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Abstract. We introduce the idea of gradually (λ, μ)-statistical convergence with regards to the n-norm in gradual n-normed linear space (GNnLS) in this study. We look at certain inclusion relations with regards to the n-norm between the sets of gradually statistically convergent and gradually (λ, μ)-statistically convergent double sequences. We discover its relationship to gradually strongly Cesàro summability, gradually strongly (V, λ, μ)-summability, and gradually statistical convergence with regards to the n-norm in GNnLS.

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1. Introduction

The idea of fuzzy sets was first proposed by Zadeh [1] in 1965. It now has several applications across many fields of engineering and research. In the field of fuzzy set theory, the idea of "fuzzy number" is crucial. Intervals, not numbers, were effectively generalized to create fuzzy numbers. In fact, a few algebraic characteristics of the classical numbers are absent from fuzzy numbers. Due to its varied behavior, many scholars disagree with the idea of a "fuzzy number". Several authors frequently substitute the idea of "fuzzy intervals" for fuzzy numbers. Researchers were perplexed, but Fortin et. al. [2] proposed the idea of gradual real numbers as components of fuzzy intervals to clear things up. Gradual real numbers (GRNs) are most commonly identified by the domain of the corresponding assignment function, which is the range $(0, 1]$. As every real number has a constant assignment function, they may all be thought of as a gradual numbers. The GRNs have been used in optimization and computing issues and provide all the algebraic properties of the traditional real numbers.

Sadeqi and Azari [3] were the first to investigate the idea of GNLS in 2011. They investigated many characteristics both topologically and algebraically. Ettefagh et. al. [4, 5] have contributed to further advancement in this approach. One may cosult [6, 7, 8, 9, 10, 11] for a thorough research on GRNs.

On the other hand, Fast [12] independently developed the concept of statistical convergence using the notion of natural density in 1951. Mursaleen and Osama [13] found relationships between statistical convergence and strongly Cesàro summable double sequences by extending the aforementioned concept from single to double sequences of scalars. The notion of statistical convergence was initiated and studied by extending upto some extent in the environment up uncertainty via single [14, 15, 16], double [17], triple [18, 19] sequences of complex uncertain variable. You can consult [20, 21, 22, 23, 24, 25, 26, 27] and several other mathematicians from all around the world for a thorough study of statistical convergence.

Mursaleen [28] expanded statistical convergence to include λ-statistical convergence in 2000 and did so as follows:

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} - \lambda_n \leq 1, \quad \lambda_1 = 1.$$

The generalized de la Vallèe-Poussin mean is defined by

$$t_n((w_u)) = \frac{1}{\lambda_n} \sum_{u \in I_n} w_u,$$

where $I_n = [n - \lambda_n + 1,\ n]$.

A sequence (w_u) is called to be (V, λ)-*summable to a number* w_0 (see [29] for details), if

$$t_n((w_u)) \to w_0, \text{ as } n \to \infty.$$

We write,

$$[V, \lambda] = \left\{ (w_u) : \text{for some } w_0, \lim_n \left(\frac{1}{\lambda_n} \sum_{u \in I_n} |w_u - w_0| \right) = 0 \right\}$$

for the sets of sequences (w_u), which are strongly (V, λ)-summable to w_0, i.e., $w_u \to w_0[V, \lambda]$.

A sequence $w = (w_u)$ is said to be λ-*statistically convergent to* w_0, if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{u \in I_n : |w_u - w_0| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\lambda - \lim w = w_0$ or $w_u \to w_0(S_\lambda)$. It is obvious that if $\lambda_n = n$, then S_λ is coincident with S, where S is the set of all statistical convergent sequences (for more details one may see [30]). Furthermore, double λ-statistical convergence was studied and examined by several authors ([31, 32, 33, 34, 35]).

The concept of 2-normed space was initially introduced by Gähler [36], in the mid 1960's, while that of n-normed spaces can be found in Misiak [37]. Since then, many others authors have studied this concept and obtained various results (see, for instance, Esi [38, 39], Esi and Açıkgöz [40], Esi and Özdemir [41], Sharma and Esi [42], Raj and Esi [43] and Fistikci et al. [44].

2. Preliminaries

We discuss some current definitions and findings that are essential to our conclusions in this section.

The gradual numbers and the gradual operations between the elements of $\mathcal{G}(\mathbb{R})$ were investigated by Fortin et al. [2] as follows:

Definition 2.1. A GRN $\tilde{s}$ is identified by an assignment function $\mathcal{F}_{\tilde{s}} : (0,1] \to \mathbb{R}$. A GRN $\tilde{s}$ is called to be a *non-negative number*, provided that for each $\gamma \in (0,1]$, $\mathcal{F}_{\tilde{s}}(\gamma) \geq 0$. The set of all GRNs and non-negative GRNs are demonstrated by $\mathcal{G}(\mathbb{R})$ and $\mathcal{G}^*(\mathbb{R})$ respectively.

Definition 2.2. Suppose that $*$ be any operation in $\mathbb{R}$ and assume $\tilde{u}_1, \tilde{u}_2 \in \mathcal{G}(\mathbb{R})$ with assignment functions $\mathcal{F}_{\tilde{u}_1}$ and $\mathcal{F}_{\tilde{u}_2}$ respectively. Then $\tilde{u}_1 * \tilde{u}_2 \in G(\mathbb{R})$ is determined with the assignment function $\mathcal{F}_{\tilde{u}_1 * \tilde{u}_2}$ defined by:

$$\mathcal{F}_{\tilde{u}_1 * \tilde{u}_2}(\tau) = \mathcal{F}_{\tilde{u}_1}(\tau) * \mathcal{F}_{\tilde{u}_2}(\tau) \ \forall \tau \in (0,1].$$

Especially, the *gradual addition* $\tilde{u}_1 + \tilde{u}_2$ and the *gradual scalar multiplication* $p\tilde{u}(p \in \mathbb{R})$ are given as follows:

$$\mathcal{F}_{\tilde{u}_1 + \tilde{u}_2}(\tau) = \mathcal{F}_{\tilde{u}_1}(\tau) + \mathcal{F}_{\tilde{u}_2}(\tau) \quad \text{and} \quad \mathcal{F}_{p\tilde{u}}(\tau) = p\mathcal{F}_{\tilde{u}}(\tau) \ \ \forall \tau \in (0,1].$$

Sadeqi and Azeri [3] created the GNLS and obtained the following conclusions about gradual convergence using the gradual numbers.

Definition 2.3. Let Y be a real vector space. Then, the function $\|\cdot\|_{\mathcal{G}} : Y \to \mathcal{G}^*(\mathbb{R})$ is called a *gradual norm* on Y, provided that for each $\tau \in (0,1]$, following situations are correct: for any $w, v \in Y$,

(i) $\mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau) = \mathcal{F}_{\tilde{0}}(\tau)$ iff $w = 0$,
(ii) $\mathcal{F}_{\|\mu w\|_{\mathcal{G}}}(\tau) = |\mu| \, \mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau)$ for any $\mu \in \mathbb{R}$,
(iii) $\mathcal{F}_{\|w+v\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|v\|_{\mathcal{G}}}(\tau)$.

