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Abstract. The main purpose of this paper is to introduce and study the notion of state relative annihilator in the framework of state residuated lattices (SRLs), along with some related properties. We show that this concept of state relative annihilator in SRLs is a generalization of the existing one in De Morgan state residuated lattices. Among many other results, we prove that state relative annihilators are a particular type of state ideals of state residuated lattices. Furthermore, we establish some links between state-morphism operators and annihilators in state residuated lattices.

2020 AMS Classification: 06B75, 03G10, 03G25, 08A35

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State residuated lattice, State operator, State ideal, State relative annihilator, De Morgan state residuated lattice.

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1. Introduction

The genesis of residuated lattices is in Mathematical Logic without contraction. Apart from their logical interest, residuated lattices have important algebraic properties, and it is well-known that the algebraic study of logical systems plays a prominent role with considerable applications in artificial intelligence.

In order to provide an algebraic foundation for reasoning about probabilities of fuzzy events inside Łukasiewicz infinite-valued logic, Flaminio and Montagna ([1, 2]) added a unary operation φ to the language of MV-algebras as an internal state, also called a *state operator* which generalize and preserves the usual properties of states. Therefore, the concept of state operator has been extended to many commutative and non-commutative algebraic structures such as BL-algebras [3], pseudo BL-algebra [4], Rl-monoids [5], residuated lattices ([6, 7, 8]). However, most of these studies have mainly focused on state filters, whereas the notion of state ideal has recently been investigated in state De Morgan residuated lattices ([9]), and in state residuated lattices [10]. Now on, since state filters and state ideals are not dual notions in state residuated lattices and knowing the importance of ideal theory in classification problems, data organization, formal concept analysis, it is meaningful to deepen the notion of state ideal in the framework of state residuated lattices. In appropriate way, almost akin to annihilators in commutative rings, in [11], the authors investigated the notion of f-relative annihilators in residuated lattices, where f is an endomorphism. Due to the fact that a state operator is not always an endomorphism, it becomes reasonable to regard what happens when the endomorphism f is replaced by a state operator φ. This paper seeks to expand our study of state relative annihilators done in state De Morgan residuated lattices ([9]) to the framework of state residuated lattices.

The article is divided into 3 sections: in the first one, we present some preliminaries comprising the basic definitions, some rules of calculus and theorems that are needed in the sequel. Section 2 is devoted to the main results. We introduce the notion of state annihilator of a nonempty set X with respect to a state ideal I in a SRL, analyze some of its properties and present some examples. We demonstrate that state relative annihilators are a particular type of state ideals. In Section 3, we introduce the notion of state-morphism operator in residuated lattices and establish a link with annihilators.

2. Preliminaries

We summarize here some fundamental definitions and results about residuated lattices. For more details, we refer the reader to the papers [12, 13, 14, 15].

A nonempty set L with four binary operations $\wedge, \vee, \odot, \to$ and two constants $0, 1$ is called a *bounded integral commutative residuated lattice* or shortly *residuated lattice*, if the following axioms are verified:

(C1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(C2) $(L, \odot, 1)$ is a commutative monoid (with the unit element 1),
(C3) For all $x, y \in L$, $x \odot y \leq z$ iff $x \leq y \to z$.

A residuated lattice satisfying the De Morgan property $(x \wedge y)' = x' \vee y'$ is called a *De Morgan residuated lattice*.
We shall notice from [16] that Boolean algebras, BL-algebras, MTL-algebras, Stonean residuated lattices and regular residuated lattices (MV-algebras, IMTL-algebras) are particular important subclasses of De Morgan residuated lattices.

The following notations of residuated lattices will be used:

L will stand for a residuated lattice $(L, \wedge, \vee, \odot, \to, 0, 1)$. For any $x \in L$ and $n \in \mathbb{N}^*$, $x' := x \to 0$, $x'' := (x')'$, $x^0 := 1$ and $x^n := x^{n-1} \odot x$.

The following basic arithmetic of residuated lattices will be used for any $x, y, z \in L$ (See [8, 12]):

(RL1) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$, $0 \to x = 1$,
(RL2) $x \leq y \Leftrightarrow x \to y = 1$,
(RL3) $x \to y = y \to x = 1 \Leftrightarrow x = y$,
(RL4) if $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x \odot z \leq y \odot z$ and $y' \leq x'$,

(RL5) $x \odot (x \to y) \leq y$, $x \odot (x \to y) \leq x \wedge y$,
(RL6) $x \odot y \leq x \wedge y \leq x$, $y \leq x \vee y$, $x \leq y \to x$, $x \odot y \leq x \to y$ $y \to x$,
(RL7) $(x \odot y)'' = x'' \odot y''$, $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' \geq x' \vee y'$,
(RL8) $0' = 1$, $1' = 0$,
(RL9) $x \leq x'' \leq x' \to x$,
(RL10) $x \to y \leq y' \to x'$,
(RL11) $x''' = x'$, $(x \odot y)' = x \to y' = y \to x' = x'' \to y'$,
(RL12) $x \odot x' = 0$, $x \odot y = 0 \Leftrightarrow x \leq y'$, $x \odot 0 = 0$,
(RL13) $x' \to y \leq (x' \odot y')'$, $x' \odot y' \leq (x' \to y)'$, $x' \odot y' \leq (x \odot y)'$,
(RL14) $x \to (x \wedge y) = x \to y$,
(RL15) $x \odot y = x \odot (x \to x \odot y)$,
(RL16) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z)$,
$x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$.

For every $x,\ y \in L$, we set $x \oslash y = x' \to y$.

Definition 2.1 ([13], Definition 3.1)**.** A nonempty subset I of a residuated lattice L is called an *ideal*, if the following conditions are satisfied: for every $x, y \in L$,

(I1) if $y \in I$ and $x \leq y$, then $x \in I$,
(I2) if $x, y \in I$, then $x \oslash y \in I$.

By the fact that $\oslash$ is neither commutative nor associative in residuated lattices, in order to get an operation with properties closed to addition properties, D. Buşneag et al. (See Lemma 1, [17]) defined a commutative and associative operation $\oplus$ in residuated lattices as follows: $x \oplus y = (x' \odot y')'$ for every $x,\ y \in L$.
Here are some properties of the operation $\oplus$ (See [9, 14, 17]).
(P1) $x \oplus y = x' \to y'' = y' \to x''$,
(P2) $x \oplus x' = 1$, $x \oplus 0 = x''$, $x \oplus 1 = 1$,
(P3) $x \oplus y = y \oplus x$, $x, y \leq x \oplus y$,
(P4) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(P5) if $x \leq y$, then $x \oplus z \leq y \oplus z$,
(P6) if $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$.

