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Abstract. In this paper, we develop a systematic theory for the ideals of an L-ring $L(\mu, R)$. Recently, the authors have introduced the concepts of prime ideals, semiprime ideals and the radical of an ideal in an L-ring. Moreover, they have also introduced the notion of maximal ideals in L-setting. In this paper, we introduce the concept of a primary ideal of an L-ring and establish a necessary and sufficient condition for an ideal to be a primary in term of level subring. We establish some results pertaining to the notions of radical of an ideal of an L-ring which are versions of corresponding results of classical ring theory. Besides this we prove that for a commutative ring R, the radical $\sqrt{\eta}$ of a primary ideal η of an L-ring $L(\mu, R)$ is a prime ideal of μ provided η has sup-property.

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1. Introduction

The notion of a maximal ideal of an L-ring $L(\mu, R)$ has been introduced and discussed by the authors in [1, 2]. In paper [3], the concepts of prime ideals, semiprime ideals and the radical of an ideal of an L-ring have been studied in L-setting. In another paper [4], the concept of right (left) quotient (or residual) of an ideal η by an ideal ν of an L-ring μ is introduced and discussed. Thus a systematic development of the theory of ideals came into fore in an L-ring. This machinery has been effectively applied in a forthcoming [5] wherein the notions of primary decomposition and reduced primary decomposition of an ideal in an L-ring have been introduced. Moreover in the same paper [5], necessary and sufficient conditions for the existence of a primary decomposition of an ideal of an L-ring have been provided.

In this paper, we introduce the concept of primary ideal of an L-ring and establish a necessary and sufficient condition for an ideal to be a primary in term of level subring. We prove several results pertaining to these notions which are versions of their counterpart in classical ring theory. Besides this we prove that for a commutative ring R, the radical $\sqrt{\eta}$ of a primary ideal η of an L-ring $L(\mu, R)$ is a prime ideal of μ provided η has sup-property.

The concept of radical of an ideal in an L-ring is introduced in paper [3]. We will establish some results pertaining to the notions of radical of an ideal of an L-ring which are versions of corresponding results of classical ring theory. It is also prove that every semiprime ideal of an L-ring which is also primary is a prime ideal of the L-ring. In classical ring theory, it is well known that if the radical I of an ideal I of a ring R is maximal, then I is primary ideal. We have established the corresponding result in an L-ring.

2. Preliminaries

In this section, we recall some of the basic definitions and concepts which are used in the sequel. For details we refer to [6, 7, 8].

In this paper, L denotes a lattice, '$\leq$' denotes the partial ordering on L, and '$\vee$' and '$\wedge$' denote the join and the meet of the elements of L respectively. Let X be a non-empty set. An *L-subset* of X is a function from X into L. The set of L-subsets of X is called the *L-power set* of X and is denoted by L^X. For $\mu \in L^X$, the set $\{\mu(x) \mid x \in X\}$ is called the *image* of μ and is denoted by $Im\mu$. An L-subset μ of X is said to *be contained in* an L-subset η of X, if $\mu(x) \leq \eta(x)$ for all $x \in X$. This is denoted by $\mu \subseteq \eta$. If $\nu \subseteq \mu$ and $\nu \neq \mu$, then ν is said to *be properly contained in* μ and we write $\nu \subsetneq \mu$. Throughout the paper, R will denote an ordinary ring and L will denote a lattice, unless otherwise specifically mentioned. Also, $\mathbb{Z}^+$ will denote the set of positive integers and ϕ will denote an empty set.

Definition 2.1 ([7])**.** Let $\mu \in L^R$. Then μ is called an *L-subring* of R, if it satisfies the following conditions: for any $x,\ y \in R$,
(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
(ii $\mu(xy) \geq \mu(x) \wedge \mu(y)$.

The set of all L-subrings of R is denoted by $L(R)$. It is obvious that if μ is an L-subring of R, then $\mu(x) \leq \mu(0)\ \forall\ x \in R$. For convenience, we use the notation $L(\mu, R)$ for the L-subring μ of R and we shall refer to it here as an L-ring $L(\mu, R)$.

