in A, F(e) \subseteq G(e)$.

Definition 2.4 ([29])**.** Two soft sets (F, A) and (G, B) over a common universe set U are said to be *soft equal*, denoted by $(F, A) = (G, B)$, if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.5 ([29])**.** A soft set (F, A) over U is said to be a *null soft set*, denoted by Φ, if for all $e \in A, F(e) = \varnothing$.

If $A = E$, then Φ will denoted by $\widetilde{\Phi}$.

Definition 2.6 ([29])**.** A soft set (F, A) over U is said to be an *absolute soft set*, denoted by $\widetilde{A}$, if for all $e \in A, F(e) = U$.

If $A = E$, then $\widetilde{A}$ will denoted by $\widetilde{U}$.

Definition 2.7 ([29])**.** The *union* of two soft sets (F, A) and (G, B) over the common universe set U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & e \in A - B, \\ G(e) & e \in B - A, \\ F(e) \cup G(e) & e \in A \cap B. \end{cases}$$

We write $(F, A) \widetilde{\cup} (G, B) = (H, C)$.

Definition 2.8 ([29])**.** The *intersection* (H, C) of two soft sets (F, A) and (G, B) over a common universe set U, denoted $(F, A) \widetilde{\cap} (G, B)$, is defined as follows:

$$C = A \cap B \text{ and } H(e) = F(e) \cap G(e) \text{ for all } e \in C.$$

Definition 2.9 ([29])**.** The *relative complement* of a soft set (F, A), denoted by $(F, A)^c$, is defined by $(F, A)^c = (F^c, A)$, where $F^c : A \longrightarrow P(U)$ is a mapping given by:

$$F^c(e) = U - F(e) \text{ for all } e \in A.$$

The following is an immediate consequence of the above definitions.

Proposition 2.10. *Let (F, E) and (G, E) be the two soft sets over U. Then*

(1) $((F, E) \widetilde{\cap} (G, E))^c = (F, E)^c \widetilde{\cup} (G, E)^c$,

(2) $((F, E) \widetilde{\cup} (G, E))^c = (F, E)^c \widetilde{\cap} (G, E)^c$.

Definition 2.11. The soft set $(\widetilde{H}, E)$ or simply $\widetilde{H}$, where $H \subseteq U$ is said to be a *uniform soft set* over U, if for all $e \in E$, $\widetilde{H}(e) = H$.

Notation 2.12. In our study, the set of all soft sets and uniforms soft sets over a common universe U parameterized by the set E will be denoted by **S(U,E)** and **UNIS(U,E)** respectively.

Definition 2.13. The soft set $(F, A) = H_e$ is called a *soft point* over U, if $\phi \neq H \subseteq U$ and for all $e^{'} \in A$,

$$F(e^{`}) = \begin{cases} H & e^{'} = e, \\ \phi & e^{'} \neq e. \end{cases}$$

Definition 2.14. The soft point H_e over U is called a *soft singleton* over U, if $H = \{x\}$ for some $x \in U$.

Definition 2.15. The soft point H_e over U *is contained in* a soft set (F, A) over U, if $H \subseteq F(e)$ and we write $H_e \widetilde{\in} (F, A)$.

Notation 2.16. In our study, the set of all soft points over a common universe U parameterized by the set E will be denoted by **SP(U,E)**.

For more information about the properties of operations on the set **S(U,E)**, see [10].

Definition 2.17 ([10])**.** Let U and Y be two initial universal sets and let E and M be two sets of parameters, $f : U \longrightarrow Y$, and $g : E \longrightarrow M$, then we denote the soft mapping (f, g) : **S(U,E))** $\longrightarrow$ **S(Y,M)** for which

(i) for each $(F,E) \in \mathbf{S(U,E))}$ and for every $m \in M$, $(f,g)^{\rightarrow}(F,E) = (G,M)$, where

$$G(m) = \begin{cases} \bigcup\{f(F(e)) : e \in g^{-1}(\{m\})\} & g^{-1}(\{m\})\} \neq \phi, \\ \phi & g^{-1}(\{m\})\} = \phi, \end{cases}$$

(ii) for each $(G,M) \in \mathbf{S(Y,M))}$ and for every $e \in E$, $(f,g)^{\leftarrow}(G,E) = (F,E)$, where $F(e) = f^{-1}(G(g(e)))$.