In this case, $(Y, \|\cdot\|_{\mathcal{G}})$ is called a *GNLS*.

Example 2.4. Take $Y = \mathbb{R}^\alpha$ and for $w = (w_1, w_2, ..., w_\alpha) \in \mathbb{R}^\alpha, \gamma \in (0,1]$, determine $\|\cdot\|_{\mathcal{G}}$ as

$$\mathcal{F}_{\|w\|_{\mathcal{G}}}(\gamma) = e^\gamma \sum_{j=1}^{\alpha} |w_j|.$$

Then $\|\cdot\|_{\mathcal{G}}$ is a gradual norm on $\mathbb{R}^\alpha$ and $(\mathbb{R}^\alpha, \|\cdot\|_{\mathcal{G}})$ is a GNLS.

Nevertheless, Ettefagh et al. [5] were the first to identify a sequence's gradual boundedness in a GNLS and explore how it relates to gradual convergence.

Definition 2.5 ([5])**.** Suppose that $(Y, \|\cdot\|_{\mathcal{G}})$ be a GNLS. Then, a sequence (w_u) in Y is said to be *gradual bounded*, provided that for each $\tau \in (0,1]$, there is an $M = M(\tau) > 0$ such that $\mathcal{F}_{\|w_u\|_{\mathcal{G}}}(\tau) < M \ \forall u \in \mathbb{N}$.

Definition 2.6 ([3])**.** Let (w_u) be a sequence in the GNLS $(Y, \|\cdot\|_{\mathcal{G}})$. Then (w_u) is called to be *gradual convergent to* $w_0 \in X$, provided that for each $\tau \in (0,1]$ and $\kappa > 0$, there is an $N(= N_\kappa(\tau)) \in \mathbb{N}$ such that
$\mathcal{F}_{\|w_{uv} - w_0\|_{\mathcal{G}}}(\tau) < \kappa$, $\forall u, v \geq N$.

Symbolically, $w_{uv} \xrightarrow{\|\cdot\|_G} w_0$.

Definition 2.7 ([10])**.** Let (w_u) be a sequence in the GNLS $(Y, \|\cdot\|_G)$. Then (w_u) is said to be *gradual statistically convergent to* $w_0 \in Y$, if for every $\xi \in (0,1]$ and $\varepsilon > 0$,

$$\lim_{q \to \infty} \frac{1}{q} \left| \left\{ u \leq q : \mathcal{F}_{\|w_u - w_0\|_{\mathcal{G}}}(\tau) \geq \kappa \right\} \right| = 0.$$

Symbolically, $w_u \xrightarrow{st - \|\cdot\|_G} w_0$. The set $S(\mathcal{G})$ denotes the set of all gradually statistical convergent sequences.

Definition 2.8 ([11])**.** Let (w_u) be a sequence in the GNLS $(Y, ||\cdot||_G)$. Then (w_u) is said to be *gradually λ-statistical convergent to* $w_0 \in Y$, if for every $\xi \in (0,1]$ and $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \mid \{u \in I_n : \mathcal{F}_{||w_u - w_0||_G}(\xi) \geq \varepsilon\} \mid = 0.$$

Equivalently, $\mathcal{F}_{||w_u - w_0||_G}(\xi) < \varepsilon$ a.a.u.

In this case, w_0 is called the *gradual λ-statistical limit* of the sequence (w_u) and we write

$$S_\lambda - ||\cdot||_G \lim w_u = w_0 \text{ or } w_u \xrightarrow{S_\lambda - ||\cdot||_G} w_0.$$

We shall also use $S_\lambda(G)$ to denote the set of all gradually λ-statistical convergent sequences.

Definition 2.9. [33] Let $\lambda = (\lambda_m)$ and $\mu = (\mu_n)$ be two non-decreasing sequences of positive real numbers, each tending to ∞ and such that

$$\lambda_{m+1} \leq \lambda_m + 1, \ \lambda_1 = 1; \ \mu_{n+1} \leq \mu_n + 1, \ \mu_1 = 1.$$

Let $I_m = [m - \lambda_m + 1, m]$, $I_n = [n - \mu_n + 1, n]$ and $I_{m,n} = I_m \times I_n$. For any set $Q \subseteq \mathbb{N} \times \mathbb{N}$,

$$\delta_{(\lambda,\mu)}(Q) = P - \lim_{m,n \to \infty} \frac{1}{\lambda_m \mu_n} |\{(p,q) \in I_m \times I_n : (p,q) \in Q\}|$$

is called a (λ, μ)-*density* of the set Q, provided the limit exists.

If we take, $\lambda_m = m, \mu_n = n$, then the above definition reduces to the definition of double natural density.

Definition 2.10 ([13])**.** A real double sequence $w = (w_{uv})$ is said to be *statistically convergent to the number* w_0, if for each $\varepsilon > 0$, the set

$$\{(u,v), \ u \leq m, v \leq n : |w_{uv} - w_0| \geq \varepsilon\}$$

has double natural density zero. In this case, we write $st_2 - \lim_{u,v} w_{uv} = w_0$.
We denote the set of all statistically convergent double sequences by st_2.

Definition 2.11 ([37]). Let n be a non negative integer and Y be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\|.,\ldots,.\|$ from Y^n into $\mathbf{R}$ satisfying the following conditions:

(i) $\|w_1, w_2, \ldots, w_n\| = 0$ if and only if $w_1, w_2, \ldots, w_n$ are linearly dependent,
(ii) $\|w_1, w_2, \ldots, w_n\|$ is invariant under permutation,
(iii) $\|\alpha w_1, w_2, \ldots, w_n\| = |\alpha| \, \|w_1, w_2, \ldots, w_n\|$ for any $\alpha \in \mathbf{R}$,
(iv) $\|w + \bar{w}, w_2, \ldots, w_n\| \leq \|w, w_2, \ldots, w_n\| + \|\bar{w}, w_2, \ldots, w_n\|$

is called an *n-norm* on Y and the pair $(Y, \|.,\ldots,\|.)$ is called an *n-normed space.*

Throughout the paper, we indicate $\lambda_{m,n} = \lambda_n \mu_n$ and the collection of such sequences λ will be showed by Δ_2. Furthermore, $\delta_2(P)$ denotes the double natural density of the set $P \subseteq \mathbb{N}$.

3. Main results

Here, we share our research results. The following definitions, which will be used throughout the study, serve as our starting point.

