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Fixed point theorems for generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-contractions in Branciari distance spaces

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Abstract. In this paper, the notion of generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-contractions is introduced and a new fixed point theorem for such contractions is established. An example for illustrating the main theorem is given.

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Keywords: Fixed point, $\mathcal{L}$-contraction, Suzuki type $\mathcal{L}$-contraction, Suzuki-Berinde type $\mathcal{L}$-contraction, metric space, Branciari distance space

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1. Introduction and preliminaries

Berinde [1] introduced the concept of almost contractions:

A map $T : X \to X$, (X, d) is a metric space, is called an *almost contraction*, if it satisfies

$$d(Tx, Ty) \leq kd(x, y) + Ld(y, Tx),$$

where $k \in (0, 1)$ and $L \geq 0$.

Berinde [1] obtained a generalization of the Banach contraction principle by proving existence of fixed point for almost contractions defined on complete metric spaces.

Suzuki [2] generalized Banach contraction principle by using the notion of contractive map $T : X \to X$, where (X, d) is compact metric space, as follows:

$$\forall x, y \in X (x \neq y),\ \frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y).$$

On the one hand, Branciari [3] extended Banach contraction principle to Branciari distance spaces, which is a generalization of metric spaces.

After that, many researchers ([4, 5, 6, 7, 8, 9, 10, 11] and references therein) extended fixed point results in metric spaces to Branciari distance spaces despite the topological disadvantages of the branchiari distance ([9, 10, 12, 13, 14]) as follows.

- · Branchiari distance is not necessarily continuous in each coordinates;
- · An open ball doesn't have to be open, and hence there is no topology which is compatible with the Branchiari distance;
- · A convergent sequence doesn't have to be Cauchy.

Given function $\vartheta : (0,\infty) \to (1,\infty)$, we consider the following conditions:

(ϑ1) ϑ is non-decreasing,

(ϑ2) $\forall \{h_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} h_n = 0 \Leftrightarrow \lim_{n\to\infty} \vartheta(h_n) = 1,$$

(ϑ3) $\exists r \in (0,1) \wedge l \in (0,\infty)$:

$$\lim_{t\to 0^+} \frac{\vartheta(t)-1}{t^r} = l,$$

(ϑ4) ϑ is continuous on $(0,\infty)$.

Jleli and Samet [15] gave the concept of ϑ-contractions in Branciari distance spaces and obtained related fixed point result with conditions (ϑ1), (ϑ2) and (ϑ3). Ahmad *et al.* [16] proved the existence of fixed points by introducing the concept of Suzuki-Berinde type ϑ-contractions in metric spaces with conditions (ϑ1), (ϑ2) and (ϑ4).

Very recently, Cho [17] gave the notion of $\mathcal{L}$-contractions, which is a more generalized notion than some existing concept of contractions. He proved the existence of fixed points for such contractions. And then, many researchers, for example [12, 13, 14, 18, 19, 20, 21, 22, 23], generalized the result of [17].

In the paper, we introduce the new concept of generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-contractions which is a generalization of the concept of $\mathcal{L}$-contractions, and we establish a new fixed point theorem for such contraction mappings in the setting of Branciari distance spaces. We give an example to support main theorem.

A function $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ is called an *$\mathcal{L}$-simulation* [17], if it satisfies the following conditions:

(ξ1) $\xi(1,1) = 1$,

(ξ2) $\xi(t,s) < \frac{s}{t} \ \ \forall s,t > 1$,

(ξ3) for any sequence $\{t_n\}, \{s_n\} \subset (1,\infty)$ with $t_n \leq s_n \ \ \forall n = 1,2,3,\cdots$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1 \Rightarrow \lim_{n\to\infty} \sup \xi(t_n, s_n) < 1.$$

Denote $\Gamma[1,\infty)$ the family of all non-decreasing functions $\gamma : [1,\infty) \to [1,\infty)$ such that

$$\gamma^{-1}(\{1\}) = 1.$$

A function $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$ is called an *$\mathcal{L}_\gamma$-simulation* [13], provided that it satisfies (ξ1), (ξ3) and the following condition (ξ4):

(ξ4) $\xi(t,s) < \frac{\gamma(s)}{\gamma(t)} \ \ \forall s,t > 1$, where $\gamma \in \Gamma[1,\infty)$.