For any $x \in L$ and $n \in \mathbb{N}$, we define $0x = 0$, $1x = x$ and $nx = (n-1)x \oplus x$ for $n \geq 2$.

The following relations hold: for any $x,\ y \in L$ and $m,\ n \in \mathbb{N}^*$,
(P7) $m \leq n \Rightarrow mx \leq nx$, in particular, $x \leq nx$,
(P8) $x \leq y \Rightarrow mx \leq my$,
(P9) $n(x \oplus y) = nx \oplus ny$,
(P10) $x \oplus ny \leq n(x \oplus y)$,
(P11) $x, y \leq x \oslash y \leq x \oplus y$,
(P12) $[(x')^n]' = nx$,
(P13) $(x \oplus y)'' = x \oplus y = x'' \oplus y''$,
(P14) $x \wedge (y_1 \oplus ... \oplus y_n) \leq (x'' \wedge y_1'') \oplus \cdots \oplus (x'' \wedge y_n'')$,
(P15) if L is De Morgan , then $(x \wedge y)'' = x'' \wedge y''$.

Note that in (Theorem 3.5, [16]), it was shown that a nonempty subset I of a residuated lattice L is an ideal of L if and only if for any $x,\ y \in L$,

(I3) if $y \in I$ and $x \leq y$, then $x \in I$,
(I4) if $x, y \in I$, then $x \oplus y \in I$.

The set of all ideals of a residuated lattice L will be denoted by $\mathcal{I}(L)$.

Remark 2.2. From the above definition, it is easy to see that for all $I \in \mathcal{I}(L)$, $0 \in I$, and, $x \in I$ if and only if $x'' \in I$ for any $x \in L$.

For a nonempty subset X of a residuated lattice L, the ideal generated by X is $\langle X \rangle := \{a \in L : a \leq x_1 \oplus x_2 \oplus \cdots \oplus x_n,\ for\ some\ n \in \mathbb{N}^*,\ x_i \in X,\ for\ 1 \leq i \leq n\}$ (See Proposition 6, [17]).

The concepts of state operators and state residuated lattices were introduced in [8] by Pengfei et al. as follows.

Definition 2.3 (Definition 3.1, [8])**.** A map $\varphi : L \to L$ is said to be a *state operator* on L, if the following conditions hold: for any $x,\ y \in L$,

(SO1) $\varphi(0) = 0$,
(SO2) $x \to y = 1$ implies $\varphi(x) \to \varphi(y) = 1$,
(SO3) $\varphi(x \to y) = \varphi(x) \to \varphi(x \wedge y)$,
(SO4) $\varphi(x \odot y) = \varphi(x) \odot \varphi(x \to (x \odot y))$,
(SO5) $\varphi(\varphi(x) \odot \varphi(y)) = \varphi(x) \odot \varphi(y)$,
(SO6) $\varphi(\varphi(x) \to \varphi(y)) = \varphi(x) \to \varphi(y)$,
(SO7) $\varphi(\varphi(x) \vee \varphi(y)) = \varphi(x) \vee \varphi(y)$,
(SO8) $\varphi(\varphi(x) \wedge \varphi(y)) = \varphi(x) \wedge \varphi(y)$.

The pair (L, φ) is said to be a *state residuated lattice*, or more precisely, a *residuated lattice with internal state*.

From now on, unless othewise specified, (L, φ) will always denote a state residuated lattice $(L, \vee, \wedge, \odot, \to, 0, 1)$, that is, L is a residuated lattice and φ is a state operator on L.

We shall notice that, id_L is a state operator on L which is an endomorphism but in general a state operator φ is not an endomorphism.
The *kernel* of φ is the set $\ker(\varphi) := \{x \in L : \varphi(x) = 1\}$. φ is said to be *faithful*, if $\ker(\varphi) = \{1\}$.

Analogously, the *co-kernel* of φ is the set $coker(\varphi) := \{x \in L : \varphi(x) = 0\}$. φ is called *cofaithful*, if $coker(\varphi) = \{0\}$ and *uncofaithful* otherwise.

Definition 2.4 ([8, 9])**.** An ideal I of L is said to be a *state ideal* of (L, φ), if $\varphi(I) \subseteq I$, (i.e., for all $x \in L$, $x \in I \Rightarrow \varphi(x) \in I$).

$\mathcal{SI}(L)$ will stand for the set of all state ideals of (L, φ). It is obvious that $\{0\}, L \in \mathcal{SI}(L) \subseteq \mathcal{I}(L)$.

Remark 2.5. (L, id_L) is a state residuated lattice, that is, a residuated lattice L can be view as a state residuated lattice. One can see that, each ideal of L is a state ideal of (L, id_L).

For computational issues, we will use the following properties (See [8, 9]).

For any $x,\ y \in L$, for all $n \geq 1$,

(SO9) $\varphi(1) = 1$,
(SO10) $x \leq y$ implies $\varphi(x) \leq \varphi(y)$,
(SO11) $\varphi(x') = (\varphi(x))'$,
(SO12) $\varphi(x \odot y) \geq \varphi(x) \odot \varphi(y)$ and if $x \odot y = 0$, then $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$,
(SO13) $\varphi(x \odot y') \geq \varphi(x) \odot (\varphi(y))'$,
(SO14) $\varphi(x \to y) \leq \varphi(x) \to \varphi(y)$, particularly, if $x,\ y$ are comparable, then $\varphi(x \to y) = \varphi(x) \to \varphi(y)$,
(SO15)] if φ is faithful, then $x < y$ implies $\varphi(x) < \varphi(y)$,
(SO16) $\varphi^2(x) = \varphi(\varphi(x)) = \varphi(x)$,
(SO17) $\varphi(L) = Fix(\varphi)$, where $Fix(\varphi) = \{x \in L : \varphi(x) = x\}$,
(SO18) $\varphi(L)$ is a subalgebra of L,
(SO19) if $x \leq y$, then $\varphi(x \odot y') = \varphi(x) \odot (\varphi(y))'$,
(SO20) $coker(\varphi)$ is a state ideal of (L, φ),
(SO21) $(\varphi(x))'' = \varphi(x'')$,
(SO22) $\varphi(x \oplus y) \leq \varphi(x) \oplus \varphi(y)$,
(SO23) if $x, y \in \varphi(L)$, then $x \oplus y \in \varphi(L)$,
(SO24) $\varphi(nx) \leq n\varphi(x)$.