Remark 2.18. In the above definition of soft mapping, the function $g : E \longrightarrow M$ is called a *parameters function*.

Proposition 2.19. *Let $(f,g) : \boldsymbol{S(U,E))} \longrightarrow \boldsymbol{S(Y,M)}$ be a soft mapping. Then*

(1) *for each $H_e \in \boldsymbol{SP(U,E))}$,*

$$(f,g)^{\rightarrow}(H_e) = f(H)_{g(e)} \in \boldsymbol{SP(Y,M))},$$

(2) *for each $L_m \in \boldsymbol{SP(Y,M))}$,*

$$(f,g)^{\leftarrow}(L_m) = \{(f^{-1}(L))_e \in \boldsymbol{SP(U,E))} \ : \ g(e) = m\} \subset \boldsymbol{SP(U,E))}.$$

Remark 2.20. Proposition 2.19 shows that the inverse image of any soft point over Y under the soft mapping (f,g) equal the family of soft points over U. This family of soft points will be restricted to only one soft point if the function g is injective function.

Proposition 2.21. *Let $(f_1,g_1) : \boldsymbol{S(X,E)} \longrightarrow \boldsymbol{S(Y,M)}$ and $(f_2,g_2) : \boldsymbol{S(Y,M)} \longrightarrow \boldsymbol{S(Z,K)}$ be two soft mappings. Then the composition soft mapping*

$$(f_2,g_2) \circ (f_1,g_1) = (f_2 \circ f_1, g_2 \circ g_1) : \boldsymbol{S(X,E)} \longrightarrow \boldsymbol{S(Z,K)}.$$

Proposition 2.22. *Let $(f_1,g_1) : \boldsymbol{S(X,E)} \longrightarrow \boldsymbol{S(Y,M)}$ and $(f_2,g_2) : \boldsymbol{S(Y,M)} \longrightarrow \boldsymbol{S(Z,K)}$ be two soft mappings. Then for each $(L,K) \in \boldsymbol{S(Z,K)}$,*

$$((f_2,g_2) \circ (f_1,g_1))^{\leftarrow}(L,K) = (f_1,g_1)^{\leftarrow}((f_2,g_2)^{\leftarrow}(L,K))).$$

Proposition 2.23. *Let $(f,g) : \boldsymbol{S(U,E))} \longrightarrow \boldsymbol{S(Y,M)}$ be a soft mapping, then for all $(G,M) \in \boldsymbol{S(Y,M)}$, $((f,g)^{\leftarrow}(G,M))^c = (f,g)^{\leftarrow}((G,M)^c)$.*

For more information about the properties of images and inverse images of soft sets under soft mappings, see [10].

Definition 2.24 ([23])**.** Let U be a non-empty universal set, then we call $\tau \subseteq P(U)$ a *generalized topology* on U, if it satisfies the following conditions:

(i) $\phi \in \tau$,
(ii) $\forall\ A_i \subseteq U, i \in \Delta,\ if\ A_i \in \tau,\ then\ \bigcup_{i \in \Delta} A_i \in \tau$, where Δ is an index set.

For more information about the properties of generalized topological space, see [23].

Definition 2.25 ([19])**.** $\widetilde{\tau} \subseteq \mathbf{S(U,E)}$ is said to be a *soft generalized topology* on U, if it satisfies the following conditions:

(i) $\widetilde{\Phi} \in \widetilde{\tau}$,
(ii) the union of any number of soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

The triplet $(U, \widetilde{\tau}, E)$ is called a *soft generalized topological space* over U. The soft generalized topology $\widetilde{\tau}$ on U is called a *soft topology* on U, if the intersection of any two soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.

Definition 2.26. The soft generalized topology $\widetilde{\tau}$ on U is called a *uniform soft generalized topology* on U, if $\widetilde{\tau} \subseteq$ **UNIS(U,E)**.

Definition 2.27 ([19])**.** Let $(U, \widetilde{\tau}, E)$ be a soft generalized topological space over U. Then the members of $\widetilde{\tau}$ are said to be *soft generalized open sets in* U.

Definition 2.28 ([19])**.** Let $(U, \widetilde{\tau}, E)$ be a soft generalized topological space over U. A soft set (F, A) over U is said to be a *soft generalized closed set in* U, if its relative complement $(F, E)^c$ belongs to $\widetilde{\tau}$.

