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Abstract. In this paper we study interrelation between fuzzy graphs and fuzzy topological spaces. We have introduced various fuzzy topological spaces on simple connected fuzzy graphs by using adjacency relation. Some characterization theorems for fuzzy topologies related to isomorphic fuzzy graphs, weak isomorphic fuzzy graphs and co-week isomorphic fuzzy graphs are given. Interior, closure properties, T_1 and T_2 separation axioms for these fuzzy topological spaces are studied. In last section we define neighborhood fuzzy topological spaces on simple fuzzy graph and establish a linkage between fuzzy topology and fuzzy graph.

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Keywords: Fuzzy vertex topology, Fuzzy edge topology, Fuzzy neighborhood topology.

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1. Introduction

Graph theory has wide applications in various fields like digital image processing, computer technology, communication science, networking problems, mechanism analysis, civil engineering, electric engineering, graphics, medical field, traffic problems, genetics etc. Graph is the pictorial representation of objects and binary relation between them. Various real life situations can be modeled using graphs and their practical solution can be obtained. Topology is a study of objects that are invariant under certain deformations. Topology has vast applications in fields like digital machine learning, image processing, graphics, robotics, biology, civil engineering, geographic information system, data analysis, remote sensing, networking, traffic problems, artificial intelligence, economics etc. Both graph theory and topology are originated from Euler's " Seven Bridge of Konigsberg problem " in 1736 and because of their common applications in many fields, the study of interrelation between them is of great interest. Amiri et al. [1] defined graphic topology on crisp graph. Kilicman and Abdulkalek [2] introduced a incidence topology with a set of vertices for any simple crisp graph. Sari and Kopuzlu [3] studied topologies generated by simple undirected crisp graphs without isolated vertices and discussed the condition for homeomorphic topological spaces generated by crisp graphs. Ibraheem and Nagim [4] generates topology on crisp graph by defining relations on the edges set of graph. Gholap and Nikumbh [5] introduced topological spaces on simple crisp graph by using adjacency relation and non adjacency relation on set of vertices. Many researchers introduced and studied topological spaces associated with graphs using various relations.

In 1968, Chang [6] introduced fuzzy topological spaces and in 1975, Rosenfeld [7] introduced fuzzy graphs and various results analogue to crisp graphs. Pramanik et al. [8] introduced interval valued fuzzy competition graph (IVFC) and discussed its properties. Rashmanlou et al. [9] defined direct product, degree of vertex in cubic graph and introduced complete cubic graphs. The concept of reinforcement number with respect to half-domination introduced by Muhiuddin [10]. Amanathulla et al. [11] studied surjective $L(2,1)$ labeling problems for paths and interval graphs. Atef et al. [12] introduced a new kind of fuzzy topological structures in terms of fuzzy graphs called fuzzy topological graphs, which is study of fuzzy graphs on certain class of fuzzy subsets.

Since fuzzy graphs and fuzzy topological spaces have some common applications in various fields hence to find relation between them is important. Many authors studied topological spaces on various crisp graphs but there in no literature available where fuzzy topological spaces on fuzzy graphs are studied. Using fuzzy topological structure we can study different aspect of various fuzzy graphs, this is our motivation to introduce fuzzy topological spaces on certain fuzzy graphs. In this paper we introduce fuzzy vertex topology and fuzzy edge topology on simple connected fuzzy graphs using adjacency relation and discussed related results. In the last section we define fuzzy neighborhood topology on simple fuzzy graphs.

2. Preliminaries

Definition 2.1 ([13])**.** Let S be a set. A map $\sigma : S \to [0,1]$ is called a *fuzzy subset* of S. A *fuzzy relation* on S is a fuzzy subset of $S \times S$.

Let S be a set , A and B be fuzzy sets on S. Then the *join* and the *meet* of A and B, denoted by $A \vee B$ and $A \wedge B$, are defined as follows: for each $x \in S$,

$$(A \vee B)(x) = max(A(x), B(x)) \text{ and } (A \wedge B)(x) = min(A(x), B(x)).$$

Definition 2.2 ([14])**.** A *fuzzy graph* is a triplet $G = (V, \sigma, \mu)$, where V is a finite nonempty set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ satisfying

$$\mu(a, b) \leq \sigma(a) \wedge \sigma(b) \ \forall a,\ b \in V.$$

The fuzzy set σ and μ are called the *fuzzy vertex set* of G and the *fuzzy edge set* of G respectively. Clearly, μ is a fuzzy relation on σ. A fuzzy graph $H = (U, \rho, \nu)$ is called a *fuzzy partial fuzzy subgraph* of $G = (V, \sigma, \mu)$, if $\rho \leq \sigma$ and $\nu \leq \mu$. We call $H = (U, \rho, \nu)$ a *spanning fuzzy subgraph* of $G = (V, \sigma, \mu)$, if $\rho = \sigma$. A *path* in a fuzzy graph $G = (V, \sigma, \mu)$ is a sequence of distinct vertices $a_0, a_1, a_2, ..., a_n$ such that $\mu(a_{i-1}, a_i) > 0, 1 \leq i \leq n$. Two vertices are said to *be connected*, if they are joined by path. The *strength* of path is defined as $\bigwedge_{i=1}^{n} \mu(a_{i-1}, a_i)$. The *strength of connectedness* between two vertices a and b is defined as the maximum of the strengths of all paths between a and b and it is denoted by $\mu^\infty(a, b)$. A fuzzy graph $G = (V, \sigma, \mu)$ is said to *be connected*, if $\mu^\infty(a, b) > 0$ for every a, b in V.