Definition 3.1. Let Y be a real vector space. Then the function $\|.,...,.\|_{\mathcal{G}} : Y^n \to \mathcal{G}^*(\mathbb{R})$ is called to be a *gradual n-norm* on Y^n, provided that following statements are correct: for any $w,\ v \in Y$,

(i) $\mathcal{F}_{\|w_1, w_2, ..., w_n\|_{\mathcal{G}}}(\tau) = \mathcal{F}_{\tilde{0}}(\tau)$ iff $w_1, w_2, ..., w_n$ are linear dependent,
(ii) $\mathcal{F}_{\|\mu w_1, w_2, ..., w_n\|_{\mathcal{G}}}(\tau) = |\mu| \, \mathcal{F}_{\|w_1, w_2, ..., w_n\|_{\mathcal{G}}}(\tau)$ for any $\mu \in \mathbb{R}$,
(iii) $\mathcal{F}_{\|w+v, w_1, w_2, ..., w_n\|_{\mathcal{G}}}(\tau) \leq \mathcal{F}_{\|w, w_1, w_2, ..., w_n\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|v, w_1, w_2, ..., w_n\|_{\mathcal{G}}}(\tau)$.

In this case, $(Y, \|.,...,.\|_{\mathcal{G}})$ is called an *n-normed GNnLS.*

Definition 3.2. A double sequence $w = (w_{uv})$ in $(Y, \|.,...,.\|_{\mathcal{G}})$ is called to be *statistically gradual convergent* (in short $st_2(\mathcal{G})$-convergent) *to* $w_0 \in Y$ *with respect to the n-norm*, provided that for each $\kappa > 0$ and $\tau \in (0, 1]$,

$$\delta_2 \left(\left\{ (u, v) \in \mathbb{N} \times \mathbb{N} : \mathcal{F}_{\|w_{uv} - w_0, z_1, z_2, ..., z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa \right\} \right) = 0,$$

for all $z_1, z_2, ..., z_{n-1} \in Y$.

In this case, we write $st_2^{nN}(\mathcal{G}) - \lim w_{uv} = w_0$ or $w_{uv} \to w_0(st_2^{nN}(\mathcal{G}))$.

Definition 3.3. A double sequence $w = (w_{uv})$ is called to be *gradually* (λ, μ)-*statistically convergent* (or shortly $S_{(\lambda,\mu)}(\mathcal{G})$-convergent) *to* $w_0 \in Y$ *with respect to the n-norm*, provided that for each $\kappa > 0$ and $\tau \in (0, 1]$,

$$\lim_{m,n \to \infty} \frac{1}{\lambda_{m,n}} \left| \left\{ (u, v) \in I_{m,n} : \mathcal{F}_{\|w_{uv} - w_0, z_1, z_2, ..., z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa \right\} \right| = 0,$$

for all $z_1, z_2, ..., z_{n-1} \in Y$. Namely, the set

$$T(\kappa) := \left\{ (u, v) \in I_{m,n} : \mathcal{F}_{\|w_{uv} - w_0, z_1, z_2, ..., z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa \right\}$$

has (λ, μ)-density zero.

Symbolically, we write $S_{(\lambda,\mu)}^{nN}(\mathcal{G}) - \lim w_{uv} = w_0$ or $w_{uv} \to w_0(S_{(\lambda,\mu)}^{nN}(\mathcal{G}))$.

In addition, we use $S^{nN}_{(\lambda,\mu)}(Y)$ to denote the collection of all gradually $S_{(\lambda,\mu)}(\mathcal{G})$-convergent double sequences in Y, and

$$S^{nN}_{(\lambda,\mu)}(Y) := \left\{ w = (w_{uv}) : \exists w_0 \in Y,\, S^{nN}_{(\lambda,\mu)}(\mathcal{G}) - \lim w_{uv} = w_0 \right\}.$$

If $\lambda_m = m$, $\mu_n = n$ for all m, n then the space $S^{nN}_{(\lambda,\mu)}(Y)$ is reduced to the space $st_2^{nN}(Y)$ and since $\delta_2(T) \leq \delta_{(\lambda,\mu)}(T)$, we have $S^{nN}_{(\lambda,\mu)}(Y) \subset st_2^{nN}(Y)$.

Example 3.4. Let $Y = \mathbb{R}^\alpha$ and $\|\cdot\|_{\mathcal{G}}$ be the norm defined in Example 2.4. Consider the sequence $(\lambda_{m,n})$ determined by

$$\lambda_{m,n} = \begin{cases} 1 & mn = 1, \\ \frac{mn}{2} & mn \geq 2. \end{cases}$$

Then the sequence (w_{uv}) in $\mathbb{R}^\alpha$ defined as

$$w_{uv} = \begin{cases} (0,0,...,0,\alpha) & \text{if } u = p^2, v = q^2,\, p, q \in \mathbb{N}, \\ (0,0,.....,0,0) & \text{otherwise} \end{cases}$$

is gradually (λ,μ)-statistical convergent to $\mathbf{0}$ with respect to the n-norm in $\mathbb{R}^\alpha$ where $\mathbf{0}$ demonstrates the α-tuple $(0,0,....0,0)$.
Justification. We get

$$\begin{aligned}
&\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-\mathbf{0},z_1,z_2,...,z_{n-1}\|_G}(\tau) \geq \kappa\right\}\right| \\
&= 2 \lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{u \in \left[\tfrac{m}{2}+1, m\right], v \in \left[\tfrac{n}{2}+1, n\right] : \mathcal{F}_{\|w_{uv}-\mathbf{0},z_1,z_2,...,z_{n-1}\|_G}(\tau) \geq \kappa\right\}\right| \\
&\leq 2 \lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{u \leq m, v \leq n : \mathcal{F}_{\|w_{uv}-\mathbf{0},z_1,z_2,...,z_{n-1}\|_G}(\tau) \geq \kappa\right\}\right| \\
&\leq 2 \lim_{m,n\to\infty} \frac{[\sqrt{mn}]}{mn}, \textit{ where } [q] \textit{ demonstrates the largest integer } \leq q \\
&= 0.
\end{aligned}$$

Then we conclude that $w_{uv} \to \mathbf{0}(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$.

Example 3.5. Let $Y = \mathbb{R}$ and for any $w_0 \in \mathbb{R}$, let $\|.\|_{\mathcal{G}}$ be the norm defined as $\mathcal{F}_{\|w_0\|_G} = e^\tau |w_0|$.

Consider the sequence $(\lambda_{m,n})$ determined in Example 3.4. Then the sequence $w = (w_{uv})$ in Y identified as $w_{uv} = u^2 v^2$ is not gradually (λ,μ)-statistical convergent with respect to the n-norm.
Justification. For any $w_0 \in \mathbb{R}$, we have $w_0 \leq 0$ or $w_0 > 0$. Then for all of the following situations, w will not gradually (λ,μ)-statistical convergent to w_0 with respect to the n-norm.

Case-I: When $w_0 \leq 0$, we select $\kappa = \frac{1}{2}e^\tau$. Then we have

$$\begin{aligned}
&\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,...,z_{n-1}\|}(\tau) \geq \kappa\right\}\right| \\
&= \lim_{m,n\to\infty} \frac{2}{mn} \left|\left\{u \in \left[\tfrac{m}{2}+1, m\right], v \in \left[\tfrac{n}{2}+1, n\right] : \mathcal{F}_{\|u^2v^2-w_0,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \tfrac{1}{2}e^\xi\right\}\right| \\
&= \begin{cases} \lim_{m,n\to\infty} \frac{2}{mn}\left(\frac{mn}{2} - 1\right); & \textit{when } m, n \textit{ is even} \\ \lim_{m,n\to\infty} \frac{2}{mn}\left(\frac{mn+1}{2} - 1\right); & \textit{when } m, n \textit{ is odd} \end{cases} \\
&= 1 \\
&\neq 0.
\end{aligned}$$

Case-II: If $w_0 > 0$, then there are $u_0 \in \mathbb{N}$ such that $w_{u_0-1} \leq w_0 \leq w_{u_0}$.