Definition 2.2 ([7])**.** Let $\mu \in L^R$. Then μ is called an *L-ideal* of R, if it satisfies the following conditions: for any $x,\ y \in R$,
(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

We denote the set of all L-ideals of R by $LI(R)$. It is obvious that if R has identity 1 and $\mu \in LI(R)$, then $\mu(x) \geq \mu(1)\ \forall\ x \in R$.

Definition 2.3 ([7])**.** Let X be a nonempty set. For $\mu \in L^X$ and $\alpha \in L$, we define the *level subset* μ_α and the *strong level subset* $\mu_\alpha^>$ of μ are defined respectively as follows:

$$\mu_a = \{x \in X \mid \mu(x) \geq a\} \text{ and } \mu_a^> = \{x \in X \mid \mu(x) > a\}.$$

Obviously, $\mu_\alpha^> \subseteq \mu_\alpha$ and for $\alpha \leq \beta$, $\mu_\beta \subseteq \mu_\alpha$ and $\mu_\beta^> \subseteq \mu_\alpha^>$.

Definition 2.4 (Definition 3.2.11 [7]). Let $\nu \in L^R$ and $\mu \in L(R)$ with $\nu \subseteq \mu$. Then ν is called an *L-ideal* of μ (or in μ), if it satisfies the following conditions: for any $x,\ y \in R$,

(i) $\nu(x-y) \geq \nu(x) \wedge \nu(y)$,
(ii) $\nu(xy) \quad \geq \{\nu(y) \wedge \mu(x)\} \vee \{\nu(x) \vee \mu(y)\}$.

For convenience, ν is called an *ideal* of μ (or L-ring $L(\mu, R)$). Clearly, for $\mu \in L(R)$, a non-empty level subset μ_a is an ordinary subring of R, called a *level subring* of μ.

Definition 2.5 ([7]). Let $L(\mu, R)$ be an L-ring and let $\nu \in L(R)$. If $\nu \subseteq \mu$, then ν is called a *subring* of μ (or L -ring $L(\mu, R)$).

Clearly, if ν is a subring of μ, then $\nu(x^n) \geq \nu(x) \ \ \forall\ n \in \mathbb{Z}^+$

Theorem 2.6 ([4]). *Let $L(\mu, R)$ be an L-ring and $\eta \in L^R$ with $\eta \subseteq \mu$. Then η is an ideal of μ if and only if each non-empty level subset η_a is an ideal of level subring μ_a.*

Definition 2.7 ([7]). Let L be a complete lattice and $\eta, \nu \in L^R$. Then we define $\eta + \nu,\ \eta\nu$ and $\eta \circ \nu$ by

$$\eta + \nu(x) = \bigvee_{x=y+z} \{\eta(y) \wedge \nu(z)\},$$

$$\eta\nu(x) = \bigvee \left\{ \bigwedge_{i=1}^{n} \{\eta(y_i) \wedge \nu(z_i) : x = y_i z_i\} \right\},$$

$$\eta \circ \nu(x) = \bigvee_{x=yz} \{\eta(y) \wedge \nu(z)\}.$$

Clearly, if η and ν are subrings of an L-ring $L(\mu, R)$ with $\eta(0) = \nu(0)$, then η and $\nu \subseteq \eta + \nu$.

Lemma 2.8 ([7]). *Let L be a complete lattice and $\eta,\ \nu,\ \xi \in L^R$. Then the following assertions hold :*

(1) $\eta \circ \eta \subseteq \eta\nu$,
(2) $\xi \circ (\eta + \nu) \subseteq \xi \circ \eta + \xi \circ \nu$,
(3) *if* $\eta \subseteq \nu$, *then* $\eta\xi \subseteq \nu\xi$ *and* $\eta \circ \xi \subseteq \nu \circ \xi$,
(4) $\eta\ (\nu\xi) = (\eta\nu)\xi$,
(5) $\eta\nu(x+y) \geq \eta\nu(x) \wedge \eta\nu(y)\ \forall\ x,\ y \in R$.

The following lemma is easy to verify:

Lemma 2.9 ([4]). *Let L be a complete lattice, $L(\mu, R)$ be an L-ring and let η be a subring of μ. Then*

(1) $\eta \circ \eta \subseteq \eta\eta \subseteq \eta$,
(2) $\eta + \eta = \eta$.