Definition 2.3 ([14]). A fuzzy graph $G = (V, \sigma, \mu)$ is said to be a *complete fuzzy graph*, if $\mu(a, b) = \sigma(a) \wedge \sigma(b)\ \forall\ a,\ b \in V$.

Definition 2.4 ([15]). A fuzzy graph $G = (V, \sigma, \mu)$ is said to be *complete bipartite*, if vertex set V can be partitioned into two non empty sets V_1 and V_2 such that $\mu(a, b) = 0$ if a and b both belongs to V_1 or V_2 and $\mu(a, b) = \sigma(a) \wedge \sigma(b)$, if $a \in V_1$ and $b \in V_2$.

Definition 2.5 ([7]). Let $G = (V, \sigma, \mu)$ and $G' = (V', \sigma', \mu')$ be two fuzzy graphs. A map $f : G \to G'$ is said to be a *homomorphism* from G to G', if $\sigma(a) \leq \sigma'(f(a))$ for all $a \in V$ and $\mu(a, b) \leq \mu'(f(a), f(b))$ for all $a,\ b \in V$. A bijective map $f : G \to G'$ is said to be an *isomorphism* from G to G' if $\sigma(a) = \sigma'(f(a))$ for all $a \in V$ and $\mu(a, b) = \mu'(f(a), f(b))$ for all $a,\ b \in V$. A bijective homomorphism $f : G \to G'$ is said to be a *weak isomorphism* from G to G', if $\sigma(a) = \sigma'(f(a))$ for all $a \in V$. A bijective homomorphism $f : G \to G'$ is said to be a *co-weak isomorphism* from G to G' if $\mu(a, b) = \mu'(f(a), f(b))$ for all $a, b \in V$.

Definition 2.6 ([16]). Let (X, τ) be a fuzzy topological space. Then a subfamily β of τ is called a *base* for τ, if every member of τ can be written as a union of members of β. A subfamily S of τ is called a *subbase* for τ, if the family of finite intersections of its members forms a base for τ. If τ_1 and τ_2 are fuzzy topologies on X and $\tau_1 \subseteq \tau_2$, then we say that τ_1 *is coarser than* τ_2 or τ_2 *is finer than* τ_1.

Definition 2.7 ([17]). A fuzzy topology τ is said to be *generated* by a subfamily S of fuzzy sets in X, if every member of τ is a union of finite intersections of members of S.

Definition 2.8 ([18]). Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then the *complement* of G, denoted by $\bar{G} = (V, \sigma, \bar{\mu})$, is defined as:

$$\bar{\mu}(a, b) = \sigma(a) \wedge \sigma(b) - \mu(a, b)\ \forall a,\ b \in V.$$

Definition 2.9 ([18]). A fuzzy graph $G = (V, \sigma, \mu)$ is said to be *self complementary*, if G is isomorphic to its complement $\bar{G}$.

Definition 2.10 ([17]). A fuzzy topological space (X, τ) is called a *fuzzy T_1- topological space*, if for every pair of distinct fuzzy points $p,\ q \in X$, we can find fuzzy open sets U and V in (X, τ) such that $p \in U, p \notin V$ and $q \in V,\ q \notin U$.

Definition 2.11 ([17]). A fuzzy topological space (X, τ) is said to be *fuzzy T_2* or *Hausdroff*, if for every pair of distinct fuzzy points $p,\ q \in X$, there exist two fuzzy open sets U and V in (X, τ) such that $p \in U$, $q \in V$ and $U \cap V = 0$.

3. Fuzzy vertex topology generated by fuzzy graphs

Definition 3.1. Let $G = (V, \sigma, \mu)$ be the simple connected fuzzy graph and let $V(G)$ be a fuzzy set of vertices. On fuzzy set $V(G)$ we define an *adjacency relation*

R as $((u, \sigma(u)), (v, \sigma(v))) \in R$, if $\mu(u, v) > 0$, where $(u, \sigma(u))$ and $(v, \sigma(v)) \in V(G)$. Now for $(u, \sigma(u)) \in V(G)$, we define

$$R[u] = \{(v, \sigma(v)) \in V(G)/\mu(u, v) > 0\}$$

Then the set $S = \{R[u] : (u, \sigma(u)) \in V(G)\}$ forms a subbasis for a topology on $V(G)$. Let β be the finite intersection of members of subbasis S. Then clearly, β forms a basis. The collection τ of all union of members of β is a topology on $V(G)$ called as a *fuzzy vertex topology* generated by fuzzy graph G and the ordered pair $(V(G), \tau)$ as a *fuzzy vertex topological space* generated by fuzzy graph G. The members of τ are called *v-open fuzzy sets* and the complement of an v-open fuzzy set is called a *v-closed fuzzy set*.