Subcase-I: If $0 < w_0 < 1$, then select $\kappa = \frac{e^\tau}{2}\min\{w_0, 1-w_0\}$. Thus it is easy to verify that

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| = 1 \neq 0.$$

Subcase-II: If $w_0 \geq 1$, then select $\kappa = \frac{e^\tau}{2}\min\{w_0 - w_{u_0-1}, w_{u_0} - w_0\}$. Thus it is easy to verify that

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| = 1 \neq 0.$$

From the above case study, we can conclude that w is not gradually (λ,μ)-statistical convergent with respect to n-norm.

Definition 3.6. Assume $(Y, \|.,...,.\|_{\mathcal{G}})$ be any n-normed GNLS. We establish the new sequence spaces $[C,1,1]^{nN}(\mathcal{G})$ and $[V,\lambda,\mu]^{nN}(\mathcal{G})$ as follows:

$$[C,1,1]^{nN}(\mathcal{G}) = \left\{\begin{array}{c} (w_{uv}) : \text{for some } w_0 \in Y \text{ and for all } \tau \in (0,1], \\ \lim\limits_{m,n\to\infty} \frac{1}{mn}\left(\sum\limits_{u,v=1,1}^{m,n} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)\right) = 0 \end{array}\right\}$$

and

$$[V,\lambda,\mu]^{nN}(\mathcal{G}) = \left\{\begin{array}{c} (w_{uv}) : \text{for some } w_0 \in Y \text{ and for all } \tau \in (0,1], \\ \lim\limits_{m,n\to\infty} \frac{1}{\lambda_{m,n}}\left(\sum\limits_{(u,v)\in I_{m,n}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)\right) = 0 \end{array}\right\}$$

for the sets of Y-valued double sequences $w = (w_{uv})$ which are gradually strongly Cesàro summable and gradually strongly (V,λ,μ)-summable to w_0 with respect to n-norm in $(Y, \|.,...,.\|_{\mathcal{G}})$. In this case, we denote $w_{uv} \to w_0([C,1,1]^{nN}(\mathcal{G}))$ and $w_{uv} \to w_0([V,\lambda,\mu]^{nN}(\mathcal{G}))$respectively.
When $\lambda_{m,n} = mn$, then gradually $[V,\lambda,\mu]^{nN}$-summability reduces to gradually $[C,1,1]^{nN}$-summability in the GNLS.

Theorem 3.7. *Assume $w = (w_{uv})$ be a double sequence in the n-normed GNLS $(Y, \|.,...,.\|_{\mathcal{G}})$.*

(1) *If $w_{uv} \to w_0([V,\lambda,\mu]^{nN}(\mathcal{G}))$, then $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$ but the converse is not true.*

(2) *$[V,\lambda,\mu]^{nN}(\mathcal{G})$ is a proper subset of $S^{nN}_{(\lambda,\mu)}(\mathcal{G})$ in $(Y, \|.,...,.\|_{\mathcal{G}})$.*

(3) *If (w_{uv}) is gradually bounded ($w \in l^2_\infty(Y)$) and $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$, then $w_{uv} \to w_0([V,\lambda,\mu]^{nN}(\mathcal{G}))$ and thus $w_{uv} \to w_0([C,1,1]^{nN}(\mathcal{G}))$.*

(4) *$S^{nN}_{(\lambda,\mu)}(\mathcal{G}) \cap l^2_\infty(Y) = [V,\lambda,\mu]^{nN}(\mathcal{G}) \cap l^2_\infty(Y)$.*

Proof. (1) Suppose $\kappa > 0$ be arbitrary $\tau \in (0,1]$ and $w_{uv} \to w_0([V,\lambda,\mu]^{nN}(\mathcal{G}))$. Then we can write

$$\begin{aligned}\sum_{(u,v)\in I_{m,n}} &\mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\\ &\geq \sum_{\substack{(u,v)\in I_{m,n}\\ \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\geq\kappa}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\\ &\geq \kappa\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\geq\kappa\right\}\right|.\end{aligned}$$

Thus

$$\begin{aligned}\tfrac{1}{\kappa.\lambda_{m,n}}\sum_{(u,v)\in I_{m,n}} &\mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\\ &\geq \tfrac{1}{\lambda_{m,n}}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\geq\kappa\right\}\right|.\end{aligned}$$

This finalizes the proof.

(2) To demonstrate that the inclusion $[V,\lambda,\mu]^{nN}(\mathcal{G}) \subset S^{nN}_{(\lambda,\mu)}(\mathcal{G})$ is proper, we define a sequence $w = (w_{uv})$ by

$$w = (w_{uv}) = \begin{cases} uv & \text{if } m - [\sqrt{\lambda_m}] + 1 \leq u \leq m \text{ and } n - [\sqrt{\mu_n}] + 1 \leq v \leq n,\\ 0 & \text{otherwise.}\end{cases}$$

It is obvious that $w \notin l^2_\infty(Y)$. Then for each $\kappa > 0$ with $0 < \kappa e^\kappa \leq 1$, we obtain

$$\frac{1}{\lambda_{m,n}}\sum_{(u,v)\in I_{m,n}} \mathcal{F}_{\|w_{uv}-\mathbf{0},z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \leq \frac{[\sqrt{\lambda_m}][\sqrt{\mu_n}]}{\lambda_m\mu_n} \to 0 \text{ as } m,n\to\infty,$$

namely, $w_{uv} \to \mathbf{0}([V,\lambda,\mu]^{nN}(\mathcal{G}))$. On the other hand,

$$\frac{1}{\lambda_{m,n}}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-\mathbf{0},z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\geq\kappa\right\}\right| \to \infty \text{ as } m,n\to\infty,$$

in Pringsheim sense, i.e., w_{uv} does not gradually (λ,μ)-statistically convergent to $\mathbf{0}$ with respect to the n-norm.