The set of all soft generalized closed sets in $(U, \widetilde{\tau}, E)$ will be denoted by $\widetilde{\tau}^c$.

Proposition 2.29 ([19])**.** *Let $(U, \widetilde{\tau}, E)$ be a soft generalized topological space over U. Then*

(1) $\widetilde{U} \in \widetilde{\tau}^c$,
(2) *the intersection of any number of soft generalized closed sets is an element in $\widetilde{\tau}^c$.*

Proposition 2.30 ([19])**.** *Let $(U, \widetilde{\tau}, E)$ be a soft generalized topological space over U. Then the collection $\widetilde{\tau}_e = \{F(e) \subseteq U : (F, E) \in \widetilde{\tau}\}$ for each $e \in E$, defines a generalized topology on U.*

Proposition 2.31 ([19])**.** *Let $(U, \widetilde{\tau}_i, E)$ for each i be an arbitrary family of soft generalized topological spaces over the same universe U. Then $(U, \bigcap_i \widetilde{\tau}_i, E)$ is a soft generalized topological space over U.*

Definition 2.32 ([19])**.** Let $(U, \widetilde{\tau}, E)$ be a soft generalized topological space and let $\varnothing \neq Y \subset U$. Then $(Y, \widetilde{\tau}_Y, E)$ is called a *soft generalized topological subspace* of $(U, \widetilde{\tau}, E)$, where $\widetilde{\tau}_Y = \{(F, A) \widetilde{\cap} \widetilde{Y} : (F, A) \in \widetilde{\tau}\}$.

It is known that for every generalized topological space (U, τ) and $Y \subseteq U$, there exists a generalized subspace (Y, τ_Y), where τ_Y is defined by:

$$\tau_Y = \{A \cap Y : A \in \tau\}.$$

Now, one can show that there exists another generalized topological space $(Y, \tau_{\downarrow Y})$, which is a restriction generalized topological space of (Y, τ_Y) and is defined by:

$$\tau_{\downarrow Y} = \{A \subseteq Y : A \in \tau\}.$$

Conversely, for every generalized topological space $(Y, \omega), Y \subseteq U$, there exists an extended generalized topological space $(U, \omega_{\uparrow U})$ of (Y, ω), where

$$\omega_{\uparrow U} = \{A \subseteq U : A \cap Y \in \omega\}.$$

Similar, for every soft generalized topological space $(U, \widetilde{\tau}, E)$ over U, and for all $\varnothing \neq Y \subseteq U$, there exists a restricted soft generalized topological subspace $(Y, \widetilde{\tau}_{\downarrow Y}, E)$ of $(Y, \widetilde{\tau}_Y, E)$, where

$$\widetilde{\tau}_{\downarrow Y} = \{(F, A) \in \mathbf{S(Y,E)} : (F, A) \in \widetilde{\tau}\}.$$

Conversely, for every soft generalized topological space $(Y, \widetilde{\omega}, E), Y \subseteq U$, there exists an extended soft generalized topological space $(U, \widetilde{\omega}_{\uparrow U}, E)$ of $(Y, \widetilde{\omega}, E), Y \subseteq U$, where

$$\widetilde{\omega}_{\uparrow U} = \{(F, A) \in \mathbf{S(U,E)} : (F, A) \widetilde{\cap} \widetilde{Y} \in \widetilde{\omega}\}.$$

Proposition 2.33. *Let $(Y, \widetilde{\tau}_{\downarrow Y}, E)$ be a restriction soft generalized topological space over Y of a soft generalized topological space $(U, \widetilde{\tau}, E)$ over U. Then $(Y, (\widetilde{\tau}_e)_{\downarrow Y})$ is a restriction generalized topological space of $(U, \widetilde{\tau}_e)$ for each $e \in E$.*

Proof. Since $(Y, \widetilde{\tau}_{\downarrow Y}, E)$ is a soft generalized topological space over Y, $(Y, (\widetilde{\tau}_e)_{\downarrow Y})$ is a generalized topological space for each $e \in E$. Then for any $e \in E$, we have that

$$\begin{aligned} (\widetilde{\tau}_e)_{\downarrow Y} &= \{C \subseteq Y : C \in \widetilde{\tau}_e\} \\ &= \{F(e) \subseteq Y : (F, E) \in \widetilde{\tau}\} \\ &= \{F(e) \subseteq Y : F(e) \in \widetilde{\tau}_e\}. \end{aligned}$$