Example 3.2. From figure 1, let G_1 be fuzzy graph with vertex set and let

$$V(G_1) = \{(a, 0.2), (b, 0.5), (c, 0.4), (d, 0.7)\}.$$

Then we have

$R[a] = \{(b, 0.5)\},\ R[b] = \{(a, 0.2),\ (c, 0.4), (d, 0.7)\},\ R[c] = \{(b, 0.5)\},\ R[d] = \{(b, 0.5)\}.$

Moreover, we get

$$S_1 = \big\{\{(b, 0.5)\}, \{(a, 0.2), (c, 0.4), (d, 0.7)\}\big\}$$

and

$$\beta_1 = \big\{0, \{(b, 0.5)\}, \{(a, 0.2), (c, 0.4), (d, 0.7)\}\big\}.$$

Thus $\tau_1 = \big\{0, \{(b, 0.5)\}, \{(a, 0.2), (c, 0.4), (d, 0.7)\}, V(G_1\big\}$ is a fuzzy vertex topology generated by G_1.

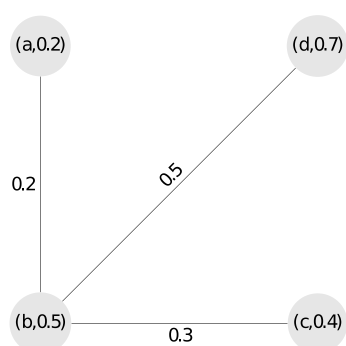

Figure 1. A simple connected fuzzy graph G_1

Theorem 3.3. *If H is a simple connected spanning fuzzy subgraph of fuzzy graph G, then the fuzzy vertex topology generated by fuzzy graph H is finer than fuzzy vertex topology generated by fuzzy graph G.*

Proof. Let $H = (U, \tau, \nu)$ be a simple connected spanning fuzzy subgraph of fuzzy graph $G = (V, \sigma, \mu)$, where $U = V$. Then by 3.1, $S_1 = \{R[u] : (u, \tau(u)) \in U\}$ and $S_2 = \{R[u] : (u, \sigma(u)) \in V\}$ forms a subbasis of vertex topology τ_1 on H and fuzzy vertex topology τ_2 on G respectively. Clearly, $S_1 \subseteq S_2$. Thus $\tau_2 \subseteq \tau_1$. □

Theorem 3.4. *If two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are isomorphic, then the fuzzy vertex topologies τ_1 and τ_2 generated by G_1 and G_2 respectively are homeomorphic.*

Proof. Since $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are isomorphic. Then there exists a bijective mapping $f : G_1 \to G_2$ such that $\sigma_1(a) = \sigma_2(f(a))\ \forall a \in V_1$ and $\mu_1(a, b) = \mu_2(f(a), f(b))\ \forall a, b \in V_1$. Thus the sets $S_1 = \{R[a] : (a, \sigma_1(a)) \in V_1\}$ and $S_2 = \{R[f(a)] : (f(a)\ \sigma_2(f(a))) \in V_2\}$ are equivalent. So the fuzzy vertex topologies τ_1 and τ_2 generated by subbasis S_1 and S_2 respectively are homeomorphic. □

Example 3.5. From figure 2, G_1 and G_2 are isomorphic fuzzy graphs. Let τ_1 be the fuzzy vertex topology generated by G_1 and let τ_2 be the fuzzy vertex topology generated by G_2. Then we have

$$\tau_1 = \big\{0, \{(b, 0.5)\}, \{(d, 0.2)\}, \{(b, 0.5), (d, 0.2)\}, \{(a, 0.6), (c, 0.9)\},$$
$$\{(a, 0.6), (c, 0.9), (d, 0.2)\}, \{(a, 0.6), (b, 0.5), (c, 0.9)\}, V(G_1\big\}$$

and

$$\tau_2 = \big\{0, \{(x, 0.5)\}, \{(z, 0.2)\}, \{(y, 0.9), (w, 0.6)\}, \{(x, 0.5), z, 0.2)\},$$
$$\{(y, 0.9), (z, 0.2), (w, 0.6)\}, \{(x, 0.5), (y, 0.9), (w, 0.6)\}, V(G_2\big\}.$$

So τ_1 and τ_2 are homeomorphic.

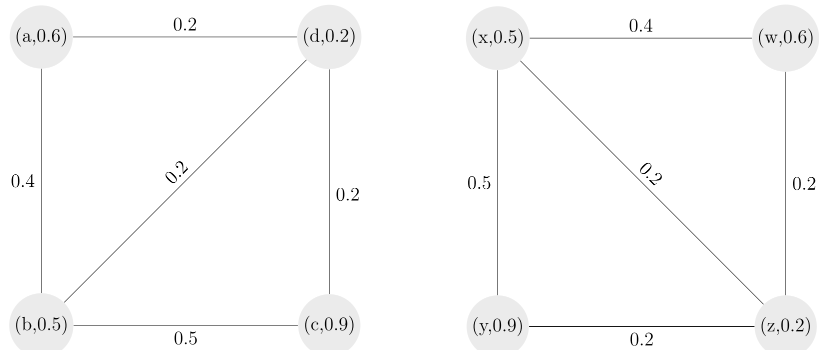

Figure 2. Isomorphic fuzzy graphs G_1 and G_2

Corollary 3.6. *If two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are weak isomorphic then the fuzzy vertex topologies τ_1 and τ_2 generated by G_1 and G_2 respectively are homeomorphic.*

Remark 3.7. If two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are co-weak isomorphic then the fuzzy vertex topology τ_1 generated by G_1 and the fuzzy vertex topology τ_2 generated by G_2 may not be homeomorphic.