(3) Let $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$ and (w_{uv}) is gradually bounded, say

$$\mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \leq M,$$

for all $u, v \in \mathbb{N}$. Then for any $\kappa > 0$, we get

$$\frac{1}{\lambda_{m,n}} \sum_{(u,v)\in I_{m,n}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$= \frac{1}{\lambda_{m,n}} \sum_{\substack{(u,v)\in I_{m,n} \\ \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)\geq\kappa}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$+ \frac{1}{\lambda_{m,n}} \sum_{\substack{k\in I_n \\ \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)<\kappa}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$\leq \frac{M}{\lambda_{m,n}} \left| \left\{ (u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)) \geq \frac{\kappa}{2} \right\} \right| + \frac{\kappa}{2},$$

which consequently means that $w_{uv} \to w_0([V,\lambda,\mu]^{nN}(\mathcal{G}))$.
Again, we obtain

$$\frac{1}{mn} \sum_{u,v=1,1}^{m,n} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}$$

$$\leq \frac{1}{mn} \sum_{m,n=1,1}^{m-\lambda_m,n-\mu_n} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$+ \frac{1}{mn} \sum_{(u,v)\in I_{m,n}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$\leq \frac{1}{\lambda_{m,n}} \sum_{m,n=1,1}^{m-\lambda_m,n-\mu_n} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$+ \frac{1}{\lambda_{m,n}} \sum_{(u,v)\in I_{m,n}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau)$$

$$\leq \frac{2}{\lambda_{m,n}} \sum_{(u,v)\in I_{m,n}} \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\dots,z_{n-1}\|_{\mathcal{G}}}(\tau).$$

As a result, $w_{uv} \to w_0([C,1,1]^{nN}(\mathcal{G}))$, because $w_{uv} \to w_0([V,\lambda,\mu]^{nN}(\mathcal{G}))$.

(4) This is an immediate consequence of (1), (2) and (3). □

If we take $\lambda_{m,n} = mn$ in Theorem 3.7, we get the following corollary.

Corollary 3.8. *Assume $w = (w_{uv})$ be a double sequence in the GNnLS $(Y, \|., ..., .\|_{\mathcal{G}})$.*

(1) *If $w_{uv} \to w_0([C,1,1]^{nN}(\mathcal{G}))$, then $w_{uv} \to w_0(st_2^{nN}(\mathcal{G}))$ but the converse is not true.*

(2) *$[C,1,1]^{nN}(\mathcal{G})$ is a proper subset of $st_2^{nN}(\mathcal{G})$ in GNnLS $(Y, \|., ..., .\|_{\mathcal{G}})$.*

(3) *If (w_{uv}) is gradually bounded ($w \in l_\infty^2(Y)$) and $w_{uv} \to w_0(st_2^{nN}(\mathcal{G}))$, then $w_{uv} \to w_0([C,1,1]^{nN}(\mathcal{G}))$.*

(4) *$st_2^{nN}(\mathcal{G}) \cap l_\infty^2(Y) = [C,1,1]^{nN}(\mathcal{G}) \cap l_\infty^2(Y)$.*

Theorem 3.9. *Let* (w_{uv}) *be a double sequence in the GNnLS* $(Y, \|., ..., .\|_{\mathcal{G}})$. *Then,* $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$ *iff there is an*

$$M = \{(m_u, n_v) : m_1 < m_2 < ... < m_u < ...; n_1 < n_2 < ... < n_v < ...\} \subset \mathbb{N} \times \mathbb{N}$$

such that $\delta_{(\lambda,\mu)}(M) = 1$ *and* $(w_{m_u n_v}) \overset{\|.,...,.\|_{\mathcal{G}}}{\to} w_0$.

Proof. Firstly, we assume that there is a set

$$M = \{(m_u, n_v) : m_1 < m_2 < ... < m_u < ...; n_1 < n_2 < ... < n_v < ...\} \subset \mathbb{N} \times \mathbb{N}$$

supplying

$$\delta_{(\lambda,\mu)}(M) = 1 \text{ and } (w_{m_u n_v}) \overset{\|.,...,.\|_{\mathcal{G}}}{\to} w_0.$$

Then for each $\kappa > 0$ and $\tau \in (0, 1]$, there are $M(= M_\kappa(\tau))$, $N(= N_\kappa(\tau)) \in \mathbb{N}$ such that

$$\mathcal{F}_{\|w_{m_u n_v} - w_0, z_1, z_2, ..., z_{n-1}\|}(\tau) < \kappa, \forall u \geq M, v \geq N.$$

Consider

$$B(\tau, \kappa) = \left\{(u, v) \in \mathbb{N} \times \mathbb{N} : \mathcal{F}_{\|w_{uv} - w_0, z_1, z_2, ..., z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}.$$

Then the inclusion

$$B(\tau, \kappa) \subset (\mathbb{N} \times \mathbb{N}) \setminus \{(m_{M+1}, n_{N+1}), (m_{M+2}, n_{N+2}), ...\}$$

holds. Thus $\delta_{(\lambda,\mu)}(B(\tau, \kappa)) = 0$. So $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$.

For the converse part, suppose that $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$ supplies. Then for each $\tau \in (0, 1]$ and $j \in \mathbb{N}$, $\delta_{(\lambda,\mu)}(M_j) = 1$, where

$$M_j = \left\{(u, v) \in \mathbb{N} \times \mathbb{N} : \mathcal{F}_{\|w_{uv} - w_0, z_1, z_2, ..., z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{1}{j}\right\}.$$

From the construction of M_j's, it is obvious that

$$M_1 \supset M_2 \supset ... \supset M_j \supset M_{j+1} \supset ... \tag{3.1}$$

Let us select $q_1 \in M_1$ be an arbitrary element. Then there is an $q_2 \in M_2$ such that for all $m, n \geq q_2$,

$$\frac{1}{\lambda_{m,n}} |\{(u, v) \in I_{m,n} : (u, v) \in M_2\}| > \frac{1}{2}$$

holds. In a similar way, there exists $q_3 \in M_3$ such that for all $m, n \geq q_3$,

$$\frac{1}{\lambda_{m,n}} |\{(u, v) \in I_{m,n} : (u, v) \in M_3\}| > \frac{2}{3}$$

holds. Proceeding like this, we can establish a increasing sequence (q_j) of positive integers such that $q_j \in M_j$ and for all $m, n \geq q_j$,

$$\frac{1}{\lambda_{m,n}} |\{(u, v) \in I_{m,n} : (u, v) \in M_j\}| > 1 - \frac{1}{j} \tag{3.2}$$

is true.

Let us examine M as follows: each natural number of the interval $[1, q_1]$ belong to M and any natural number of the interval $[q_j, q_{j+1}]$ belongs to M iff it belongs to

M_j, $(j \in \mathbb{N})$.
From (3.1) and (3.2), we have for each $q_j \leq m, n < q_{j+1}$,

$$\frac{|\{(u,v) \in I_{m,n} : (u,v) \in M\}|}{\lambda_{m,n}} \geq \frac{|\{(u,v) \in I_{m,n} : (u,v) \in M_j\}|}{\lambda_{m,n}} > 1 - \frac{1}{j}.$$

As a result, $\delta_{(\lambda,\mu)}(M) = 1$. Let $\kappa > 0$ be given. By Archimedean property, select $j \in \mathbb{N}$ such that $\frac{1}{j} < \kappa$. Further, assume $u, v \in M$ be such that $u \geq q_j$ and $v \geq q_j$. Then there exists $t \geq j$ such that $q_t \leq u \leq q_{t+1}$, $q_t \leq v \leq q_{t+1}$. But by the definition of M, $(u,v) \in M_t$. Thus we have

$$\mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) < \frac{1}{t} \leq \frac{1}{j} < \kappa.$$