3. State relative annihilators in state residuated lattices

In this section, we bring in the concept of state relative annihilator of a nonempty set X with respect to a state ideal I in a state residuated lattice (L, φ), and study some of its properties.

Definition 3.1. Let (L, φ) be a state residuated lattice and I be a state ideal of (L, φ). Given a nonempty subset X of L, the set

$$X_I^{\perp_\varphi} := \{a \in L : x'' \wedge \varphi(a)'' \in I, \text{ for all } \ x \in X\}$$

is called the *state relative annihilator* of X with respect to I.

We have the following.

(1) If $I = \{0\}$, then $X^{\perp_\varphi} = X_{\{0\}}^{\perp_\varphi} := \{a \in L : x'' \wedge \varphi(a)'' = 0,\ \textit{for all } x \in X\}$,

(2) If $X = \{x\}$, we denote $x_I^{\perp_\varphi} = \{x\}_I^{\perp_\varphi} := \{a \in L : x'' \wedge \varphi(a)'' \in I\}$,

(3) If $\varphi = id_L$, we obtain $X_I^{\perp_{id_L}} := \{a \in L : x'' \wedge a'' \in I, \text{ for all } \ x \in X\} = X_I^{\perp}$ and it is called the *relative annihilator* of X with respect to I (See Definition 3.1, [11]);

(4) If $\varphi = id_L$ and $I = \{0\}$, we denote $X_{\{0\}}^{\perp_{id_L}} := \{a \in L : x'' \wedge a'' = 0, \text{ for all } x \in X\} = X^{\perp}$, which is called the *annihilator* of X in L (See [11, 18]).

For the convenience of the reader, we present some illustrative examples.

Example 3.2. Let $L = \{0, a, b, c, 1\}$ be a poset with $0 < a,\ b < c < 1$ and a, b not comparable. Consider the Cayley tables of $\odot$ and $\to$ presented in Figure 1.

Then $L = (L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice which is not De Morgan, since $(a \wedge b)' = 1 \neq c = a' \vee b'$. Now, we define a map φ on the residuated lattice L as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

FIGURE 1. Cayley tables of the operations $\odot$ and $\rightarrow$

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in \{0, a\}, \\ 1 & \text{for } x \in \{b, c, 1\}. \end{cases}$$

One can easily check that (L, φ) is a state residuated lattice and $I = coker(\varphi) = \{0, a\}$ is a state ideal of (L, φ). We have $0_I^{\perp_\varphi} = a_I^{\perp_\varphi} = L$ and $b_I^{\perp_\varphi} = c_I^{\perp_\varphi} = 1_I^{\perp_\varphi} = \{0, a\} = I$. For $X = \{a, b\}$, we obtain $X_I^{\perp_\varphi} = \{0, a\} = I$.

Remark 3.3. If L is De Morgan , then $X_I^{\perp_\varphi} = \{a \in L : x \wedge \varphi(a) \in I,\ for\ all\ x \in X\}$, which coincides with the definition of a state relative annihilator in state De Morgan residuated lattices given in (Definition 7, [9]. In addition, if $\varphi = id_L$, $X_I^{\perp_{id_L}} := \{a \in L : x \wedge a \in I,\ for\ all\ x \in X\} = X_I^{\perp}$, which is the definition of relative annihilator in De Morgan residuated lattices given in (Definition 4.23, [16]). Indeed, suppose that L is De Morgan, then we have

$$\begin{aligned} X_I^{\perp_\varphi} &= \{a \in L : x'' \wedge \varphi(a)'' \in I \text{ for all } x \in X\} \\ &= \{a \in L : (x \wedge \varphi(a))'' \in I, \text{ for all } x \in X\} \text{ [By (P15)]} \\ &= \{a \in L : x \wedge \varphi(a) \in I, \text{ for all } x \in X\}. \text{ [By Remark 2.2]} \end{aligned}$$

Remark 3.4. Let L be a residuated lattice. Then the following holds: for any $x,\ y \in L$,

(SO25) $\varphi(x \oslash y) \leq \varphi(x) \oslash \varphi(y)$.

This follows from the fact that

$$\begin{aligned} \varphi(x \oslash y) &= \varphi(x' \rightarrow y) \\ &\leq \varphi(x') \rightarrow \varphi(y) \text{ [By (SO14)]} \\ &= \varphi(x)' \rightarrow \varphi(y) \text{ [By (SO11)]} \\ &= \varphi(x) \oslash \varphi(y). \end{aligned}$$

Theorem 3.5. *Let I be a state ideal of a state residuated lattice (L, φ). Given a nonempty subset X of L, $X_I^{\perp_\varphi}$ is a state ideal of (L, φ).*

Proof. Let $a \in L$ and $b \in X_I^{\perp_\varphi}$ such that $a \leq b$.

Then, from (SO10), $\varphi(a) \leq \varphi(b)$, which implies that $\varphi(a)'' \leq \varphi(b)''$ from (RL4). It follows that $x'' \wedge \varphi(a)'' \leq x'' \wedge \varphi(b)''$ for all $x \in X$. Since $\varphi(b)'' \wedge x'' \in I$ and I is an ideal, we deduce that $\varphi(a)'' \wedge x'' \in I$ for all $x \in X$. Thus $a \in X_I^{\perp_\varphi}$.

Moreover, let $a, b \in X_I^{\perp_\varphi}$. Then we have $x'' \wedge \varphi(a)'' \in I$ and $x'' \wedge \varphi(b)'' \in I$ for all $x \in X$. Since I is an ideal, we have $(x'' \wedge \varphi(a)'') \oplus (x' \wedge \varphi(b)'') \in I$. We will show that $a \oslash b \in X_I^{\perp_\varphi}$, that is, $\varphi(a \oslash b)'' \wedge x'' \in I$ for all $x \in X$. From (RL13), we have $\varphi(a)' \odot \varphi(b)' \leq (\varphi(a)' \rightarrow \varphi(b))'$. It follows that

$$\begin{aligned}\varphi(a\oslash b)'' &\leq (\varphi(a)\oslash\varphi(b))'' \text{ [By (SO25) and (RL4)]}\\ &= (\varphi(a)'\to\varphi(b))''\\ &\leq (\varphi(a)'\odot\varphi(b)')', \text{ [By (RL4)]}\end{aligned}$$

that is, we have: for all $x \in X$,

$$\begin{aligned}x''\wedge\varphi(a\oslash b)'' &\leq x''\wedge(\varphi(a)'\odot\varphi(b)')'\\ &= [x'\vee(\varphi(a)'\odot\varphi(b)')]' \text{ [By (RL7)]}\\ &\leq [(x'\vee\varphi(a)')\odot(x'\vee\varphi(b)')]' \text{ [By (RL4) and (RL16)]}\\ &= [(x'\vee\varphi(a)')\odot(x'\vee\varphi(b)')]''' \text{ [By (RL11)]}\\ &= [(x'\vee\varphi(a)')''\odot(x'\vee\varphi(b)')'']' \text{ [By (RL7)]}\\ &= [(x''\wedge\varphi(a)'')'\odot(x'\wedge\varphi(b)'')']' \text{ [By (RL7)]}\\ &= (x''\wedge\varphi(a)'')\oplus(x'\wedge\varphi(b)'')\in I.\end{aligned}$$