Example 3.8. From figure 3, G_1 and G_2 are co-weak isomorphic fuzzy graphs. Let τ_1 be the fuzzy vertex topology generated by G_1 and let τ_2 be the fuzzy vertex topology generated by G_2. Then we have

$$\tau_1 = \{0, \{(b, 0.5)\}, \{(d, 0.2)\}, \{(b, 0.5), (d, 0.2)\}, \{(a, 0.6), (c, 0.9)\}, \{(a, 0.6), (c, 0.9), (d, 0.2)\}, \{(a, 0.6), (b, 0.5), (c, 0.9)\}, V(G_1)\}$$

and

$$\tau_2 = \{0, \{(x, 0.6)\}, \{(z, 0.4)\}, \{(y, 1), (w, 0.6)\}, \{(x, 0.6), z, 0.4)\}, \{(y, 1), (z, 0.4), (w, 0.6)\}, \{(x, 0.6), (y, 1), (w, 0.6)\}, V(G_2\}.$$

Since τ_1 and τ_2 contains different fuzzy sets, τ_1 is not homeomorphic to τ_2.

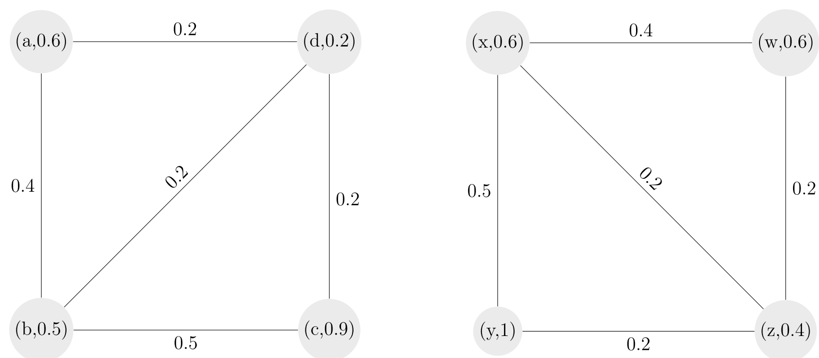

FIGURE 3. Co-weak isomorphic fuzzy graphs G_1 and G_2

Theorem 3.9. *Let X be the set of all simple connected fuzzy graphs. Then the relation $\sim$ defined on X as $G_1 \sim G_2$ if and only if $\tau_1 \simeq \tau_2$ is an equivalence relation on X, where τ_1 and τ_2 are fuzzy vertex topologies generated by G_1 and G_2 respectively.*

Proof. 1. Reflexive: Since $\tau_1 \simeq \tau_1$, $G_1 \sim G_1$.

2. Symmetry: As $\tau_1 \simeq \tau_2$, clearly, $\tau_2 \simeq \tau_1$. Then $G_1 \sim G_2$ implies $G_2 \sim G_1$.

3. Transitive: As $G_1 \sim G_2$ and $G_2 \sim G_3$, $\tau_1 \simeq \tau_2$ and $\tau_2 \simeq \tau_3$ which implies $\tau_1 \simeq \tau_3$. Then $G_1 \sim G_3$. Thus the relation $\sim$ defined on X is an equivalence relation. □

Theorem 3.10. *If $G = (V, \sigma, \mu)$ is a complete fuzzy graph, then the fuzzy vertex topology τ generated by G is T_1.*

Proof. As $G = (V, \sigma, \mu)$ is a complete fuzzy graph, $\mu(a, b) = \sigma(a) \wedge \sigma(b)\ \forall\ a,\ b \in V$. Let $a,\ b \in V$ be any two distinct elements. Then by 3.1, $a \in R[b]$ but $a \notin R[a]$ and $b \in R[a]$ but $b \notin R[b]$. Since $R[a]$ and $R[b]$ are elements of τ, they are open sets. thus for every pair of distinct fuzzy points $a,\ b \in V$, there exist fuzzy open sets $R[a]$ and $R[b]$ in $(V(G), \tau)$ such that $a \in R[b]$ but $a \notin R[a]$ and $b \in R[a]$ but $b \notin R[b]$. So τ is T_1. □

Remark 3.11. Converse of above theorem need not be true. That is if fuzzy vertex topology τ generated by G is T_1, then that fuzzy graph $G = (V, \sigma, \mu)$ may not be complete. For example the fuzzy graph G in figure 4 is not complete but its fuzzy vertex topology $\tau = \big\{0, \{(a, 0.4)\}, \{(b, 0.6)\}, \{(c, 1)\}, \{(a, 0.4), (b, 0.6)\}, \{(a, 0.4), (c, 1)\}, \{(b, 0.6), (c, 1)\}, V(G)\big\}$ is T_1.