In particular, $\mu \circ \mu \subseteq \mu\mu \subseteq \mu$ and $\mu + \mu = \mu$.

Definition 2.10 ([1])**.** A proper ideal η of an L-ring $L(\mu, R)$ is called a *maximal ideal* of μ, if for any ideal θ of μ, whenever $\eta \subseteq \theta \subseteq \mu$, then either $\eta = \theta$ or $\theta = \mu$.

Theorem 2.11 ([1])**.** *Let L be a chain, let $L(\mu, R)$ be an L-ring and let η be a maximal ideal of μ. Then there is exactly one pair (η_{t_0}, μ_{t_0}) such that $\eta_{t_0} \subsetneq \mu_{t_0}$ and for all other pairs (η_t, μ_t), we have $\eta_t = \mu_t$.*

Lemma 2.12 ([7])**.** *Let L be a complete lattice and let $L(\mu, R)$ be an L-ring. Then the intersection of an arbitrary family of ideals of μ is an ideal of μ.*

Lemma 2.13 ([3])**.** *Let $L(\mu, R)$ be an L-ring. If η is an ideal of μ, then for all $x, y \in R$,*

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(x) \wedge \mu(y)$$

and

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(y) \wedge \mu(x).$$

Definition 2.14 ([3])**.** Let R be a commutative ring and let $L(\mu, R)$ be an L-ring. An ideal $\eta \neq \mu$ of μ is called a *prime ideal* of μ, if for all $x, y \in R$,
either

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y)$$

or

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(y) \wedge \mu(xy).$$

Definition 2.15 ([3])**.** Let R be a commutative ring and let $L(\mu, R)$ be an L-ring. An ideal $\eta \neq \mu$ of μ is called a *semiprime ideal* of μ, if

$$\eta(x^n) \wedge \mu(x) = \eta(x) \ \forall x \in R \text{ and } \forall n \in \mathbb{Z}^+.$$

Theorem 2.16 ([3])**.** *Let R be a commutative ring and let $L(\mu, R)$ be an L-ring and let η be a prime ideal of μ. Then η is a semiprime ideal of μ.*

Definition 2.17 ([3])**.** Let R be a commutative ring, let L be a complete lattice,let $L(\mu, R)$ be an L-ring and let η be an ideal of μ. The *radical* of η, denoted by $\sqrt{\eta}$, is defined by

$$\sqrt{\eta}(x) = \bigvee_{n \in \mathbb{Z}^+} \{\eta(x^n) \wedge \mu(x)\} \ \forall x \in R.$$

Clearly, $\eta \subseteq \sqrt{\eta} \subseteq \mu$.

Theorem 2.18 ([3])**.** *Let R be a commutative ring, let L be a complete lattice and let $L(\mu, R)$ be an L-ring. An ideal η of μ is a semiprime ideal of μ if and only if $\sqrt{\eta} = \eta$.*

Here we recall the definition of sup-property:

Definition 2.19 ([7])**.** Let $\mu \in L^X$. Then, μ is said to *have sup-property*, if for each $A \subseteq X$, there exists $a_0 \in A$ such that $\bigvee_{a \in A} \mu(a) = \mu(a_0)$.

Lemma 2.20 ([3])**.** *Let R be a commutative ring, let L be a complete lattice, let $L(\mu, R)$ be an L-ring and let η be an ideal of μ such that η has sup-property. Then $(\sqrt{\eta})_t = \sqrt{\eta_t} \cap \mu_t \ \forall \ t \in L$.*

Theorem 2.21 ([3]). *Let R be a commutative ring, let L be a complete lattice, let $L(\mu, R)$ be an L-ring and let η be an ideal of μ having sup-property. Then $\sqrt{\eta}$ is an ideal of μ.*

Theorem 2.22 ([3]). *Let R be a commutative ring, let L be a complete Heyting algebra, let $L(\mu, R)$ be an L-ring and let η be an ideal of μ. Then $\sqrt{\eta}$ is an ideal of μ.*