Denote $\mathcal{L}$ by the set of all $\mathcal{L}$-simulation functions $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$, and Denote $\mathcal{L}_\gamma$ by the family of all $\mathcal{L}_\gamma$-simulation functions $\xi : [1,\infty) \times [1,\infty) \to \mathbb{R}$.

Remark 1.1. We have the following:

(1) $\mathcal{L} \subset \mathcal{L}_\gamma$,
(2) $\xi(t,t) < 1 \ \forall t > 1$, whenever $\xi \in \mathcal{L}$.

Example 1.2 ([13])**.** Let $\xi_b, \xi_w, \xi_c, \xi_i : [1,\infty) \times [1,\infty) \to \mathbb{R}$, $i = 1,\ 2,\ 3$, be functions defined as follows respectively:

(1) $\xi_b(t,s) = \frac{[\gamma(s)]^k}{\gamma(t)} \ \ \forall t, s \geq 1$, where $k \in (0,1)$,

(2) $\xi_w(t,s) = \frac{\gamma(s)}{\gamma(t)\phi(\gamma(s))} \ \ \forall t, s \geq 1$ where $\phi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$,

(3) $\xi_c(t,s) = \begin{cases} 1 & \text{if } (s,t) = (1,1) \\ \frac{\gamma(s)}{2\gamma(t)} & \text{if } s < t \\ \frac{[\gamma(s)]^\lambda}{\gamma(t)} & \text{otherwise} \end{cases}$

$\forall s, t \geq 1$, where $\lambda \in (0,1)$,

(4) $\xi_1(t,s) = \frac{\gamma(\psi(s))}{\gamma(\varphi(t))} \ \forall t, s \geq 1$, where $\psi, \varphi : [1,\infty) \to [1,\infty)$ are continuous functions such that $\psi(t) = \varphi(t) = 1$ if and only if $t = 1$, $\psi(t) < t \leq \varphi(t) \ \forall t > 1$ and φ is an increasing function,

(5) $\xi_2(t,s) = \frac{\gamma(\eta(s))}{\gamma(t)} \ \forall s, t \geq 1$, where $\eta : [1,\infty) \to [1,\infty)$ is upper semi-continuous with $\eta(t) < t \ \forall t > 1$ and $\eta(t) = 1$ if and only if $t = 1$,

(6) $\xi_3(t,s) = \frac{\gamma(s)}{\gamma(\int_0^t \phi(u)du)} \ \forall s, t \geq 1$, where $\phi : [0,\infty) \to [0,\infty)$ is a function such that for each $t \geq 1$, $\int_0^t \phi(u)du$ exists and $\int_0^t \phi(u)du > t$ and $\int_0^1 \phi(u)du = 1$.

Then $\xi_b, \xi_w, \xi_c, \xi_1, \xi_2$ and ξ_3 are $\mathcal{L}_\gamma$-simulation functions.

Note that if $\gamma(t) = t,\ \forall t \geq 1$, then $\xi_b, \xi_w, \xi_c, \xi_1, \xi_2, \xi_3 \in \mathcal{L}$ (See [13, 17, 20]).

Example 1.3. Let functions $\xi_n, \xi_r, \xi_g : [1,\infty) \times [1,\infty) \to \mathbb{R}$ be defined as follows respectively:

(1) $\xi_n(t,s) = \frac{\gamma(s)}{[\gamma(t)]^\lambda} \ \ \forall t, s \geq 1$, where $\lambda > 1$,

(2) $\xi_r(t,s) = \frac{s\phi(s)}{t} \ \ \forall t, s \geq 1$, where $\phi : [1,\infty) \to [1, \vartheta(1))$ and $\vartheta : (0,\infty) \to (1,\infty)$ is non-decreasing such that

$$\lim_{t \to s} \sup \phi(t) < \vartheta(1),$$

(3) $\xi_g(t,s) = \frac{s\alpha(s)}{t} \ \ \forall t, s \geq 1$, where $\alpha : [1,\infty) \to [1, \vartheta(1))$ and $\vartheta : (0,\infty) \to (1,\infty)$ is non-decreasing such that

$$\lim_{n \to \infty} \alpha(t_n) = \vartheta(1) \Longleftrightarrow \lim_{n \to \infty} t_n = 1.$$

Then $\xi_n,\ \xi_r$ and ξ_g are $\mathcal{L}_\gamma$-simulation functions.