Thus $(Y, (\widetilde{\tau}_e)_{\downarrow Y})$ is a restricted generalized topological space of $(U, \widetilde{\tau}_e)$. □

Proposition 2.34. *Let $(U, \widetilde{\omega}_{\uparrow U}, E)$ be an extended soft generalized topological space over U of a soft generalized topological space $(Y, \widetilde{\omega}, E)$ over Y. Then $(U, (\widetilde{\omega}_e)_{\uparrow U})$ is an extended generalized topological space of $(Y, \widetilde{\omega}_e)$ for each $e \in E$.*

Proof. Since $(U, \widetilde{\omega}_{\uparrow U}, E)$ is a soft generalized topological space over U, $(U, (\widetilde{\omega}_e)_{\uparrow U})$ is a generalized topological space for each $e \in E$. Then for any $e \in E$, we have that

$$\begin{aligned} (\widetilde{\omega}_e)_{\uparrow U} &= \{C \subseteq U : C \cap Y \in \widetilde{\omega}_e\} \\ &= \{F(e) \subseteq U : (F, E) \widetilde{\cap} \widetilde{Y} \in \widetilde{\omega}\} \\ &= \{F(e) \subseteq U : F(e) \cap Y \in \widetilde{\omega}_e\}. \end{aligned}$$

Thus $(U, (\widetilde{\omega}_e)_{\uparrow U})$ is an extended generalized topological space of $(Y, \widetilde{\omega}_e)$. □

3. Categories GTOP and ESGTOP

From now, in this section, we use the fixed parameter set E.

Definition 3.1 ([23])**.** Let (U, σ_1) and (Y, σ_2) be two generalized topological spaces over U and Y respectively. The mapping $f : (U, \sigma_1) \longrightarrow (Y, \sigma_2)$ is said to be *generalized continuous*, if

$$f^{-1}(A) \in \sigma_1, \ \textit{for all} \ A \in \sigma_2.$$

It is clear that for any given generalized topological space (U, σ) over U, the identity mapping

$$Id_U : (U, \sigma) \longrightarrow (U, \sigma)$$

is generalized continuous mapping. Moreover, the composition of any two generalized continuous mappings is a generalized continuous mapping. Hence the family of all generalized topological spaces and all generalized continuous mappings form a category that will be denoted by **GTOP**.

Proposition 3.2. *The category of all topological spaces and all continuous mappings* ***TOP*** *is embedded in the category* ***GTOP****.*

Proof. The proof is obvious since for any topological space (U, σ) and for any continuous mapping f on topological spaces, the inclusion functor

$$I : \mathbf{TOP} \longrightarrow \mathbf{GTOP},\ where\ I(U, \sigma) = (U, \sigma),\ I(f) = f,$$

is embedded functor. □

Definition 3.3 ([19])**.** Let $(U, \widetilde{\tau}_1, E)$ and $(Y, \widetilde{\tau}_2, E)$ be two soft generalized topological spaces over U and Y respectively. The soft mapping $(f, g) : (U, \widetilde{\tau}_1, E) \longrightarrow (Y, \widetilde{\tau}_2, E)$ is said to be *soft generalized continuous*, if

$$(f, g)^{\leftarrow}(G, E) \in \widetilde{\tau}_1,\ for\ all\ (G, E) \in \widetilde{\tau}_2.$$

It is clear that for any given soft generalized topological space $(U, \widetilde{\tau}, E)$ over U, the soft identity mapping

$$(Id_U, Id_E) : (U, \widetilde{\tau}, E) \longrightarrow (U, \widetilde{\tau}, E)$$

is soft generalized continuous mapping.