So $(w_{m_u n_v}) \overset{\|.,\ldots,.\|_{\mathcal{G}}}{\rightarrow} w_0$ holds. The proof is complete. □

Theorem 3.10. *Assume $(Y, \|.,\ldots,.\|_{\mathcal{G}})$ be any n-normed GNLS and let $(\lambda_{m,n}) \in \Delta_2$. Then $st_2^{nN}(\mathcal{G}) \subset S_{(\lambda,\mu)}^{nN}(\mathcal{G})$ iff $\liminf_{m,n} \frac{\lambda_{m,n}}{mn} > 0$.*

Proof. Assume first that $\liminf_{m,n} \frac{\lambda_{m,n}}{mn} > 0$. Then for given $\kappa > 0$ and $\tau \in (0,1)$, we obtain

$$\begin{aligned}
&\tfrac{1}{mn}\left|\left\{u \leq m, v \leq n : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) \geq \kappa\right\}\right| \\
&\quad \geq \tfrac{1}{mn}\left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) \geq \kappa\right\}\right| \\
&\quad \geq \tfrac{\lambda_{m,n}}{mn} \cdot \tfrac{1}{\lambda_{m,n}}\left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) \geq \kappa\right\}\right|.
\end{aligned}$$

It follows that

$$w_{uv} \rightarrow w_0(st_2^{nN}(\mathcal{G})) \Rightarrow w_{uv} \rightarrow w_0(S_{(\lambda,\mu)}^{nN}(\mathcal{G})).$$

Thus $st_2(\mathcal{G}) \subset S_{(\lambda,\mu)}(\mathcal{G})$.

Conversely, assume that $\liminf_{m,n} \frac{\lambda_{m,n}}{mn} = 0$. Then we can choose a subsequence $(m(p), n(r))_{p,r=1,1}^{\infty,\infty}$ such that

$$\frac{\lambda_{m(p),n(r)}}{m(p)\,n(r)} < \frac{1}{pr}.$$

We consider a double sequence $w = (w_{uv})$ as follows:

$$w = (w_{uv}) = \begin{cases} 1 & \text{if } u, v \in I_{m(p),n(r)}\ , \quad p, r = 1, 2, 3, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Then w is gradually statistically convergent with respect to n-norm in GNLS. Thus $w \in st_2^{nN}(\mathcal{G})$. However, $w \notin [V, \lambda, \mu]^{nN}(\mathcal{G})$. So by Theorem 3.7 (3), $w \notin S_{(\lambda,\mu)}^{nN}(\mathcal{G})$. This finalizes the proof. □

Theorem 3.11. *Assume $(Y, \|.,\ldots,.\|_{\mathcal{G}})$ be any GNnLS. If $(\lambda_{m,n}) \in \Delta_2$ such that $\lim_{m,n} \frac{\lambda_{m,n}}{mn} = 1$, then $st_2^{nN}(\mathcal{G}) = S_{(\lambda,\mu)}^{nN}(\mathcal{G})$.*

Proof. Since $\lim \frac{\lambda_{m,n}}{mn}=1$, for each $\kappa>0$ and $\tau\in(0,1)$, we examine that

$$\frac{1}{mn}\left|\left\{u\le m, v\le n : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau)\ge\kappa\right\}\right|$$
$$\le \frac{1}{mn}\left|\left\{u\le m-\lambda_m, v\le n-\mu_n : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau)\ge\kappa\right\}\right|$$
$$+\frac{1}{mn}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau)\ge\kappa\right\}\right|$$
$$\le \frac{(m-\lambda_m)(n-\mu_n)}{mn}+\frac{1}{mn}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau)\ge\kappa\right\}\right|$$
$$= \frac{(m-\lambda_m)(n-\mu_n)}{mn}+\frac{\lambda_{m,n}}{mn}.\frac{1}{\lambda_{m,n}}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau)\ge\kappa\right\}\right|.$$

This means that w is gradually statistically convergent with respect to the n-norm in GNLS, if (w_{uv}) is gradually (λ,μ)-statistically convergent with respect to the n-norm. As a result, we get $S^{nN}_{(\lambda,\mu)}(\mathcal{G})\subset st^{nN}_2(\mathcal{G})$. Since $\lim_{m,n}\frac{\lambda_{m,n}}{mn}=1$, means that $\liminf_{m,n}\frac{\lambda_{m,n}}{mn}>0$, according to Theorem 3.10, we get $st^{nN}_2(\mathcal{G})\subset S^{nN}_{(\lambda,\mu)}(\mathcal{G})$. Then we have $st^{nN}_2(\mathcal{G})=S^{nN}_{(\lambda,\mu)}(\mathcal{G})$. □

Open Question: We do not know whether the condition $\lim_{m,n}\frac{\lambda_{m,n}}{mn}=1$, in the Theorem 3.11 is necessary and leave it as an open problem.

Theorem 3.12. *Let (w_{uv}) be a double sequence in the n-normed GNLS $(Y,\|.,...,.\|_{\mathcal{G}})$ such that $w_{uv}\to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$. Then w_0 is unique.*

Proof. Suppose that there exist elements w_0, w_1 $(w_0\ne w_1)$ in Y such that

$$w_{uv}\to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}));\; w_{uv}\to w_1(S^{nN}_{(\lambda,\mu)}(\mathcal{G})).$$

Since $w_0\ne w_1$, $w_0-w_1\ne 0$. Thus there exist $z_1,z_2,...,z_{n-1}\in Y$ such that w_0-w_1 and $z_1,z_2,...,z_{n-1}$ are linearly independent. So

$$\mathcal{F}_{\|w_0-w_1,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau)=2\kappa>0.$$

Since $w_{uv}\to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$ and $w_{uv}\to w_1(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$, it follows that

$$\lim_{m,n\to\infty}\frac{1}{\lambda_{m,n}}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\ge\kappa\right\}\right|=0$$

and

$$\lim_{m,n\to\infty}\frac{1}{\lambda_{m,n}}\left|\left\{(u,v)\in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_1,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\ge\kappa\right\}\right|=0$$

for each $\kappa>0$, $\tau\in(0,1]$ and for all $z_1,z_2,...,z_{n-1}\in Y$.
There are $(u,v)\in I_{m,n}$ such that

$$\mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)<\kappa \text{ and } \mathcal{F}_{\|w_{uv}-w_1,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)<\kappa.$$

Further, for these $u,\ v$, we have

$$\begin{aligned}\mathcal{F}_{\|w_0-w_1,z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) &< \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}\\ &+\mathcal{F}_{\|w_{uv}-w_1,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)<2\kappa.\end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 3.13. *Let $(Y, \|.,...,.\|_{\mathcal{G}})$ be n-normed GNLS, (w_{uv}) and (t_{uv}) be two double sequences in Y. Then*
(1) $w_{uv} + t_{uv} \to w_0 + t_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$,
(2)$pw_{uv} \to pw_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$ *for any* $p \in \mathbb{R}$.