Theorem 2.23 ([3]). *Let R be a commutative ring, let L be a complete lattice, let η and let θ be ideals of μ. If $\eta \subseteq \theta$, then $\sqrt{\eta} \subseteq \sqrt{\theta}$.*

Theorem 2.24 ([3]). *Let R be a commutative ring, let L be a complete Heyting algebra, let $L(\mu, R)$ be an L-ring and let η be an ideal of μ. Then $\sqrt{\sqrt{\eta}} = \eta$.*

Theorem 2.25 ([4]). *Let L be a complete lattice, let $L(\mu, R)$ be an L-ring and let $\eta \in L^R$ with $\eta \subseteq \mu$. Then η is an ideal of μ if and only if*
(1) $\eta(x-y) \geq \eta(x) \wedge \eta(y)\ \forall\ x,\ y \in R,$
(2) $\eta \circ \mu,\ \mu \circ \eta \subseteq \eta.$

Theorem 2.26 ([4]). *Let L be a complete lattice, let $L(\mu, R)$ be an L-ring and let $\eta \in L^R$ with $\eta \subseteq \mu$. Then η is an ideal of μ if and only if*
(1) $\eta(x-y) \geq \eta(x) \wedge \eta(y)\ \forall\ x,\ y \in R,$
(2) $\eta\mu,\ \mu\eta \subseteq \eta.$

Theorem 2.27 ([4]). *Let L be a complete lattice and let $L(\mu, R)$ be an L-ring. If η and ν are ideals of μ with $\eta(0) = \nu(0)$, then $\eta + \nu$ is an ideal of μ and $\eta \subseteq \eta + \nu$, $\nu \subseteq \eta + \nu$.*

Theorem 2.28 ([4]). *Let L be a complete lattice and let $L(\mu, R)$ be an L-ring. If η and ν are ideals of μ, then $\eta\nu$ is an ideal of μ.*

3. Radicals of an ideal and a primary ideal

Theorem 3.1. *Let R be a commutative ring, let L be a completely distributive lattice and let $L(\mu, R)$ be an L-ring. If η and θ are ideals of μ, then $\sqrt{\eta \cap \theta} = \sqrt{\eta}\ \cap \sqrt{\theta}$.*

Proof. Let $x \in R$. Then

$$\begin{aligned}\sqrt{\eta\ \cap \theta}(x) &= \bigvee_{n \in Z^+} \{(\eta\ \cap \theta)(x^n) \wedge \mu(x)\} \\ &= \bigvee_{n \in Z^+} \{\eta(x^n) \wedge \theta(x^n) \wedge \mu(x)\} \\ &= \left\{\bigvee_{n \in Z^+} \{\eta(x^n) \wedge \mu(x)\}\right\} \bigwedge \left\{\bigvee_{n \in Z^+} \{\theta(x^n) \wedge \mu(x)\}\right\} \\ &\quad \text{[Since } L \text{ is a completely distributive lattice]} \\ &= \sqrt{\eta}\ (x)\ \cap\ \sqrt{\theta}\ (x)\ = (\sqrt{\eta}\ \cap\ \sqrt{\theta})\ (x).\end{aligned}$$

Thus $\sqrt{\eta \cap \theta} = \sqrt{\eta} \cap \sqrt{\theta}$. Since η and θ are ideals of μ, by Theorem 2.28, $\eta\theta$ is an ideal of μ. Also by Lemma 2.8 and Theorem 2.26, $\eta\theta \subseteq \eta\mu \subseteq \eta$. By Theorem 2.23, $\sqrt{\eta\theta} \subseteq \sqrt{\eta}$. Similarly, $\sqrt{\eta\theta} \subseteq \sqrt{\theta}$. So $\sqrt{\eta\theta} \subseteq \sqrt{\eta} \cap \sqrt{\theta} = \sqrt{\eta \cap \theta}$.