Thus $x''\wedge\varphi(a\oslash b)''\in I$ for all $x\in X$. So $a\oslash b\in X_I^{\perp_\varphi}$. Hence $X_I^{\perp_\varphi}$ is an ideal of L.

In addition, let $a\in X_I^{\perp_\varphi}$. Since $(\varphi(\varphi(a)))'' \overset{(SO21)}{=} \varphi(\varphi(a)'') \overset{(SO21)}{=} \varphi(\varphi(a'')) \overset{(SO16)}{=} \varphi(a'') \overset{(SO21)}{=} \varphi(a)''$, it follows that $(\varphi(\varphi(a)))''\wedge x'' = \varphi(a)''\wedge x''\in I$ for all $x\in X$. Then $\varphi(a)\in X_I^{\perp_\varphi}$. We conclude that $X_I^{\perp_\varphi}$ is a state ideal of (L,φ). □

The subsequent case illustrates that the converse of Theorem 3.5 does not necessarily hold.

Indeed, consider the state residuated lattice (L,φ) defined in Example 3.2, and let $J=\{0,b\}$ be a subset of L. Then, $a_J^{\perp_\varphi}=L$ which is a state ideal of (L,φ), though $J=\{0,b\}$ is not a state ideal of (L,φ), since $b\in J$ but $\varphi(b)=1\notin J$.

Corollary 3.6. *Let I be a state ideal of (L,φ). Given a nonempty subset X of L, we have:*

(1) *$X^{\perp_\varphi}$ is a state ideal of (L,φ),*
(2) *$X_I^{\perp}$ and $X^{\perp}$ are state ideals of L.*

Lemma 3.7. *Let L be a residuated lattice. Then, for any $x,\ y,\ z,\ x_1,\cdots x_n\in L$, $n\in\mathbb{N}^*$,*

(P16) $x\wedge(y\oplus z)\leq x''\wedge(x\oplus z)''\leq(x''\wedge y'')\oplus(x''\wedge z'')$,
(P17) $x\wedge(x_1\oplus ...\oplus x_n)\leq x''\wedge(x_1\oplus ...\oplus x_n)''\leq(x''\wedge x_1'')\oplus ...\oplus(x''\wedge x_n'')$.

Proof. (P16) Let $x,\ y,\ z\in L$. From (RL9), we have $x\leq x''$ and $y\oplus z\leq(y\oplus z)''$. Then $x\wedge(y\oplus z)\leq x''\wedge(x\oplus z)''$. In addition, we get

$$\begin{aligned}x''\wedge(y\oplus z)'' &= x''\wedge(y'\odot z')'''\\ &= x''\wedge(y'\odot z')' \text{ [By (RL11)]}\\ &= [x'\vee(y'\odot z')]' \text{ [By (RL7)]}\\ &\leq [(x'\vee y')\odot(x'\vee z')]' \text{ [By (RL4) and (RL16)]}\\ &= [(x'\vee y')\odot(x'\vee z')]''' \text{ [By (RL11)]}\\ &= [(x'\vee y')''\odot(x'\vee z')'']' \text{ [By (RL7)]}\\ &= (x''\wedge y'')'\odot(x''\wedge z'')']' \text{ [By (RL7)]}\\ &= (x''\wedge y'')\oplus(x''\wedge z').\end{aligned}$$

Thus $x''\wedge(x\oplus z)''\leq(x''\wedge y'')\oplus(x''\wedge z'')$. So we have

$$x\wedge(y\oplus z)\leq x''\wedge(x\oplus z)''\leq(x''\wedge y'')\oplus(x''\wedge z'').$$

(P17) Let $x,\ x_1,\ \cdots,\ x_n \in L$, $n \in \mathbb{N}^*$. From (RL9), we have $x \leq x''$ and $x_1 \oplus .... \oplus x_n \leq (x_1 \oplus ... \oplus x_n)''$. Then $x \wedge (x_1 \oplus ... \oplus x_n) \leq x'' \wedge (x_1 \oplus ... \oplus x_n)''$.

The proof of $x'' \wedge (x_1 \oplus \cdots \oplus x_n)'' \leq (x'' \wedge x_1'') \oplus \cdots \oplus (x'' \wedge x_n'')$ will be done by mathematical induction.

If $n = 1$, then we have the equality.

If $n = 2$, then by (P16), we have $x'' \wedge (x_1 \oplus x_2)'' \leq (x'' \wedge x_2'') \oplus (x'' \wedge x_2'')$.

Now suppose that for $n = k \in \mathbb{N}^*$, $k \geq 2$,

$$x'' \wedge (x_1 \oplus \cdots \oplus x_k)'' \leq (x'' \wedge x_1'') \oplus \cdots \oplus (x'' \wedge x_k'')$$

holds. We will prove that it also holds for $n = k + 1$, that is,

$$x'' \wedge (x_1 \oplus \cdots \oplus x_k \oplus x_{k+1})'' \leq (x'' \wedge x_1'') \oplus \cdots \oplus (x'' \wedge x_k'') \oplus (x'' \wedge x_{k+1}'').$$