Example 3.4. Let $U = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}, E = \{e_1, e_2\}$, and $f : U \longrightarrow Y$, $g : E \longrightarrow E$ be mappings defined as follows:

$$f(x_1) = f(x_2) = y_1,\ f(x_3) = y_3,\ g(e_1) = e_2,\ g(e_2) = e_1.$$

Define the soft generalized topological spaces $(U, \widetilde{\tau}_1, E)$ and $(Y, \widetilde{\tau}_2, E)$ over U and Y respectively as follows:

$$\widetilde{\tau}_1 = \{\widetilde{\Phi}, \widetilde{U}, (F_1, E), (F_2, E), (F_3, E)\},\ \widetilde{\tau}_2 = \{\widetilde{\Phi}, \widetilde{Y}, (G_1, E), (G_2, E), (G_3, E)\}$$

and

$$\begin{array}{ll} F_1(e_1) = U, & F_1(e_2) = \{x_3\}, \\ F_2(e_1) = U, & F_2(e_2) = \{x_1, x_2\}, \\ F_3(e_1) = \{x_1, x_2\}, & F_3(e_2) = U, \\ G_1(e_1) = \{y_2, y_3\}, & G_1(e_2) = \{y_1, y_3\}, \\ G_2(e_1) = \{y_1, y_2\}, & G_2(e_2) = Y, \\ G_3(e_1) = \{y_1, y_3\}, & G_3(e_2) = \{y_1, y_2\}. \end{array}$$

Then we have

$$(f, g)^{\leftarrow}(\widetilde{Y}) = \widetilde{U},\ (f, g)^{\leftarrow}(G_1, E) = (F_1, E),$$
$$(f, g)^{\leftarrow}(G_2, E) = (F_2, E),\ (f, g)^{\leftarrow}(G_3, E) = (F_3, E).$$

Thus the soft mapping (f, g) is a soft generalized continuous mapping from $(U, \widetilde{\tau}_1, E)$ into $(Y, \widetilde{\tau}_2, E)$.

On the other hand, if $g : E \longrightarrow E$ is the function defined by: $g(e_1) = e_1,\ g(e_2) = e_2$. Then the mapping (f, g) is not a soft generalized continuous mapping from $(U, \widetilde{\tau}_1, E)$ into $(Y, \widetilde{\tau}_2, E)$, since $(f, g)^{\leftarrow}(G_1, E) = (F, E)$, where $F(e_1) = \{x_3\},\ F(e_2) = U$ and $(F, E) \notin \widetilde{\tau}_1$.

Proposition 3.5 ([19])**.** *Let $(U, \widetilde{\tau}_1, E)$ and $(Y, \widetilde{\tau}_2, E)$ be two soft generalized topological spaces over U and Y respectively. Then the soft mapping $(f, g) : (U, \widetilde{\tau}_1, E) \longrightarrow (Y, \widetilde{\tau}_2, E)$ is a soft generalized continuous if and only if*

$$(f, g)^{\leftarrow}(G, E) \in \widetilde{\tau}_1^c\ for\ all\ (G, E) \in \widetilde{\tau}_2^c.$$

Proposition 3.6 ([19]). *Let $(X, \widetilde{\tau}_1, E), (Y, \widetilde{\tau}_2, E)$ and $(Z, \widetilde{\tau}_3, E)$ be soft generalized topological spaces over X, Y and Z respectively. And let*

$$(f_1, g_1) : (X, \widetilde{\tau}_1, E) \longrightarrow (Y, \widetilde{\tau}_2, E)$$

and

$$(f_2, g_2) : (Y, \widetilde{\tau}_2, E) \longrightarrow (Z, \widetilde{\tau}_3, E)$$

be two soft generalized continuous mappings. Then the composition soft mapping

$$(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1) : (X, \widetilde{\tau}_1, E) \longrightarrow (Z, \widetilde{\tau}_3, E)$$

is a soft generalized continuous mapping.

Since for any given soft generalized topological space $(U, \widetilde{\tau}, E)$ over U, the soft identity mapping

$$(Id_U, Id_E) : (U, \widetilde{\tau}, E) \longrightarrow (U, \widetilde{\tau}, E)$$

is soft generalized continuous mapping. Moreover, the composition of any two soft generalized continuous mappings on soft generalized topological spaces is a soft generalized continuous mapping. Hence the family of all soft generalized topological spaces and all soft generalized continuous mappings form a category that will be denoted by **ESGTOP**. Moreover, the family of all uniform soft generalized topological spaces and all soft generalized continuous mappings is a subcategory of **ESGTOP** and will be denoted by **UNIESGTOP**. In addition to that the subcategory of **UNIESGTOP** with soft generalized continuous mapping, having all parameters functions as the identity function Id_E on the parameter set E will be denoted by **UNIIDESGTOP**.