Now, let $x \in R$. Then

$$\sqrt{\eta\theta}(x) = \bigvee_{n \in Z^+} \{\eta\theta(x^n) \wedge \mu(x)\}$$
$$\geq \bigvee_{n \geq 2} \left\{ \left[\bigvee_{r=1}^{n-1} \left(\eta(x^r) \wedge \theta(x^{n-r})\right) \right] \wedge \mu(x) \right\}.$$

On the other hand,

$$\bigvee_{r=1}^{n-1} \left\{\eta(x^r) \wedge \theta(x^{n-r})\right\} \geq \left\{\eta(x^{n-1}) \wedge \theta(x)\right\} \vee \left\{\eta(x) \wedge \theta(x^{n-1})\right\}$$
$$= \left\{\eta(x^{n-1}) \vee \eta(x)\right\} \wedge \left\{\theta(x^{n-1}) \vee \theta(x)\right\}$$

[Since L is a completely distributive lattice]

$$= \eta(x^{n-1}) \wedge \theta(x^{n-1}).$$

[Since $\eta(x^{n-1}) \geq \eta(x)$ and $\theta(x^{n-1}) \geq \theta(x)$]

$$= (\eta \cap \theta)(x^{n-1}).$$

Thus $\sqrt{\eta\theta}(x) \geq \bigvee_{n \geq 2} \left\{(\eta \cap \theta)\left(x^{n-1}\right) \wedge \mu(x)\right\} = \sqrt{\eta \cap \theta}(x)$. So $\sqrt{\eta \cap \theta} \subseteq \sqrt{\eta\theta}$. Hence $\sqrt{\eta \cap \theta} = \sqrt{\eta\theta}$. □

Theorem 3.2. *Let R be a commutative ring. let L be a complete Heyting Algebra and let $L(\mu, R)$ be an L-ring. If η and θ are ideals of μ with $\eta(0) = \theta(0)$, then*

$$\sqrt{\eta} + \sqrt{\theta} \subseteq \sqrt{\sqrt{\eta} + \sqrt{\theta}} = \sqrt{\eta + \theta}.$$

Proof. By Theorem 2.22, $\sqrt{\eta}$ and $\sqrt{\theta}$ are ideals of μ. By Theorem 2.27, $\eta + \theta$ and $\sqrt{\eta} + \sqrt{\theta}$ are an ideal of μ. Clearly, $\sqrt{\eta} + \sqrt{\theta} \subseteq \sqrt{\sqrt{\eta} + \sqrt{\theta}}$. Since $\eta \subseteq \sqrt{\eta}$ and $\theta \subseteq \sqrt{\theta}$, $\eta + \theta \subseteq \sqrt{\eta} + \sqrt{\theta}$. Then by Theorem 2.23, $\sqrt{\eta + \theta} \subseteq \sqrt{\sqrt{\eta} + \sqrt{\theta}}$. By Theorem 2.22, $\sqrt{\eta + \theta}$ is an ideal of μ. Thus by Lemma 2.9, $\sqrt{\eta + \theta} + \sqrt{\eta + \theta} = \sqrt{\eta + \theta}$. Since $\eta \subseteq \eta + \theta$, by Theorem 2.23, $\sqrt{\eta} \subseteq \sqrt{\eta + \theta}$. Similarly, $\sqrt{\theta} \subseteq \sqrt{\eta + \theta}$. So we have

$$\sqrt{\eta} + \sqrt{\theta} \subseteq \sqrt{\eta + \theta} + \sqrt{\eta + \theta} = \sqrt{\eta + \theta}.$$

By Theorem 2.23 and Theorem 2.24, $\sqrt{\sqrt{\eta} + \sqrt{\theta}} \subseteq \sqrt{\sqrt{\eta + \theta}} = \sqrt{\eta + \theta}$. Hence $\sqrt{\sqrt{\eta} + \sqrt{\theta}} = \sqrt{\eta + \theta}$. □

Definition 3.3. Let R be a commutative ring and let $L(\mu, R)$ be an L-ring. An ideal $\eta \neq \mu$ of μ is said to be *primary ideal* of μ, if for all $x, y \in R$, we have either

$$\eta(x) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \tag{3.1}$$

or

$$\eta(y) \wedge \mu(x) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \tag{3.2}$$

or

$$\eta(x^n) \wedge \mu(x) \wedge \eta(y^m) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y), \tag{3.3}$$

for some integers $m,\ n > 1$.