Theorem 3.12. *If $G = (V, \sigma, \mu)$ is a complete fuzzy graph, then fuzzy vertex topology τ generated by G is T_2.*

Proof. As $G = (V, \sigma, \mu)$ is a complete fuzzy graph, for each $a \in V(G), R[a] = V - \{(a, \sigma(a))\}$. Then $S = \{V - \{(a, \sigma(a))\} : (a, \sigma(a)) \in V(G)\}$ is a subbase for τ. Let β be the finite intersection of members of S. Then for each $(a, \sigma(a) \in V(G)$, $\{(a, \sigma(a))\} \in \beta$ because $(a, \sigma(a)) \notin R[a]$ but $(a, \sigma(a)) \in R[b]$ for all $(a, \sigma(a)), (b, \sigma(b)) \in V(G)$ such that $(a, \sigma(a)) \neq (b, \sigma(b))$; so that

$$\cap R[b] = \{(a, \sigma(a))\},$$

where intersection is taken over all $(b, \sigma(b)) \in V(G)$ such that $(b, \sigma(b)) \neq (a, \sigma(a))$. Thus the fuzzy vertex topology τ generated by G must contains singleton set of the form $\{(a, \sigma(a))\}$ for each $(a, \sigma(a)) \in V(G)$. So for any $(a, \sigma(a)), (b, \sigma(b)) \in V(G)$ with $(a, \sigma(a)) \neq (b, \sigma(b))$, there exists open sets say $U = \{(a, \sigma(a))\}$ and $V = \{(b, \sigma(b))\}$ in τ such that $(a, \sigma(a)) \in U$, $(b, \sigma(b)) \in V(G)$ and $U \cap V = \phi$. Hence τ is T_2. □

Remark 3.13. Converse of above theorem need not be true. That is, if fuzzy vertex topology τ generated by G is T_2, then that fuzzy graph $G = (V, \sigma, \mu)$ may not be complete. For example, the fuzzy graph G in figure 4 is not complete but its fuzzy vertex topology, $\tau = \big\{0, \{(a, 0.4)\}, \{(b, 0.6)\}, \{(c, 1)\}, \{(a, 0.4), (b, 0.6)\}, \{(a, 0.4), (c, 1)\}, \{(b, 0.6), (c, 1)\}, v(G)\big\}$ is T_2.

Corollary 3.14. *If $G = (V, \sigma, \mu)$ is a complete bipartite fuzzy graph, then fuzzy vertex topology τ generated by G is T_1 and T_2.*

Theorem 3.15. *If $G = (V, \sigma, \mu)$ is a simple connected self complementary fuzzy graph, then fuzzy vertex topologies generated by G and $\bar{G}$ respectively are equal.*

Proof. As G is self complementary, G and $\bar{G}$ are isomorphic. Then by 3.4, fuzzy vertex topologies generated by G and $\bar{G}$ are homeomorphic. It is clear that vertex set V of G and $\bar{G}$ is same. Thus fuzzy vertex topologies generated by G and $\bar{G}$ are equal. □

Example 3.16. From figure 4, G is a self complementary fuzzy graph. Let τ be the fuzzy vertex topology generated by G and let $\bar{\tau}$ be the fuzzy vertex topology generated by $\bar{G}$. Then we have

$$\tau = \big\{0, \{(a, 0.4)\}, \{(b, 0.6)\}, \{(c, 1)\}, \{(a, 0.4), (b, 0.6)\}, \{(a, 0.4), (c, 1)\}, \{(b, 0.6), (c, 1)\}, V(G)\big\}$$

and

$$\bar{\tau} = \big\{0, \{(a, 0.4)\}, \{(b, 0.6)\}, \{(c, 1)\}, \{(a, 0.4), (b, 0.6)\}, \{(a, 0.4), (c, 1)\}, \{(b, 0.6), (c, 1)\}, V(\bar{G})\big\}.$$

Thus $\tau = \bar{\tau}$.

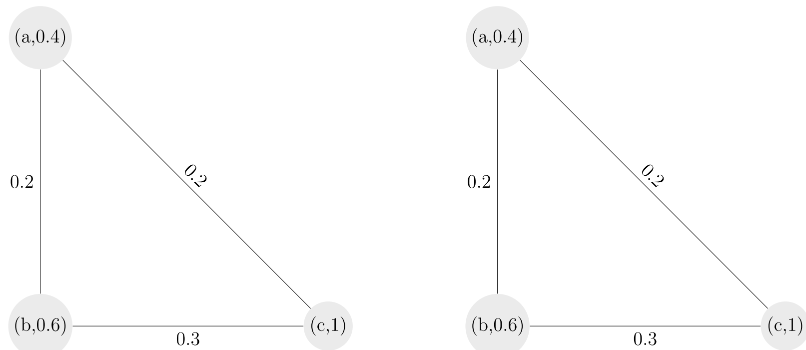

FIGURE 4. Self complementary fuzzy graphs G and $\bar{G}$

4. The interiors and the closures of a fuzzy graph

In this section, we define the interior and the closure of a subset of a vertex set of a fuzzy graph G with respect to vertex topology generated by G.

Definition 4.1. Let $G = (V, \sigma, \mu)$ be a simple connected fuzzy graph and $(V(G), \tau)$ be the corresponding fuzzy vertex topological space. Let A be any subset of $V(G)$. Then the *interior* and the *closure* of A, denoted by $int(A)$ and $cl(A)$, are defined as follows:

$$int(A) = \bigvee\{B \in \tau : B \leq A\} \text{ and } cl(A) = \bigwedge\{B \in \tau' : B \geq A\},$$

where τ' is the complement of τ.

Theorem 4.2. *Let $G = (V, \sigma, \mu)$ be the simple connected fuzzy graph and $(V(G), \tau)$ be the corresponding fuzzy vertex topological space. If A is a subset of $V(G)$, then*
(1) $int(A) \leq A$,
(2) $int(int(A)) = int(A)$
(3) $A \leq B \implies int(A) \leq int(B)$,
(4) $int(A \wedge B) = int(A) \wedge int(B)$,
(5) $A \leq cl(A)$,
(6) $cl(cl(A)) = cl(A)$,
(7) $A \leq B \implies cl(A) \leq cl(B)$,
(8) $cl(A \vee B) = cl(A) \vee cl(B)$.