We have

$$\begin{aligned}
& x'' \wedge (x_1 \oplus \cdots \oplus x_k \oplus x_{k+1})'' \\
&= x'' \wedge [(x_1 \oplus \cdots \oplus x_k)' \odot x_{k+1}']''' \\
&= x'' \wedge [(x_1 \oplus \cdots \oplus x_k)' \odot x_{k+1}']' \text{ [By (RL11)]} \\
&= [x' \vee ((x_1 \oplus \cdots \oplus x_k)' \odot x_{k+1}')]' \text{ [By (RL7)]} \\
&\leq [(x' \vee (x_1 \oplus \cdots \oplus x_k)') \odot (x' \vee x_{k+1}')]' \text{ [By (RL4) and (RL16)]} \\
&= [(x' \vee (x_1 \oplus \cdots \oplus x_k)') \odot (x' \vee x_{k+1}')]''' \text{ [By (RL11)]} \\
&= [(x' \vee (x_1 \oplus \cdots \oplus x_k)')'' \odot (x' \vee x_{k+1}')'']' \text{ [By (RL7)]} \\
&= [(x'' \wedge (x_1 \oplus \cdots \oplus x_k)'')' \odot (x'' \wedge x_{k+1}'')']' \text{ [By (RL7)]} \\
&= (x'' \wedge (x_1 \oplus \cdots \oplus x_k)'') \oplus (x'' \wedge x_{k+1}'') \\
&\leq (x'' \wedge x_1'') \oplus \cdots \oplus (x'' \wedge x_k'') \oplus (x'' \wedge x_{k+1}''). \text{ [By the hypothesis]}
\end{aligned}$$

Then it follows that, $x'' \wedge (x_1 \oplus \cdots \oplus x_n)'' \leq (x'' \wedge x_1'') \oplus ... \oplus (x'' \wedge x_n'')$ for all $n \in \mathbb{N}^*$.
Thus, $x \wedge (x_1 \oplus \cdots \oplus x_n) \leq x'' \wedge (x_1 \oplus ... \oplus x_n)'' \leq (x'' \wedge x_1'') \oplus \cdots \oplus (x'' \wedge x_n'')$. □

In the following, we collect several properties of state relative annihilators.

Proposition 3.8. *Let $I,\ J$ be state ideals of (L, φ). Given nonempty subsets $X,\ Y$ of L, we have:*

(1) $I \subseteq J \Rightarrow X_I^{\perp_\varphi} \subseteq X_J^{\perp_\varphi}$,
(2) $X \subseteq Y \Rightarrow (Y)_I^{\perp_\varphi} \subseteq X_I^{\perp_\varphi}$,
(3) $(\bigcup_{k\in K} X_k)_I^{\perp_\varphi} = \bigcap_{k\in K} (X_k)_I^{\perp_\varphi}$,
(4) $X^{\perp_\varphi}_{(\bigcap_{k\in K} I_k)} = \bigcap_{k\in K} X_{I_k}^{\perp_\varphi}$,
(5) $\langle X\rangle_I^{\perp_\varphi} = X_I^{\perp_\varphi}$, *in particular*, $\varnothing_I^{\perp_\varphi} = 0_I^{\perp_\varphi} = L$,
(6) $coker(\varphi) \subseteq X^{\perp_\varphi}, L^{\perp_\varphi} = coker(\varphi)$,
(7) $X_I^{\perp_\varphi} = \bigcap_{x\in X} x_I^{\perp_\varphi}$,
(8) $X^{\perp_\varphi} = \bigcap_{x\in X} x^{\perp_\varphi}$,
(9) $X_I^{\perp_\varphi} = L$ *iff* $X \subseteq I$, *particularly*, $0_I^{\perp_\varphi} = I_I^{\perp_\varphi} = L$,
(10) $\bigcap_{k\in K} (X_k)_I^{\perp_\varphi} \subseteq (\bigcap_{k\in K} X_k)_I^{\perp_\varphi}$,
(11) $I \subseteq X_I^{\perp_\varphi}$,
(12) $X^{\perp_\varphi} \subseteq X_I^{\perp_\varphi}$.

Proof. (1) Let $I \subseteq J$ and $a \in X_I^{\perp_\varphi}$. Then $x'' \wedge \varphi(a)'' \in I$ for all $x \in X$. Thus $x'' \wedge \varphi(a)'' \in J$ for all $x \in X$, that is, $a \in X_J^{\perp_\varphi}$. So $X_I^{\perp_\varphi} \subseteq X_J^{\perp_\varphi}$.

(2) Let $X \subseteq Y$ and $a \in (Y)_I^{\perp_\varphi}$. Then $x'' \wedge \varphi(a)'' \in I$ for all $x \in Y$. Thus $x'' \wedge \varphi(a)'' \in I$ for all $x \in X$, which implies $a \in X_I^{\perp_\varphi}$. So $(Y)_I^{\perp_\varphi} \subseteq X_I^{\perp_\varphi}$.

(3) Since $X_k \subseteq \underset{k\in K}{\cup} X_k$, it follows from (2) that $(\underset{k\in K}{\cup} X_k)_I^{\perp_\varphi} \subseteq (X_k)_I^{\perp_\varphi}$ for all $k \in K$. We deduce that $(\underset{k\in K}{\cup} X_k)_I^{\perp_\varphi} \subseteq \underset{k\in K}{\cap}(X_k)_I^{\perp_\varphi}$.

Conversely, let $a \in \underset{k\in K}{\cap}(X_k)_I^{\perp_\varphi}$. Then we have $a \in (X_k)_I^{\perp_\varphi}$ for all $k \in K$. Thus $x_k'' \wedge \varphi(a)'' \in I$ for all $x_k \in X_k$ and $k \in K$, which implies that $a \in (\underset{k\in K}{\cup} X_k)_I^{\perp_\varphi}$. So $\underset{k\in K}{\cap}(X_k)_I^{\perp_\varphi} \subseteq (\underset{k\in K}{\cup} X_k)_I^{\perp_\varphi}$. Hence $(\underset{k\in K}{\cup} X_k)_I^{\perp_\varphi} = \underset{k\in K}{\cap}(X_k)_I^{\perp_\varphi}$.

(4) We have $a \in X^{\perp_\varphi}_{(\underset{k\in K}{\cap} I_k)}$ iff $x'' \wedge \varphi(a)'' \in \underset{k\in K}{\cap} I_k$ for all $x \in X$ iff $x'' \wedge \varphi(a)'' \in I_k$ for all $x \in X$ and $k \in K$ iff $a \in X_{I_k}^{\perp_\varphi}$ for all $k \in K$ iff $a \in \underset{k\in K}{\cap} X_{I_k}^{\perp_\varphi}$. Then $X^{\perp_\varphi}_{(\underset{k\in K}{\cap} I_k)} = \underset{k\in K}{\cap} X_{I_k}^{\perp_\varphi}$.

(5) Since $X \subseteq \langle X\rangle$, we deduce by (2) that $\langle X\rangle_I^{\perp_\varphi} \subseteq X_I^{\perp_\varphi}$.