Obviously, every prime ideal of an L-ring $L(\mu, R)$ is a primary ideal of μ.

Lemma 3.4. *Let R be a commutative ring. An ideal I of R is primary if and only if, whenever $xy \in I$ we have either $x \in I$ or $y \in I$ or* ($x^n,\ y^m \in I$ for some integers $m,\ n > 1$).

Proof. Suppose that the ideal I is primary. Let $xy \in I$. Then we consider the following three cases.

Case(i) $x \notin I, y \notin I$. Since I is a primary ideal and $x \notin I$, we have $y^m \in I$ for some positive integer m. Also $m > 1$, since $y \notin I$. Similarly, we have $x^n \in I$ for some integer $n > 1$.

Case (ii) $x \notin I$ and either $x^n \notin I$ or $y^n \notin I$ for any integer $n > 1$. Again, since I is a primary ideal and $x \notin I$, we have $y^m \in I$ for some integer $m \geq 1$. We show that $y \in I$. Assume that $y \notin I$. Then $m > 1$. Thus by the hypothesis, $x^n \notin I$ for any integer $n > 1$. Since I is primary and $y \notin I$, $x^m \in I$ for some integer $m \geq 1$. As $x \notin I$, $m > 1$. So $x^m \in I$ for some integer $m > 1$, which is a contradiction. Hence $y \in I$.

Case (iii) $y \notin I$ and either $x^n \notin I$ or $y^n \notin I$ for any integer $n > 1$. The proof of this part is similar to that of case (ii).

To prove the converse part, suppose $xy \in I$ and $x \notin I$. Then either $y \in I$ or there exists integers $m,\ n > 1$ such that $x^n \in I$ and $y^m \in I$. Thus in either case $y^m \in I$ for some positive integer m. Similarly, if $y \notin I$, then $x^n \in I$ for some positive integer n. So I is a primary ideal of R. □

Theorem 3.5. *Let R be a commutative ring, let $L(\mu, R)$ be an L-ring and let η be an ideal of μ with $\eta \neq \mu$. Then η is a primary ideal of μ if and only if for each non-empty level subset η_t, either $\eta_t = \mu_t$ or η_t is a primary ideal of μ_t.*

Proof. Suppose η is a primary ideal of μ and η_t is a non-empty level subset such that $\eta_t \neq \mu_t$. Let $xy \in \eta_t$, $x, y \in \mu_t$. Then it follows that $\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t$. Since η is primary ideal of μ, one of the conditions (3.1), (3.2) and (3.3) hold.

If condition (3.1) holds, then

$$\eta(x) \geq \eta(x) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t.$$

Thus $x \in \eta_t$.

If (3.2) holds, then

$$\eta(y) \geq \eta(y) \wedge \mu(x) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t.$$

Thus $y \in \eta_t$.

If (3.3) holds, then we have

$$\eta(x^n) \wedge \mu(x) \wedge \eta(y^m) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t$$

for some integer $m, n > 1$. Thus $x^n,\ y^m \in \eta_t$. So, by Lemma 3.4Confirm it, η_t is a primary ideal of μ_t.

Conversely, suppose that for each non-empty level subset η_t, either $\eta_t = \mu_t$ or η_t is a primary ideal of μ_t. We write $\eta(xy) \wedge \mu(x) \wedge \mu(y) = t$. Then $xy \in \eta_t$, $x \in \mu_t$ and $y \in \mu_t$. If $\eta_t = \mu_t$, then $x \in \eta_t$ and $y \in \eta_t$. Thus $\eta(x) \geq t$. So

$$\eta(x) \wedge \mu(y) \geq t \wedge t = t = \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

If η_t is a primary ideal of μ_t, then $xy \in \eta_t$ implies that $x \in \eta_t$ or $y \in \mu_t$ or $x^n,\ y^m \in \eta_t$ for some integers $m,\ n > 1$. Suppose that $x \in \eta_t$. Then $\eta(x) \geq t$ implies that

$$\eta(x) \wedge \mu(y) \geq t \wedge t = t = \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

Similarly, if $y \in \eta_t$, then

$$\eta(y) \wedge \mu(x) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

Thus η is a primary ideal of μ. □

Our next result shows that every semiprime ideal of an L-ring which is also primary is a prime ideal.