5. Fuzzy edge topology generated by a fuzzy graph

Definition 5.1. Let $G = (V, \sigma, \mu)$ be a simple connected fuzzy graph and let $E(G)$ be a fuzzy set of edges for it. On fuzzy set $E(G)$, we define an adjacency relation R as $((e_1, \mu(e_1)), (e_2, \mu(e_2))) \in R$, if e_1 is adjacent to e_2 for $(e_1, \mu(e_1)), (e_2, \mu(e_2)) \in E(G)$. Now for $(e_1, \mu(e_1)) \in E(G)$, we define,

$$R[e_i] = \big\{(e_j, \mu(e_j)) \in E(G)/((e_i, \mu(e_i)), (e_j, \mu(e_j))) \in R\big\}, \text{ where } i \neq j.$$

Then the set $S = \big\{R[e]/(e, \mu(e)) \in E(G)\big\}$ forms a subbasis for a topology on $E(G)$. Let β be the finite intersection of members of subbasis S. Then clearly, β forms a basis. The collection τ of all union of members of β is a topology on $E(G)$. We called τ as a *fuzzy edge topology* generated by fuzzy graph G and the ordered pair $(E(G), \tau)$ as a *fuzzy edge topological space* generated by fuzzy graph G. The members of τ are called an *E-open fuzzy set* and the complement of E-open fuzzy set is called an *E-closed fuzzy set*.

Example 5.2. From figure 5,let G_2 be simple connected fuzzy graph with edge set and let $E(G_2) = \{(e_1, 0.4), (e_2, 0.3), (e_3, 0.6)\}$. Then we have

$$R[e_1] = \{(e_2, 0.3), (e_3, 0.6)\}, R[e_2] = \{(e_1, 0.4), (e_3, 0.6)\}, R[e_3] = \{(e_1, 0.4), (e_2, 0.3)\},$$

$$S_2 = \big\{\{(e_2, 0.3), (e_3, 0.6)\}, \{(e_1, 0.4), (e_3, 0.6)\}, \{(e_1, 0.4), (e_2, 0.3)\}\big\},$$

$$\beta_2 = \big\{0, \{(e_1, 0.4)\}, \{(e_2, 0.3)\}, \{(e_3, 0.6)\}, \{\{(e_2, 0.3), (e_3, 0.6)\}, \{(e_1, 0.4), (e_3, 0.6)\}, \{(e_1, 0.4), (e_2, 0.3)\}\big\},$$

$$\tau_2 = \big\{0, \{(e_1, 0.4)\}, \{(e_2, 0.3)\}, \{(e_3, 0.6)\}, \{(e_2, 0.3), (e_3, 0.6)\}, \{(e_1, 0.4), (e_3, 0.6)\}, \{(e_1, 0.4), (e_2, 0.3)\}, E(G_2)\big\}.$$

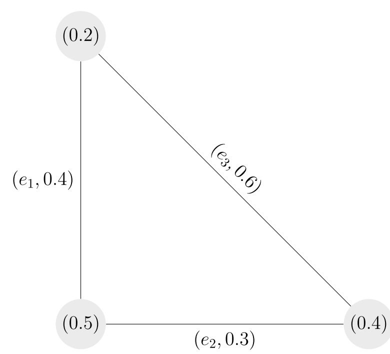

FIGURE 5. A simple connected graph G_2

Theorem 5.3. *If H is a simple connected spanning fuzzy subgraph of fuzzy graph G, then the fuzzy edge topology generated by fuzzy graph H is finer than the fuzzy edge topology generated by fuzzy graph G.*

Theorem 5.4. *If two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are isomorphic, then the fuzzy edge topologies τ_1 and τ_2 generated by G_1 and G_2 respectively are homeomorphic.*

Proof. Since $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are isomorphic. Then there exists a bijective mapping $f : G_1 \to G_2$ such that $\sigma_1(a) = \sigma_2(f(a))$ $\forall a \in V_1$ and $\mu_1((a, b)) = \mu_2(f(a), f(b))$ $\forall a, b \in V_1$. Thus $S_1 = \{R[e] : (e, \mu_1(e)) \in E_1\}$ and $S_2 = \{R[f(e)] : (f(e), \mu_2(f(e))) \in E_2\}$ are equivalent. So the fuzzy edge topologies τ_1 and τ_2 generated by subbasis S_1 and S_2 respectively are homeomorphic. □

Corollary 5.5. *If two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are co-weak isomorphic, then the fuzzy edge topologies τ_1 and τ_2 generated by G_1 and G_2 respectively are homeomorphic.*

Remark 5.6. If two fuzzy graphs $G_1 = (V_1, \sigma_1, \mu_1)$ and $G_2 = (V_2, \sigma_2, \mu_2)$ are weak isomorphic, then the fuzzy edge topology τ_1 generated by G_1 and the fuzzy edge topology τ_2 generated by G_2 may not be homeomorphic.