Proof. The proofs are straightforward. □

Theorem 3.14. *$S^{nN}_{(\lambda,\mu)}(Y) \cap l^2_\infty(Y)$ is a closed subset of $l^2_\infty(Y)$, if Y an n-Banach space.*

Proof. Suppose that $(w^i)_{i\in\mathbb{N}} = (w^i_{uv})_{u,v\in\mathbb{N}}$ is a gradual convergent double sequence in $S^{nN}_{(\lambda,\mu)}(Y) \cap l^2_\infty(Y)$ gradual converging to $w \in l^2_\infty(Y)$. We need to prove that $w \in S^{nN}_{(\lambda,\mu)}(Y) \cap l^2_\infty(Y)$. Assume that $(w^i) \overset{\|.,...,.\|_{\mathcal{G}}}{\to} p_i\left(S^{nN}_{(\lambda,\mu)}(Y)\right)$ for all $i \in \mathbb{N}$. Take a positive decreasing convergent sequence $(\kappa_i)_{i\in\mathbb{N}}$, where $\kappa_i = \frac{\kappa}{2^i}$ for a given $\kappa > 0$. Obviously, $(\kappa_i)_{i\in\mathbb{N}}$ converges to 0. Choose a positive integer i such that $\mathcal{F}_{\|w-w^i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa_i}{4}$ for given $\kappa > 0$, $\tau \in (0,1)$ and for all $z_1, z_2, ..., z_{n-1} \in Y$. Then we have

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \frac{\kappa_i}{4}\right\}\right| = 0$$

and

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\left\|w^{i+1}_{uv}-p_{i+1},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) \geq \frac{\kappa_{i+1}}{4}\right\}\right| = 0.$$

Now

$$\frac{1}{\lambda_{m,n}} \Bigg|\Bigg\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \tfrac{\kappa_i}{4} \vee \mathcal{F}_{\left\|w^{i+1}_{uv}-p_{i+1},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) \geq \tfrac{\kappa_{i+1}}{4}\Bigg\}\Bigg| < 1$$

and for all $u,\ v \in \mathbb{N}$,.

$$\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \tfrac{\kappa_i}{4}\right\} \cap \left\{(u,v) \in I_{m,n} : \mathcal{F}_{\left\|w^{i+1}_{uv}-p_{i+1},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) \geq \tfrac{\kappa_{i+1}}{4}\right\}$$

is infinite. Thus there must exist $(u,v) \in I_{m,n}$ for which we have simultanously,

$$\mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa_i}{4} \text{ and } \mathcal{F}_{\left\|w^{i+1}_{uv}-p_{i+1},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) < \frac{\kappa_{i+1}}{4}.$$

So it follows that

$$\begin{aligned}
\mathcal{F}_{\|p_i-p_{i+1},z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) &\leq \mathcal{F}_{\|p_i-w^i_{uv},z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) \\
&+\mathcal{F}_{\left\|w^i_{uv}-w^{i+1}_{uv},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\left\|w^{i+1}_{uv}-p_{i+1},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) \\
&\leq \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\left\|w^{i+1}_{uv}-p_{i+1},z_1,z_2,...,z_{n-1}\right\|_{\mathcal{G}}}(\tau) \\
&+\mathcal{F}_{\|w-w^i,z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|w-w^{i+1},z_1,z_2,...,z_{n-1}\|_{\mathcal{G}}}(\tau) \\
&\leq \tfrac{\kappa_i}{4} + \tfrac{\kappa_{i+1}}{4} + \tfrac{\kappa_i}{4} + \tfrac{\kappa_{i+1}}{4} \leq \kappa_i.
\end{aligned}$$

This means that (p_i) is a Cauchy sequence in Y. Hence there is number $p \in Y$ such that $p_i \overset{\|.,...,.\|_{\mathcal{G}}}{\to} p$ as $i \to \infty$.

We need to prove that $w \to p(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$. For any $\kappa > 0$, choose $i \in \mathbb{N}$ such that $\kappa_i < \frac{\kappa}{4}$,

$$\mathcal{F}_{\|w-w^i,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa}{4} \text{ and } \mathcal{F}_{\|p_i-p,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa}{4}.$$

Then we get

$$\begin{aligned}
&\tfrac{1}{\lambda_{m,n}}\left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-p,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| \\
&\quad \leq \tfrac{1}{\lambda_{m,n}}\left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau)\right.\right. \\
&\quad\quad \left.\left. +\mathcal{F}_{\|w_{uv}-w^i_{uv},z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|p_i-p,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa\right\}\right| \\
&\quad \leq \tfrac{1}{\lambda_{m,n}}\left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) + \tfrac{\kappa}{4} + \tfrac{\kappa}{4} \geq \kappa\right\}\right| \\
&\quad \leq \tfrac{1}{\lambda_{m,n}}\left|\left\{(u,v) \in I_{m,n} : \mathcal{F}_{\|w^i_{uv}-p_i,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \tfrac{\kappa}{2}\right\}\right| \to 0 \text{ as } m,n \to \infty
\end{aligned}$$

in Pringsheim sense. This gives that $w \to p(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$, which completes the proof. □

Lemma 3.15. *Assume $q \geq 2$ be a fixed natural number and $B = \left\{m,n \in \mathbb{N} : (mn)^{\frac{1}{q}} \in \mathbb{N}\right\}$. If $\lim_{m,n} \frac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}}$ exists, then $\delta_{(\lambda,\mu)}(B) = 0$.*

Proof. Let

$$B_{m,n} = \{(u,v) \in B : (u,v) \in I_{m,n}\} \text{ and } \lim_{m,n} \frac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}} = l.$$

Then it is easy to indicate that

$$|B_{m,n}| = \left[(mn)^{\frac{1}{q}}\right] - \left[(mn - \lambda_{m,n} + \frac{1-(-1)^{mn}}{2})^{\frac{1}{q}}\right],$$

where $[q]$ denotes the largest integer $\leq q$.

Case-I: If m,n are even, then we have

$$\begin{aligned}
&(mn)^{\frac{1}{q}} - 1 \leq \left[(mn)^{\frac{1}{q}}\right] \leq (mn)^{\frac{1}{q}} \\
&\Rightarrow \tfrac{(mn)^{\frac{1}{q}}-1}{\lambda_{m,n}} \leq \tfrac{\left[(mn)^{\frac{1}{q}}\right]}{\lambda_{m,n}} \leq \tfrac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}} \Rightarrow \lim_{m,n} \tfrac{\left[(mn)^{\frac{1}{q}}\right]}{\lambda_{m,n}} = l.
\end{aligned}$$

$$\begin{aligned}
\text{Also,}\quad &(mn - \lambda_{m,n})^{\frac{1}{q}} - 1 \leq [(mn - \lambda_{m,n})^{\frac{1}{q}}] \leq (mn - \lambda_{m,n})^{\frac{1}{q}} \\
&\Rightarrow \frac{(mn - \lambda_{m,n})^{\frac{1}{q}} - 1}{\lambda_{m,n}} \leq \frac{[(mn - \lambda_{m,n})^{\frac{1}{q}}]}{\lambda_{m,n}} \leq \frac{(mn - \lambda_{m,n})^{\frac{1}{q}}}{\lambda_{m,n}} \\
&\Rightarrow \frac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}}(1 - \frac{\lambda_{m,n}}{mn})^{\frac{1}{q}} - \frac{1}{\lambda_{m,n}} \\
&\leq \frac{[(mn - \lambda_{m,n})^{\frac{1}{q}}]}{\lambda_{m,n}} \leq \frac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}}(1 - \frac{\lambda_{m,n}}{mn})^{\frac{1}{q}}.
\end{aligned}$$