Theorem 3.7. (Main Theorem)
*The category **GTOP** is embedded in the category **ESGTOP**.*

Proof. **Step 1:** We proof that for any $(U, \sigma) \in |\mathbf{GTOP}|$, there exists $(U, (\widetilde{\tau})_\sigma, E) \in |\mathbf{ESGTOP}|$. Define $(\widetilde{\tau})_\sigma \subseteq \mathbf{UNIS(U,E)}$ as

$$(\widetilde{\tau})_\sigma = \{\widetilde{H} \in \mathbf{UNIS(U,E)} : H \in \sigma\}.$$

It is obvious that $\widetilde{\Phi} \in (\widetilde{\tau})_\sigma$, since $\phi \in \sigma$. Moreover, if

$$\{\widetilde{H}_i \in \mathbf{UNIS(U,E)} : i \in \Delta\} \subseteq (\widetilde{\tau})_\sigma,$$

then

$$\{H_i \subseteq U : i \in \Delta\} \subseteq \sigma.$$

Thus $\bigcup_{i \in \Delta} H_i \in \sigma$. So

$$\widetilde{\bigcup_{i \in \Delta} H_i} = \widetilde{\bigcup}_{i \in \Delta} \widetilde{H_i} \in (\widetilde{\tau})_\sigma.$$

Hence $(\widetilde{\tau})_\sigma$ is a soft generalized topology on U and $(U, (\widetilde{\tau})_\sigma, E)$ is a soft generalized topological space over U.

Step 2: We proof that for any $(U, \widetilde{\tau}, E) \in |\mathbf{ESGTOP}|$, there exists $(U, (\sigma)_{\widetilde{\tau}}) \in |\mathbf{GTOP}|$. It is known that for any soft generalized topology $\widetilde{\tau}$ on U and for any

$e \in E$, there exists a generalized topology $\widetilde{\tau}_e = \{G(e) \subseteq U : (G, E) \in \widetilde{\tau}\}$ on U. Define the generalized topological space $(U, (\sigma)_{\widetilde{\tau}})$ over U, where

$$(\sigma)_{\widetilde{\tau}} = \bigcap_{e \in E} \widetilde{\tau}_e.$$

Step 3: We proof that for any $(U, \widetilde{\tau}, E) \in |\mathbf{UNIESGTOP}|$ and for any $(U, \sigma) \in |\mathbf{GTOP}|$, $(\widetilde{\tau})_{(\sigma)_{\widetilde{\tau}}} = \widetilde{\tau}$. Since

$$\begin{aligned} (\sigma)_{\widetilde{\tau}} &= \textstyle\bigcap_{e \in E}\{G(e) \subseteq U : (G, E) \in \widetilde{\tau}\} \\ &= \textstyle\bigcap_{e \in E}\{H \subseteq U : \widetilde{H} \in \widetilde{\tau}\} \\ &= \{H \subseteq U : \widetilde{H} \in \widetilde{\tau}\}, \end{aligned}$$

we have

$$\begin{aligned} (\widetilde{\tau})_{(\sigma)_{\widetilde{\tau}}} &= \{\widetilde{H} \in \mathbf{UNIS(U,E)} : H \in (\sigma)_{\widetilde{\tau}}\} \\ &= \{\widetilde{H} \in \mathbf{UNIS(U,E)} : \widetilde{H} \in \widetilde{\tau}\} \\ &= \widetilde{\tau}. \end{aligned}$$

Step 4: We proof that for any $(U, \widetilde{\tau}, E) \in |\mathbf{ESGTOP}|$ and for any $(U, \sigma) \in |\mathbf{GTOP}|$, $(\sigma)_{(\widetilde{\tau})_\sigma} = \sigma$. Since we have that

$$(\widetilde{\tau})_\sigma = \{\widetilde{H} \in \mathbf{UNIS(U,E)} : H \in \sigma\},$$

we get

$$\begin{aligned} (\sigma)_{(\widetilde{\tau})_\sigma} &= \textstyle\bigcap_{e \in E}\{G(e) \subseteq U : (G, E) \in (\widetilde{\tau})_\sigma\} \\ &= \textstyle\bigcap_{e \in E}\{H \subseteq U : \widetilde{H} \in (\widetilde{\tau})_\sigma\} \\ &= \{H \subseteq U : H \in \sigma\} \\ &= \sigma. \end{aligned}$$

Step 5: We proof that for any $f \in \mathbf{Mor}((U, \sigma_1), (Y, \sigma_2))$ in the category **GTOP**, $(f, Id_E) \in \mathbf{Mor}((U, \widetilde{\tau}_{\sigma_1}, E), (Y, \widetilde{\tau}_{\sigma_2}, E))$ in the category **ESGTOP**.