Conversely, let $a \in X_I^{\perp_\varphi}$ and $z \in \langle X\rangle$. Then $x'' \wedge \varphi(a)'' \in I$ for all $x \in X$. Since $z \in \langle X\rangle$, there are $n \in \mathbb{N}^*$ and $x_1,\ x_2,\ \cdots,\ x_n \in X$ such that $z \leq x_1 \oplus x_2 \oplus \cdots \oplus x_n$. It follows that

$$\begin{aligned} z'' \wedge \varphi(a)'' &\leq \varphi(a)'' \wedge (x_1 \oplus x_2 \oplus \cdots \oplus x_n)'' \\ &\leq (\varphi(a)'' \wedge x_1'') \oplus \cdots \oplus (\varphi(a)'' \wedge x_n'') \in I. \text{ [By (P17)]} \end{aligned}$$

Thus $\varphi(a)'' \wedge z'' \in I$ for all $z \in \langle X\rangle$. So $a \in \langle X\rangle_I^{\perp_\varphi}$. Hence $\langle X\rangle_I^{\perp_\varphi} = X_I^{\perp_\varphi}$.

(6) Let $a \in coker(\varphi)$. Then $\varphi(a) = 0$, which implies that $\varphi(a)'' = 0$. Since $x'' \wedge \varphi(a)'' \leq \varphi(a)'' = 0$ for all $x \in X$, we have $x'' \wedge \varphi(a)'' = 0$, that is, $a \in X^{\perp_\varphi}$. Thus $coker(\varphi) \subseteq X^{\perp_\varphi}$. Taking $X = L$, we have $coker(\varphi) \subseteq L^{\perp_\varphi}$. Now, let $a \in L^{\perp_\varphi}$. Then $x'' \wedge \varphi(a)'' = 0$ for all $x \in L$. In particular, taking $x = \varphi(a)$, we have $\varphi(a)'' = 0$. It follows that $\varphi(a)' = \varphi(a)''' = 0' = 1$, which implies $\varphi(a)' = 1$. So $\varphi(a) = 0$. Hence $a \in coker(\varphi)$, that is, $L^{\perp_\varphi} \subseteq coker(\varphi)$. Therefore $coker(\varphi) \subseteq X^{\perp_\varphi}$ and $L^{\perp_\varphi} = coker(\varphi)$.

(7) According to (3), we have

$$X_I^{\perp_\varphi} = (\underset{x\in X}{\cup}\{x\})_I^{\perp_\varphi} = \underset{x\in X}{\cap} x_I^{\perp_\varphi}.$$

(8) Taking $I = \{0\}$ in (1), we obtain $X^{\perp_\varphi} = \underset{x\in X}{\cap} x^{\perp_\varphi}$.

(9) Assume that $X_I^{\perp_\varphi} = L$ and let $x \in X$. Then $x'' = x'' \wedge \varphi(1)'' \in I$, that is, $x'' \in I$, which implies $x \in I$ from Remark 2.2.

Conversely, suppose that $X \subseteq I$ and $a \in L$. Then for any $x \in X$, we have $x \in I$. It follows that $x'' \in I$. Since $x'' \wedge \varphi(a)'' \leq x'' \in I$, we obtain $x'' \wedge \varphi(a)'' \in I$. Thus $a \in X_I^{\perp_\varphi}$. So $X_I^{\perp_\varphi} = L$.

(10) Since $\underset{k\in K}{\cap} X_k \subseteq X_k$ for each $k \in K$, it follows from (2) that $(X_k)_I^{\perp_\varphi} \subseteq (\underset{k\in K}{\cap} X_k)_I^{\perp_\varphi}$ for all $k \in K$. Then $\underset{k\in K}{\cap}(X_k)_I^{\perp_\varphi} \subseteq (\underset{k\in K}{\cap} X_k)_I^{\perp_\varphi}$.

(11) Let $a \in I$. Since I is a state ideal, we have $\varphi(a) \in I$ which implies $\varphi(a)'' \in I$. In addition, $x'' \wedge \varphi(a)'' \leq \varphi(a)'' \in I$ for all $x \in X$. Then $x'' \wedge \varphi(a)'' \in I$ for all $x \in X$. Thus $a \in X_I^{\perp_\varphi}$. So $I \subseteq X_I^{\perp_\varphi}$.

(12) Let I be an ideal of L. We have $a \in X^{\perp_\varphi}$ implies $\varphi(a)'' \wedge x'' = 0 \in I$ for any $x \in X$. Then $\varphi(a)'' \wedge x'' \in I$ for any $x \in X$. Thus $a \in X_I^{\perp_\varphi}$. So $X^{\perp_\varphi} \subseteq X_I^{\perp_\varphi}$. □

The next result shows some affinity properties of state relative annihilators with other operations.

Proposition 3.9. *Let (L, φ) be a state residuated lattice. Then we have: for all $a,\ b \in L$,*

(1) $a \leq b \Rightarrow b_I^{\perp_\varphi} \subseteq a_I^{\perp_\varphi}$,
(2) $(a \vee b)_I^{\perp_\varphi} = a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi} = (a \oslash b)_I^{\perp_\varphi} = (b \oslash a)_I^{\perp_\varphi}$.

Proof. (1) Assume that $a \leq b$. Then $\langle a\rangle \subseteq \langle b\rangle$, which implies from Proposition 3.8 (2) and (5) that $b_I^{\perp_\varphi} \subseteq a_I^{\perp_\varphi}$.

(2) First of all, we will prove that $(a \vee b)_I^{\perp_\varphi} = a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi}$. Since $a,\ b \leq a \vee b$, it follows from (1) that $(a \vee b)_I^{\perp_\varphi} \subseteq a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi}$. Moreover, let $x \in a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi}$. Then $x \in a_I^{\perp_\varphi}$ and $x \in b_I^{\perp_\varphi}$. This implies that $\varphi(x)'' \wedge a'', \varphi(x)'' \wedge b'' \in I$. But

$$
\begin{aligned}
\varphi(x)'' \wedge (a \vee b)'' &= (\varphi(x)' \vee (a \vee b)')' \text{ [By (RL7)]}\\
&= (\varphi(x)' \vee (a' \wedge b'))' \text{ [By (RL7)]}\\
&\leq (\varphi(x)' \vee (a' \odot b'))' \text{ [By (RL4) and (RL6)]}\\
&\leq ((\varphi(x)' \vee a') \odot (\varphi(x)' \vee b'))' \text{ [By (RL4) and (RL6)]}\\
&= ((\varphi(x)' \vee a') \odot (\varphi(x)' \vee b'))''' \text{ [By (RL11)]}\\
&= ((\varphi(x)' \vee a')'' \odot (\varphi(x)' \vee b')'')' \text{ [By (RL7)]}\\
&= ((\varphi(x)'' \wedge a'')' \odot (\varphi(x)'' \wedge b'')')' \text{ [By (RL7)]}\\
&= (\varphi(x)'' \wedge a'') \oplus (\varphi(x)'' \wedge b'') \in I.
\end{aligned}
$$

Thus $x \in (a \vee b)_I^{\perp_\varphi}$. So $(a \vee b)_I^{\perp_\varphi} = a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi}$.