Theorem 3.6. *Let R be a commutative ring, let $L(\mu, R)$ be an L-ring and let η be a semiprime ideal of μ. If η is a primary ideal of μ, then η is a prime ideal of μ.*

Proof. Let $x, y \in R$. Since η is semiprime ideal of μ, we have

$$\eta(x^n) \wedge \mu(x) = \eta(x) \text{ and } \eta(y^m) \wedge \mu(y) = \eta(y)\ \forall n,\ m \in \mathbb{Z}^+.$$

Thus

$$\eta(x^n) \wedge \mu(x) \wedge \eta(y^m) \wedge \mu(y) = \eta(x) \wedge \eta(y)\ \forall n,\ m \in \mathbb{Z}^+. \tag{3.4}$$

Since η is a primary ideal of μ, one of the conditions (3.1), (3.2) and (3.3) holds. If condition (3.3) holds, then for some integers $r, s > 1$, we have

$$\eta(x^r) \wedge \mu(x) \wedge \eta(y^s) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

From this along with (3.4), we have

$$\begin{aligned}\eta(x) \wedge \mu(y) \geq \eta(x) \wedge \eta(y) &= \eta(x^r) \wedge \mu(x) \wedge \eta(y^s) \wedge \mu(y)\\ &\geq \eta(xy) \wedge \mu(x) \wedge \mu(y).\end{aligned}$$

This again gives us condition (3.1). Thus either condition (3.1) or (3.2) holds. Since η is an ideal of μ, by Lemma 2.17, we have

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(x) \wedge \mu(y) \text{ and } \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(y) \wedge \mu(x).$$

So either,

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y) \text{ or } \eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(y) \wedge \mu(x).$$

Hence η is a prime ideal of μ. □

Theorem 3.7. *Let R be a commutative ring, let L be a complete lattice and let $L(\mu, R)$ be an L-ring. If η is a primary ideal of μ having sup-property, then $\sqrt{\eta}$ is a prime ideal of μ. Also $\sqrt{\sqrt{\eta}} = \sqrt{\eta}$.*

Proof. By Theorem 2.21, $\sqrt{\eta}$ is an ideal of μ. Let $x,\ y \in R$. Since η has sup-property, there exists $m \in \mathbb{Z}^+$ such that

$$\sqrt{\eta}(xy) = \bigvee_{n \in \mathbb{Z}^+} [\eta((xy)^n) \wedge \mu(xy)] = \eta(x^m y^m) \wedge \mu(xy). \tag{3.5}$$

On the other hand,

$$\sqrt{\eta}(x) = \bigvee_{n \in \mathbb{Z}^+} [\eta(x^n) \wedge \mu(x)] \geq \eta(x^s) \wedge \mu(x)\ \forall s \in \mathbb{Z}^+.$$

Then we get

$$\sqrt{\eta}(x) \wedge \mu(y) \geq \eta(x^s) \wedge \mu(x) \wedge \mu(y) \ \forall s \in \mathbb{Z}^+. \tag{3.6}$$

Similarly, we have

$$\sqrt{\eta}(y) \wedge \mu(x) \geq \eta(y^s) \wedge \mu(x) \wedge \mu(y) \ \forall s \in \mathbb{Z}^+. \tag{3.7}$$

Since η is a primary ideal of μ, by Definition 3.3, we have either

$$\eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \leq \eta(x^m) \wedge \mu(y^m) \tag{3.8}$$

or

$$\eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \leq \eta(y^m) \wedge \mu(x^m) \tag{3.9}$$

or

$$\eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \leq \eta(x^{mk}) \wedge \mu(x^m) \wedge \eta(y^{mr}) \wedge \mu(y^m) \tag{3.10}$$

for some integers $k,\ r > 1$.
By (3.5), we have

$$\begin{aligned}\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) &= \eta(x^m y^m) \wedge \mu(xy) \wedge \mu(x) \wedge \mu(y)\\ &= \eta(x^m y^m) \wedge \mu(x) \wedge \mu(y)\\ &= \eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \wedge \mu(x) \wedge \mu(y).\end{aligned}$$