Let $f \in \mathbf{Mor}((U, \sigma_1), (Y, \sigma_2))$ in the category **GTOP** and let $\widetilde{H} \in \widetilde{\tau}_{\sigma_2}$. Then $H \in \sigma_2$ and $f^{-1}(H) \in \sigma_1$. Thus $\widetilde{f^{-1}(H)} \in \widetilde{\tau}_{\sigma_2}$. Now, let $(f, Id_E)^{\leftarrow}(\widetilde{H}) = (G, E)$. Then for all $e \in E$, we have

$$G(e) = f^{-1}(K(Id_E(e))) = f^{-1}(K(e)) = f^{-1}(H), \ where \ K(e) = H.$$

Thus $(G, E) = \widetilde{f^{-1}(H)}$. So $(f, Id_E)^{\leftarrow}(\widetilde{H}) \in \widetilde{\tau}_{\sigma_1}$. Hence (f, Id_E) is a soft generalized continuous mapping and $(f, Id_E) \in \mathbf{Mor}((U, \widetilde{\tau}_{\sigma_1}, E), (Y, \widetilde{\tau}_{\sigma_2}, E))$.

Step 6: We proof that for any $(f, Id_E) \in \mathbf{Mor}((U, \widetilde{\tau}_1, E), (Y, \widetilde{\tau}_2, E))$ in the category **UNIIDESGTOP**, $f \in \mathbf{Mor}((U, (\sigma)_{\widetilde{\tau}_1}), (Y, (\sigma)_{\widetilde{\tau}_2}))$ in the category **GTOP**. Let $(f, Id_E) \in \mathbf{Mor}((U, \widetilde{\tau}_1, E), (Y, \widetilde{\tau}_2, E))$ in the category **UNIIDESGTOP** and let

$H \in (\sigma)_{\widetilde{\tau}_2}$. Then $\widetilde{H} \in \widetilde{\tau}_2$ and $(f, Id_E)^{\leftarrow}(\widetilde{H}) = \widetilde{f^{-1}(H)} \in \widetilde{\tau}_1$. Thus $f^{-1}(H) \in (\sigma)_{\widetilde{\tau}_1}$. So f is a soft generalized continuous mapping and $f \in \mathbf{Mor}((U, (\sigma)_{\widetilde{\tau}_1}), (Y, (\sigma)_{\widetilde{\tau}_2}))$ in the category **GTOP**.

Step 7: From **Step 1** up to **Step 6**, the following two functors **R** and **Q** are well-defined as follows:

$$\mathbf{R} : \mathbf{GTOP} \longrightarrow \mathbf{ESGTOP},\ \textit{where}\ \mathbf{R}(U, \sigma) = (U, (\widetilde{\tau})_\sigma, E),\ \mathbf{R}(f) = (f, Id_E),$$

$$\mathbf{Q} : \mathbf{UNIIDESGTOP} \longrightarrow \mathbf{GTOP},\ \textit{where}\ \mathbf{Q}(U, \widetilde{\tau}, E) = (U, (\sigma)_{\widetilde{\tau}}),\ \mathbf{R}(f, Id_E) = f.$$

Moreover, **R** is injective and **Q** is bijective, since

$$\mathbf{R}(U, \sigma_1) = \mathbf{R}(U, \sigma_2) \ \Rightarrow\ (U, (\widetilde{\tau})_{\sigma_1}, E) = (U, (\widetilde{\tau})_{\sigma_2}, E) \ \Rightarrow\ (\widetilde{\tau})_{\sigma_1} = (\widetilde{\tau})_{\sigma_2}.$$