If $\frac{\lambda_{m,n}}{mn} < 1$, then from the above,

$$\frac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}} - O(\frac{\lambda_{m,n}}{mn}) - \frac{1}{\lambda_{m,n}} \leq \frac{[(mn-\lambda_{m,n})^{\frac{1}{q}}]}{\lambda_{m,n}} \leq \frac{(mn)^{\frac{1}{q}}}{\lambda_{m,n}} - O(\frac{\lambda_{m,n}}{mn}).$$

Thus $\lim\limits_{m,n} \frac{[(mn-\lambda_{m,n})^{\frac{1}{q}}]}{\lambda_{m,n}} = l$.

If $\frac{\lambda_{m,n}}{mn} = 1$, then $\lim\limits_{m,n} \frac{[(mn-\lambda_{m,n})^{\frac{1}{q}}]}{\lambda_{m,n}} = l$ is trivial. Thus we have

$$\lim_{m,n} \frac{[(mn)^{\frac{1}{q}}]}{\lambda_{m,n}} - \lim_{m,n} \frac{[(mn-\lambda_{m,n})^{\frac{1}{q}}]}{\lambda_{m,n}} = l - l = 0.$$

So if m, n are even, then $\lim\limits_{m,n} \frac{|B_{m,n}|}{\lambda_{m,n}} = 0$.

Case-II: If m, n are odd, utilizing the similar technique it can be easily shown that $\lim\limits_{m,n} \frac{|B_{m,n}|}{\lambda_{m,n}} = 0$.

Hence, from above two cases, we can conclude that $\delta_{(\lambda,\mu)}(B) = 0$. □

Remark 3.16. Every subsequence of a gradually (λ, μ)-statistical convergent sequence is not necessarily gradually (λ, μ)-statistical convergent.

Example 3.17. Assume $Y = \mathbb{R}$ and $\|\cdot\|_{\mathcal{G}}$ be the norm determined in Example 3.5. Consider the sequence $(\lambda_{m,n})$ defined by

$$\lambda_{m,n} = \begin{cases} 1 & mn = 1, \\ \frac{mn}{2} & mn \geq 2. \end{cases}$$

Take

$$w_{uv} = \begin{cases} uv & if u = p^2, v = q^2, p, q \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

Then for any $\kappa > 0$,

$$\begin{aligned} &\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left| \left\{ (u, v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-\mathbf{0},z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \kappa \right\} \right| \\ &= \delta_{(\lambda,\mu)}(B), \text{ where } B = \{mn \in \mathbb{N} : \sqrt{mn} \in \mathbb{N}\} \\ &= 0. \text{ [By Lemma 3.15, considering } q = 2] \end{aligned}$$

Thus $w_{uv} \to \mathbf{0}(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$. But the sequence considered in Example 3.5 is not gradually (λ, μ)-statistical convergent with respect to n-norm although it is a subsequence of the above sequence.

Definition 3.18. Assume $(Y, \|., ..., .\|_{\mathcal{G}})$ be any n-normed GNLS. A sequence (x_k) in X is called to be a *gradually* (λ, μ)*-statistical Cauchy sequence with respect to* n *normed*, provided that for each $\kappa > 0$ and $\tau \in (0, 1]$, there exist $P = P(\kappa) \in \mathbb{N}$ and $Q = Q(\kappa) \in \mathbb{N}$ such that

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left| \left\{ (u, v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_{PQ},z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) \geq \kappa \right\} \right| = 0$$

or equivalently, $\mathcal{F}_{\|w_{uv}-w_{PQ},z_1,z_2,\ldots,z_{n-1}\|_G}(\tau) < \kappa$ a.a.u, v.

Theorem 3.19. *Assume* $(Y, \|., ..., .\|_{\mathcal{G}})$ *be any* n*-normed GNLS. Then each gradually* (λ,μ)*-statistical convergent sequence with respect to* n*-norm is gradually* (λ,μ)*-statistical Cauchy with respect to same norm.*

Proof. Let $w_{uv} \to w_0(S^{nN}_{(\lambda,\mu)}(\mathcal{G}))$. Then for any $\kappa > 0$ and $\tau \in (0,1]$,

$$\lim_{m,n\to\infty} \frac{1}{\lambda_{m,n}} \left| \left\{ (u,v) \in I_{m,n} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \geq \frac{\kappa}{2} \right\} \right| = 0.$$

This means that $\mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa}{2}$ a.a.u, v.
i.e.,

$$\delta_{\lambda,\mu}\left(\left\{ (u,v) \in \mathbb{N}\times\mathbb{N} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}} \geq \frac{\kappa}{2} \right\} \right) = 0$$

i.e.,

$$\delta_{\lambda,\mu}\left(\left\{ (u,v) \in \mathbb{N}\times\mathbb{N} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}} < \frac{\kappa}{2} \right\} \right) \neq 0.$$

Thus the set

$$\left\{ (u,v) \in \mathbb{N}\times\mathbb{N} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa}{2} \right\}$$

is non empty.
Select $P = P(\kappa) \in \mathbb{N}$ and $Q = Q(\kappa) \in \mathbb{N}$ such that

$$P, Q \quad \in \quad \left\{ (u,v) \in \mathbb{N}\times\mathbb{N} : \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) < \frac{\kappa}{2} \right\}.$$

Then we obtain

$$\begin{aligned} \mathcal{F}_{\|w_{uv}-w_{PQ},z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) &= \mathcal{F}_{\|w_{uv}-w_0+w_0-w_{PQ},z_1,z_2,\ldots,z_{n-1}\|}(\tau) \\ &\leq \mathcal{F}_{\|w_{uv}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) + \mathcal{F}_{\|w_{PQ}-w_0,z_1,z_2,\ldots,z_{n-1}\|_{\mathcal{G}}}(\tau) \\ &< \kappa \quad a.a.u,\ v. \end{aligned}$$

Thus (w_{uv}) is gradually (λ,μ)-statistical Cauchy with respect to n-norm. □

4. Conclusion

We have outlined a few key characteristic of (λ,μ)-statistical convergence in the n-normed GNnLS in this paper. To show how the ideas are related, we also added $[V,\lambda,\mu]^{nN}$-summability to the n-normed GNnLS and created Theorem 3.7. Lastly, we have researched the idea of (λ,μ)-statistical Cauchy double sequences in the n-normed GNnLS and analyzed the interconnection between gradual (λ,μ)-statistical convergent and gradual (λ,μ)-statistical Cauchy double sequences with regards to n-norm.

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