Example 5.7. From figure 6, G_1 and G_2 are weak isomorphic. Let τ_1 be the fuzzy edge topology generated by G_1 and let τ_2 be the fuzzy edge topology generated by G_2. Then we have

$$\begin{aligned}\tau_1 = \big\{&0, \{(e_2, 0.2)\}, \{(e_1, 0.4), (e_5, 0.2)\}, \{(e_3, 0.2), (e_4, 0.5)\},\\ &\{(e_1, 0.4), (e_2, 0.2), (e_5, 0.2)\}, \{(e_2, 0.2), (e_3, 0.2), (e_4, 0.5)\},\\ &\{(e_1, 0.4), (e_3, 0.2), (e_4, 0.5), (e_5, 0.2), E(G_1)\big\}\end{aligned}$$

and

$$\begin{aligned}\tau_2 = \big\{&0, \{(e'_2, 0.2)\}, \{(e'_1, 0.4), (e'_5, 0.2)\}, \{(e'_3, 0.2), (e'_4, 0.5)\},\\ &\{(e'_1, 0.4), (e'_2, 0.2), (e'_5, 0.2)\}, \{(e'_2, 0.2), (e'_3, 0.2), (e'_4, 0.5)\},\\ &\{(e'_1, 0.4), (e'_3, 0.2), (e'_4, 0.5), (e'_5, 0.2), E(G_2)\big\}.\end{aligned}$$

Thus clearly, τ_1 is not homeomorphic to τ_2.

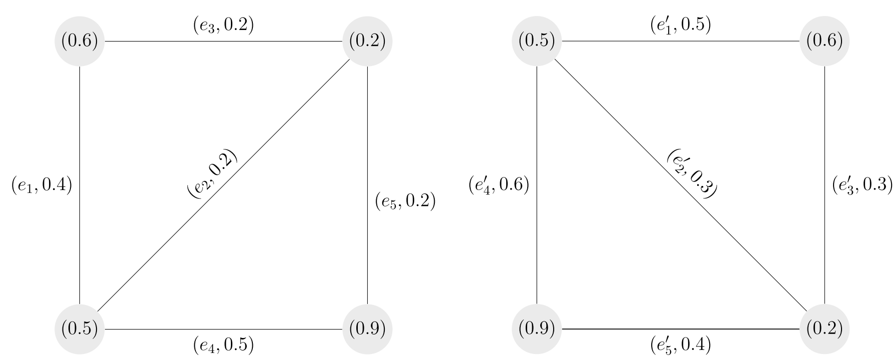

Figure 6. Weak isomorphic fuzzy graphs G_1 and G_2

Theorem 5.8. *Let X be the set of all simple connected fuzzy graphs. Then the relation $\sim$ defined on X as $G_1 \sim G_2$ if and only if $\tau_1 \simeq \tau_2$ is a equivalence relation, where τ_1 and τ_2 be the fuzzy edge topologies generated by G_1 and G_2 respectively.*

Proof. 1. Reflexive: Since $\tau_1 \simeq \tau_1$, $G_1 \sim G_1$.
2. Symmetry: As $\tau_1 \simeq \tau_2$, clearly, $\tau_2 \simeq \tau_1$. Then $G_1 \sim G_2$ implies $G_2 \sim G_1$.
3. Transitive: As $G_1 \sim G_2$ and $G_2 \sim G_3$, $\tau_1 \simeq \tau_2$ and $\tau_2 \simeq \tau_3$ which implies $\tau_1 \simeq \tau_3$. Then $G_1 \sim G_3$. Thus $\sim$ is reflexive, symmetric and transitive. So $\sim$ is equivalence relation. □

Theorem 5.9. *If $G = (V, \sigma, \mu)$ is a simple connected self complementary fuzzy graph, then fuzzy edge topologies generated by G and $\bar{G}$ respectively are equal.*

Proof. As G is self complementary, G and $\bar{G}$ are isomorphic. Then by 5.4, the fuzzy edge topologies generated by G and $\bar{G}$ are homeomorphic. It is obvious that E of G and $\bar{G}$ is same. Thus fuzzy edge topologies generated by G and $\bar{G}$ are equal. □

Theorem 5.10. *If $G = (V, \sigma, \mu)$ is a complete fuzzy graph, then fuzzy edge topology τ generated by G is T_1 and T_2.*

Corollary 5.11. *If $G = (V, \sigma, \mu)$ is a complete bipartite fuzzy graph, then fuzzy edge topology τ generated by G is T_1 and T_2.*

6. Fuzzy neighborhood topology generated by a simple fuzzy graph

Definition 6.1. Let $G = (V, \sigma, \mu)$ be the simple fuzzy graph and let $V(G)$ be a fuzzy set of vertices. Then a *neighborhood* of each $(a, \sigma(a)) \in V(G)$ is defined as set of all vertices in $(b, \sigma(b)) \in V(G)$ such that $\mu(a, b) > 0$ and containing $(a, \sigma(a))$ itself. We denote it as $N[a]$.