Then $H \in \sigma_1$ if and only if $\widetilde{H} \in (\widetilde{\tau})_{\sigma_1} = (\widetilde{\tau})_{\sigma_2}$ if and only if $H \in \sigma_1$. Thus $(U, \sigma_1) = (U, \sigma_2)$. On the other hand,

$$\mathbf{R}(f_1) = \mathbf{R}(f_2) \ \Rightarrow\ (f_1, Id_E) = (f_1, Id_E) \ \Rightarrow\ f_1 = f_2.$$

Also if **R** is defined as:

$$\mathbf{R} : \mathbf{GTOP} \longrightarrow \mathbf{UNIIDESGTOP},$$

then

$$\mathbf{Q}(\mathbf{R}((U, \sigma))) = (U, \sigma),\ \mathbf{R}(\mathbf{Q}((U, \widetilde{\tau}, E))) = (U, \widetilde{\tau}, E),$$
$$\mathbf{Q}(\mathbf{R}((f, Id_E))) = (f, Id_E),\ \mathbf{R}(\mathbf{Q}(f)) = f.$$

Thus **GTOP** isomorphic the subcategory **UNIIDESGTOP** of **ESGTOP**. So **GTOP** is embedded in **ESGTOP**. □

4. Conclusion

From the above discussions, I can advocate that logically, the category **GTOP** is the natural generalization of the category **TOP**. Moreover, the category **ESGTOP** is the natural generalization of the category **GTOP**.

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References

[1] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.

[2] L. A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.

[3] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 64 (2) (1986) 87–96.

[4] Z. Pawlak, Rough Sets, International Journal of Information and Computer Science 11 (1982) 341–356.

[5] D. A. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.

[6] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.

[7] A. Aygunoglu and H. Aygün, Some notes on soft topological spaces, Neural. Comput. Appl. 21 (2011) 113–119.

[8] N. Çağman and S. Enginoğlu, Soft set theory and uni-int decision making, European J. Oper. Res. 207 (2010) 848–855.

[9] N. Çağman, S. Karataş and S. Enginoğlu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.

[10] D. N. Georgiou and A. C. Megaritis, Soft set theory and topology, App. Gen. Topol. 15 (1) (2014) 93–109.
[11] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62 (2011) 4058–4067.
[12] W. K. Min, A note on soft topological spaces, Comput. Math. Appl. 62 (2011) 3524–3528.
[13] G. Şenel, Soft topology generated by L-soft sets. Journal of New Theory 4 (24) (2018) 88–100.
[14] G. Şenel, A new approach to Hausdorff space theory via the soft sets, Mathematical Problems in Engineering 9 (2016) 1–6.
[15] G. Şenel and N. Çağman, Soft closed sets on soft bitopological space, Journal of New Results in Science 3 (5) (2014) 57–66.
[16] G. Şenel and N. Çağman, Soft topological subspaces, Ann. Fuzzy Math. Inform. 10 (4) (2015) 525–535.
[17] G. Şenel, J. G. Lee and K. Hur, Distance and Similarity Measures for Octahedron Sets and Their Application to MCGDM Problems, Mathematics 8 (10) 1690 (2020).
[18] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[19] J. Thomas and J. S. John, On soft generalized topological spaces, J. New Results Sci. 4 (2014) 01–15.
[20] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[21] D. A. Molodtsov, V. Y. Leonov and D. V. Kovkov, Soft sets technique and its application, Nechetkie Sistemy i Myagkie Vychisleniya 1, no. 1 (2006) 8–39.
[22] I. Zorlutuna, M. Akdağ, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3 (2) (2012) 171–185.
[23] *Á*. Cs*á*sz*á*r, Generalized topology, generalized continuity, Acta Math. Hungar. 96 (4) (2002) 351–357.
[24] T. Y. Öztürk, Soft mappings on soft generalized topological spaces, Creat. Math. Inform. 27 (2) (2018) 177–190.
[25] J. Adámek, H. Herrlich and G. E. Strecker, Abstract and Concrete Categories, Wiley, New York 1990.
[26] N. Bourbaki, General Topologey, Paris 1940.
[27] S. Saleh, On category of interval valued fuzzy topological spaces, Ann. Fuzzy Math. Inform. 4 (2) (2012) 385–392.
[28] S. Saleh, R. Abu-Gdairi, T. M. AL-Shami and M. S. Abdo, On categorical property of fuzzy soft topological spaces, Appl. Math. Inf. Sci. 16 (4) (2022) 635–641.
[29] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.

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