Now, we show that $a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi} = (a \oslash b)_I^{\perp_\varphi}$. We have $a,\ b \overset{(P11)}{\leq} a \oslash b$ implies from (1) that $(a \oslash b)_I^{\perp_\varphi} \subseteq a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi}$.

Conversely, let $x \in a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi}$. Then $x \in a_I^{\perp_\varphi}$ and $x \in b_I^{\perp_\varphi}$, that is, $\varphi(x)'' \wedge a'', \varphi(x)'' \wedge b'' \in I$. It follows that $(\varphi(x)'' \wedge a'') \oplus (\varphi(x)'' \wedge b'') \in I$. But,

$$
\begin{aligned}
\varphi(x)'' \wedge (a \oslash b)'' &\leq \varphi(x)'' \wedge (a' \odot b')' \text{ [By (P11)]}\\
&= [\varphi(x)' \vee (a' \odot b')]' \text{ [By (RL7)]}\\
&\leq [(\varphi(x)' \vee a') \odot (\varphi(x)' \vee b')]' \text{ [By (RL4) and (RL16)]}\\
&= [(\varphi(x)' \vee a') \odot (\varphi(x)' \vee b')]''' \text{ [By (RL11)]}\\
&= [(\varphi(x)' \vee a')'' \odot (\varphi(x)' \vee b')'']' \text{ [By (RL7)]}\\
&= [(\varphi(x)'' \wedge a'')' \odot (\varphi(x)'' \wedge b'')']' \text{ [By (RL7)]}\\
&= (\varphi(x)'' \wedge a'') \oplus (\varphi(x)'' \wedge b'') \in I.
\end{aligned}
$$

Thus $\varphi(x)'' \wedge (a \oslash b)'' \in I$. So $x \in (a \oslash b)_I^{\perp_\varphi}$. Hence $a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi} = (a \oslash b)_I^{\perp_\varphi}$. Analogously, we get $a_I^{\perp_\varphi} \cap b_I^{\perp_\varphi} = (b \oslash a)_I^{\perp_\varphi}$. □

The following properties always hold for $\varphi = id_L$, which may not in general be the case for some state operators of residuated lattices.

Proposition 3.10. *Let (L, id_L) be a state residuated lattice, let I, J be two state ideals of (L, id_L) and let X, Y two nonempty subsets of L.*

(1) $X \cap X_I^{\perp} \subseteq I$.
(2) $I \subseteq X$ *iff* $X \cap X_I^{\perp} = I$. *Particularly,* $X_I^{\perp} \cap (X_I^{\perp})_I^{\perp} = I$.
(3) $Y_I^{\perp} \cup X_I^{\perp} \subseteq Y^{\perp}_{(X_I^{\perp})}, X^{\perp}_{(Y_I^{\perp})}$ *and* $Y^{\perp}_{(X_I^{\perp})} \cup X^{\perp}_{(Y_I^{\perp})} \subseteq (X \wedge Y)_I^{\perp}$, *where* $X \wedge Y = \{x \wedge y : x \in X,\ y \in Y\}$.
(4) $L_I^{\perp} = I$ *and* $1_I^{\perp} = I$.
(5) $X \subseteq (X_I^{\perp})_I^{\perp}$. *Particularly,* $(I_I^{\perp})_I^{\perp} = I$ *and* $(L_I^{\perp})_I^{\perp} = L$.
(6) $X_I^{\perp} = ((X_I^{\perp})_I^{\perp})_I^{\perp}$.
(7) $X^{\perp}_{(X_I^{\perp})} \subseteq X_{(X_{(Y_I^{\perp})})}$.
(8) *If* $X \subseteq I$, *then:* $X^{\perp}_{(X_I^{\perp})} = X^{\perp}_{(X^{\perp}_{(X_I^{\perp})})} = X_{(X_{(Y_I^{\perp})})}$, *whenever* $X \subseteq Y$.

Proof. The proof is similar to the one of Proposition 3.7 and Corollary 3.8 in [11]. □

Example 3.11 exhibits a state residuated lattice in which items (1) and (2) of Proposition 3.10 do not hold.

Example 3.11. Let $L = \{0, p, a, b, c, d, e, f, q, 1\}$ be a set such that $0 < p < a < c < e < q < 1$, $0 < p < b < d < f < q < 1$, $b < c$, $d < e$, $f < q$, but a and b are not comparable, as well as c and d, also e and f.

The Cayley tables of $\odot$ and $\rightarrow$ are displayed in Figure 2:

$\odot$	0	p	a	b	c	d	e	f	q	1
0	0	0	0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	0	0	0	p
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
q	0	0	a	b	c	d	e	f	q	q
1	0	p	a	b	c	d	e	f	q	1

$\rightarrow$	0	p	a	b	c	d	e	f	q	1
0	1	1	1	1	1	1	1	1	1	1
p	q	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
q	p	p	a	b	c	d	e	f	1	1
1	0	p	a	b	c	d	e	f	q	1

FIGURE 2. Cayley tables of the operations $\odot$ and $\rightarrow$

Then $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a residuated lattice. The map φ defined on L by:

$$\varphi(x) = \begin{cases} 0 & \text{for } x \in \{0, p, b, d, f\}, \\ 1 & \text{for } x \in \{a, c, e, q, 1\}. \end{cases}$$

is a state operator on L. Thus (L, φ) is a state residuated lattice. $I = \{0, p\}$ and $J = \{0, p, b, d, f\}$ are two state ideals of (L, φ).

We have $0_I^{\perp_\varphi} = p_I^{\perp_\varphi} = L$ and $a_I^{\perp_\varphi} = b_I^{\perp_\varphi} = c_I^{\perp_\varphi} = d_I^{\perp_\varphi} = e_I^{\perp_\varphi} = f_I^{\perp_\varphi} = q_I^{\perp_\varphi} = 1_I^{\perp_\varphi} = \{0, p, b, d, f\} = J$.