Definition 6.2. Let $G = (V, \sigma, \mu)$ be the simple fuzzy graph and let $V(G)$ be a fuzzy set of vertices. Then the set $S = \{N[a] : (a, \sigma(a)) \in V(G)\}$ forms a subbasis for a topology on $V(G)$. Let β be the finite intersection of members of subbasis S. Then clearly, β forms a basis. The collection τ of all union of members of β is a topology on $V(G)$. We called τ as a *fuzzy neighborhood topology* generated by simple fuzzy graph G and the ordered pair $(V(G), \tau)$ as a *fuzzy neighborhood topological space* generated by simple fuzzy graph G. The members of τ are called a *fuzzy N-open fuzzy set* and the complement of open fuzzy set is called a *fuzzy N-closed set.*

Theorem 6.3. *Let $G = (V, \sigma, \mu)$ be the simple fuzzy graph and let $V(G)$ be a fuzzy set of vertices. Then the collection $S = \big\{N[a]/(a, \sigma(a)) \in V(G)\big\}$ forms a subbasis for a fuzzy topology on $V(G)$.*

Example 6.4. From figure 7, let G_3 be fuzzy graph with vertex set
$V(G_3) = \{(a, 0.4), (b, 0.7), (c, 0.8), (d, 0.9)\}$. Then we have

$N[a] = \{(a, 0.4), (b, 0.7)\},\ N[b] = \{(a, 0.4), (b, 0.7)\}, N[c] = \{(c, 0.8)\},\ N[d] = \{(d, 0.9)\},$

$$S_3 = \big\{\{(a, 0.4), (b, 0.7)\}, \{(c, 0.8)\}, \{(d, 0.9)\}\big\},$$
$$\beta_3 = \big\{0, \{(a, 0.4), (b, 0.7)\}, \{(c, 0.8)\}, \{(d, 0.9)\}\big\},$$
$$\tau_3 = \big\{0, \{(c, 0.8)\}, \{(d, 0.9)\}, \{(a, 0.4), (b, 0.7)\}, \{(a, 0.4), (b, 0.7), (c, 0.8)\}, \{(a, 0.4), (b, 0.7), (d, 0.9)\}, V(G_3)\big\}.$$

Theorem 6.5. *If $G = (V, \sigma, \mu)$ is a null fuzzy graph and $(V(G), \tau)$ is a fuzzy neighborhood topological space generated by G, then the proper N-open sets of $(V(G), \tau)$ are singleton.*

Example 6.6. Let G_4 be a null fuzzy graph with vertex set
$$V(G_4) = \{(a, 0.4), (b, 0.7), (c, 0.8), (d, 0.9)\}.$$
Then we have
$$N[a] = \{(a, 0.4)\},\ N[b] = \{(b, 0.7)\},\ N[c] = \{(c, 0.8)\},\ N[d] = \{(d, 0.9)\},$$
$$S_4 = \big\{\{(a, 0.4)\}, \{(b, 0.7)\}, \{(c, 0.8)\}, \{(d, 0.9)\}\big\},$$
$$\beta_4 = \big\{0, \{(a, 0.4)\}, \{(b, 0.7)\}, \{(c, 0.8)\}, \{(d, 0.9)\}\big\}.$$
Thus the fuzzy neighborhood topology generated by G_2 is
$$\tau_4 = \big\{0, \{(a, 0.4)\}, \{(b, 0.7)\}, \{(c, 0.8)\}, \{(d, 0.9)\},$$

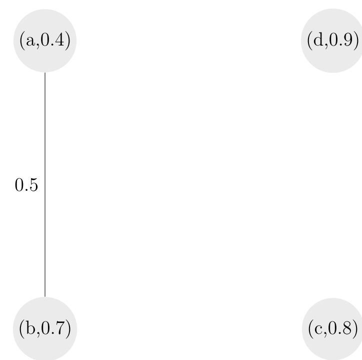

FIGURE 7. A simple fuzzy graph G_3

$$\{(a, 0.4), (b, 0.7)\}, \{(a, 0.4), (c, 0.8)\}, \{(a, 0.4), (d, 0.9)\},$$
$$\{(b, 0.7), (c, 0.8)\}, \{(b, 0.7), (d, 0.9)\}, \{(c, 0.8), (d, 0.9)\},$$
$$\{(a, 0.4), (b, 0.7), (c, 0.8)\}, \{(a, 0.4), (b, 0.7), (d, 0.9)\},$$
$$\{(a, 0.4), (c, 0.8), (d, 0.9)\}, \{(b, 0.7), (c, 0.8), (d, 0.9)\}, V(G_4)\Big\}$$

Theorem 6.7. *If $G = (V, \sigma, \mu)$ is a complete fuzzy graph, then fuzzy neighborhood topology τ generated by G is T_1 and T_2.*

Corollary 6.8. *If $G = (V, \sigma, \mu)$ is a complete bipartite fuzzy graph, then fuzzy neighborhood topology τ generated by G is T_1 and T_2.*

7. Conclusion

In this paper we have studied a relation between fuzzy graphs and fuzzy topological spaces. We have generated three types of fuzzy topological spaces on fuzzy graphs by using adjacency relation. Also fuzzy topologies related to isomorphic fuzzy graphs, weak isomorphic fuzzy graphs and co-week isomorphic fuzzy graphs are investigated. We have shown that homeomorphic fuzzy topological structure forms an equivalence relation. We have verified that the fuzzy topological spaces generated by complete fuzzy graph and complete bipartite fuzzy graph satisfies separation axioms T_1 and T_2.We have shown that using interior and closure of a set corresponding to a fuzzy graph a fuzzy topology can be formed and there interrelationship may give some useful insights. Using fuzzy topological structures as discussed in the paper one can further study various topological aspects and fuzzy topological indices of different types of fuzzy graphs. The results discussed may lead to significant applications in fields that deal with uncertainty,vagueness and imprsion like pattern recognition, image processing,decision making,control system etc. in future.

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