If (3.8) holds, then

$$\begin{aligned}\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) &\leq \eta(x^m) \wedge \mu(y^m) \wedge \mu(x) \wedge \mu(y)\\ &= [\sqrt{\eta}(x^m) \wedge \mu(x)] \wedge \mu(y)\\ &\leq \sqrt{\eta}(x) \wedge \mu(y).[By(3.6)]\end{aligned}$$

If (3.9) holds, then

$$\begin{aligned}\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) &\leq \eta(y^m) \wedge \mu(x^m) \wedge \mu(x) \wedge \mu(y)\\ &= [\eta(y^m) \wedge \mu(x)] \wedge \mu(y)\\ &\leq \eta(y) \wedge \mu(x).[By(3.7)]\end{aligned}$$

If the condition (3.10) is valid, then

$$\begin{aligned}\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) &\leq \eta(x^{mk}) \wedge \mu(x^m) \wedge \eta(y^{mr}) \wedge \mu(y^m) \wedge \mu(x) \wedge \mu(y)\\ &= \eta(x^{mk}) \wedge \eta(y^{mr}) \wedge \mu(x) \wedge \mu(y)\\ &= \left[\eta(x^{mk}) \wedge \mu(x) \wedge \mu(y)\right] \wedge [\eta(y^{mr}) \wedge \mu(y) \wedge \mu(x)]\\ &\leq [\sqrt{\eta}(x) \wedge \mu(y)] \wedge [\sqrt{\eta}(y) \wedge \mu(x)]\\ &\leq \sqrt{\eta}(x) \wedge \mu(y).\end{aligned}$$

Thus $\sqrt{\eta}$ is a prime ideal of μ. So by Theorem 2.16, η is a semiprime ideal. Hence by Theorem 2.18, $\sqrt{\sqrt{\eta}} = \sqrt{\eta}$. □

It is well-known in classical ring theory that if the radical $\sqrt{I}$ of an ideal I in a commutative ring R is a maximal ideal of R, then I itself is a primary ideal of R. Now we provide the L-version of this result.

Theorem 3.8. *Let L be a complete chain, let R be a commutative ring with unity, let $L(\mu, R)$ be an L-ring and let η be an ideal of μ having sup-property. If $\sqrt{\eta}$ is a maximal ideal of μ such that $\left(\sqrt{\eta}\right)_{t_0}$ is a maximal ideal of $\mu_{t_0}, t_0 \in Im\ \mu$ and $\mu_{t_0} = R$, then η is a primary ideal of μ.*

Proof. Let η_t be a non-empty level subset of μ_t such that $\eta_t \subsetneqq \mu_t$. We show that η_t is primary ideal of μ_t. Now, two cases arise:

Case (i) $\left(\sqrt{\eta}\right)_t = \mu_t$. Then by Lemma 2.20, we have $\sqrt{\eta_t} \cap \mu_t = \mu_t$. Let $ab \in \eta_t$, $a,\ b \in \mu_t$ and $a \notin \eta_t$. Then $b \in \mu_t = \sqrt{\eta_t} \cap \mu_t \subseteq \sqrt{\eta}_t$. Thus η_t is a primary ideal of μ_t.

Case (ii) $\left(\sqrt{\eta}\right)_t \neq \mu_t$. By Theorem 2.11, we have $\left(\sqrt{\eta}\right)_t = \left(\sqrt{\eta}\right)_{t_0}$ and $\mu_t = \mu_{t_0} = R$. Thus by Lemma 2.20, we have

$$(\sqrt{\eta})_{t_0} = (\sqrt{\eta})_t = (\sqrt{\eta})_t \bigcap \mu_t = (\sqrt{\eta})_t \bigcap R = \sqrt{\eta}_t.$$

By the hypothesis, $\left(\sqrt{\eta}\right)_{t_0}$ is a maximal ideal of μ_{t_0}. So η_t is a maximal ideal of R. Hence in view of a result of classical ring theory, η_t is a primary ideal of R, i.e., η_t is a primary ideal of μ_t. □

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