(1) For $X = \{d\}$, we obtain $X_I^{\perp_\varphi} = J$. But $X \cap X_I^{\perp_\varphi} = \{d\}$, which is not a subset of I.
(2) Let $X = \{0, p, f\}$. By Proposition 3.8 (3), we have $X_I^{\perp_\varphi} = 0_I^{\perp_\varphi} \cap p_I^{\perp_\varphi} \cap f_I^{\perp_\varphi} = L \cap L \cap J = J$. Clearly $I \subseteq X$, but $X \cap X_I^{\perp_\varphi} = X \cap J = X \neq I$.

4. State-morphism operators and annihilators in state residuated lattices

In this section, we introduce the notion of state-morphism operator in state residuated lattices and establish a relationship between annihilators and state-morphism operators.

Definition 4.1. A map $\varphi : L \longrightarrow L$ is called a *state morphism operator*, if φ is an idempotent endomorphism (that is, φ is an endomorphism of L such that $\varphi^2 = \varphi$). The couple (L, φ) is called a *state-morphism residuated lattice*.

Example 4.2. Set $L = \{0, a, b, 1\}$ with $0 < a < b < 1$. Then, L is a residuated lattice with the operations presented in Figure 3.

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Figure 3. Cayley tables of the operations $\odot$ and $\rightarrow$

Let define the unary operator φ on L by

$$\varphi(x) = \begin{cases} 0 & \text{if } x = 0, \\ a & \text{if } x = a, \\ 1 & \text{if } x \in \{b, 1\}. \end{cases}$$

Then one can easily check that φ is a state operator on L. Thus the couple (L, φ) is a SRL. In addition, $coker(\varphi) = \{0\}$ and φ verified the following properties:

$$\varphi(x \rightarrow y) = \varphi(x) \rightarrow \varphi(y) \text{ and } \varphi(x \odot y) = \varphi(x) \odot \varphi(y) \text{ for any } x,\ y \in L.$$

So φ is a cofaithful state-morphism operator. Moreover, the state ideals of (L, φ) are $\{0\}$ and L.

Example 4.3. Let L_1 and L_2 be two nontrivial residuated lattices and $h : L_1 \longrightarrow L_2$

a homomorphism. We define the map $\begin{array}{rcl}\varphi_h : L_1 \times L_2 & \longrightarrow & L_1 \times L_2 \\ (x,y) & \longmapsto & \varphi_h(x,y) = (x, h(x)).\end{array}$
Then one can check that φ_h is an idempotent endomorphism on $L_1 \times L_2$. Thus φ_h is a state-morphism operator and the couple $(L_1 \times L_2, \varphi_h)$ is a state-morphism residuated lattice. In addition, $coker(\varphi_h) = \{0\} \times L_2$. So φ_h is a uncofaithful state-morphism operator.

Proposition 4.4. *Every state-morphism operator of a residuated lattice L is a state operator of L.*

Proof. Let φ be a state-morphism operator of a residuated lattice L. We will show that φ verifies the eight conditions of a state operator from Definition 2.3.

(SO1) : Since φ is an endomorphism of L, we have $\varphi(0) = 0$.

(SO2) : Let $x,\ y \in L$ such that $x \to y = 1$. Then $\varphi(x) \to \varphi(y) = \varphi(x \to y) = \varphi(1) = 1$.

(SO3) : Let $x,\ y \in L$. Then from (RL14), we have $x \to (x \wedge y) = x \to y$. Thus $\varphi(x \to y) = \varphi(x \to (x \wedge y)) = \varphi(x) \to \varphi(x \wedge y)$.

(SO4) : Let $x,\ y \in L$. Then from (RL15), we have $x \odot y = x \odot (x \to (x \odot y))$. Thus $\varphi(x \odot y) = \varphi(x \odot (x \to (x \odot y))) = \varphi(x) \odot \varphi(x \to (x \odot y))$.

(SO5) : Let $x,\ y \in L$. Then $\varphi(\varphi(x) \odot \varphi(y)) = \varphi^2(x) \odot \varphi^2(y) = \varphi(x) \odot \varphi(y)$, because φ is idempotent.

(SO6), (SO7) and (SO8) are obtained similarly as (SO5). So φ is a state operator on L. □

In the following, we put on view some links between annihilators and state-morphism operators.

Proposition 4.5. *Let (L, φ) be a state-morphism residuated lattice and X a nonempty subset of L. Then we have the following:*

(1) $\varphi(X^\perp) \subseteq (\varphi(X))^\perp$,
(2) *If φ is surjective then,* $(\varphi^{-1}(X))^\perp \subseteq \varphi^{-1}(X^\perp)$,
(3) $\varphi(X^\perp) = (\varphi(X))^\perp$ *iff* $\varphi(X^\perp) = (\varphi(X^\perp))^{\perp\perp}$ *and* $(\varphi(X))^\perp \cap (\varphi(X^\perp))^\perp = \{0\}$,
(4) *if φ is surjective, then* $(\varphi^{-1}(X))^\perp = \varphi^{-1}(X^\perp)$ *iff* $\varphi^{-1}(X^\perp) \cap (\varphi^{-1}(X))^{\perp\perp} = \{0\}$,
(5) *if φ is bijective, then* $\varphi(X^\perp) = (\varphi(X))^\perp$ *and* $(\varphi^{-1}(X))^\perp = \varphi^{-1}(X^\perp)$.

Proof. Since a state-morphism operator is an endomorphism, the proofs of (1),(2),(3), (4) and (5) are similar to the one of Proposition 3.12, Proposition 3.13, Proposition 3.14, Proposition 3.15 and Proposition 3.16 in [11] respectively. □

From (Example 3.30 and Example 3.32, [19]), one can see that $\varphi(X^\perp)$ and $\varphi^{-1}(X^\perp)$ are not always annihilators.

Conclusion

This work was devoted to the investigation on state relative annihilators in the framework of state residuated lattices. We brought in the concept of state relative annihilator of a given nonempty subset with respect to a state ideal in a SRL, investigated some of its properties and examples. We have shown that this notion is a generalization of the one of De Morgan state residuated lattices recently studied in our previous paper [9]. Particulary, we have proved that state relative annihilators are state ideals. Also, we have presented some links between state-morphism operators and annihilators.

In the same view as the work done by Pengfei et al. in [20], we will study as future work the L-fuzzy state annihilators in state residuated lattices.

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