We recall the following definitions which are in [3].

Let X be a nonempty set, and let $d : X \times X \to [0,\infty)$ be a map such that for all $x, y \in X$ and all distinct points $u, v \in X - \{x, y\}$,

(d1) $d(x,y) = 0$ if and only if $x = y$,
(d2) $d(x,y) = d(y,x)$,
(d3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then d is called a *Branciari distance* on X and (X, d) is called a *Branciari distance space*.

Let (X, d) be a Branciari distance space. Then we say that

(i) a sequence $\{x_n\} \subset X$ is *convergent* to x, denoted by $\lim_{n\to\infty} x_n = x$, if $\lim_{n\to\infty} d(x_n, x) = 0$,

(ii) a sequence $\{x_n\} \subset X$ is *Cauchy*, if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$,

(iii) (X, d) is *complete*, if every Cauchy sequence in X is convergent to some point in X.

Lemma 1.4 ([24]). *Let (X, d) be a Branciari distance space, $\{x_n\} \subset X$ be a Cauchy sequence and $x, y \in X$. If there exists a positive integer N such that*

(1) $x_n \neq x_m\ \forall n, m > N$,

(2) $x_n \neq x\ \forall n > N$,

(3) $x_n \neq y\ \forall n > N$,

(4) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$, *then* $x = y$.

Lemma 1.5. *Let $l > 0$, and let $\{t_n\}, \{s_n\} \subset (l, \infty)$ be non-increasing sequences such that*

$$t_n \leq s_n, \forall n = 1, 2, 3, \cdots \text{ and } \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l.$$

If $\vartheta : (0, \infty) \to (1, \infty)$ is non-decreasing, then we have

$$\lim_{n\to\infty} \vartheta(t_n) = \lim_{n\to\infty} \vartheta(s_n) = \lim_{t\to l^+} \vartheta(t) > 0.$$

Proof. Since ϑ is non-decreasing and $\{t_n\}$ is non-increasing,

$$\lim_{t\to l^+} \vartheta(t) = \lim_{n\to\infty} \vartheta(t_{n+1}) \leq \lim_{n\to\infty} \vartheta(t_n) \leq \lim_{n\to\infty} \vartheta(s_n) \leq \lim_{n\to\infty} \vartheta(s_{n-1}) \leq \lim_{t\to l^+} \vartheta(t).$$

Then we have

$$\lim_{n\to\infty} \vartheta(t_n) = \lim_{n\to\infty} \vartheta(s_n) = \lim_{t\to l^+} \vartheta(t) > 1.$$

□

Lemma 1.6 ([13]). *Let $\varpi : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$ be a function such that*

$$\varpi(s, t) \leq \frac{1}{2}s - t,\ \forall s, t \in [0, \infty).$$

If $\frac{1}{2}s < t\ \ \forall s, t \in [0, \infty)$, then we have that

(1) $\varpi(s, t) < 0$,

(2) $\varpi(\min\{s, u\}, t) < 0$.

2. Fixed point theorems

Let (X, d) be a Branciari distance space.

A map $T : X \to X$ is called a generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-*contraction* with respect to $\xi \in \mathcal{L}_\gamma$, if there exist a positive real number L and a function $\vartheta : (0, \infty) \to (1, \infty)$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,

$$\begin{aligned} &\varpi(m(x, y), d(x, y)) < 0 \\ &\Longrightarrow \xi(\vartheta(d(Tx, Ty)), \vartheta(M(x, y) + Lm(x, y))) \geq 1, \end{aligned} \tag{2.1}$$

where $m(x, y) = \min\{d(x, Tx), d(y, Tx)\}$ and $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$.

Now, we prove our main result.

Theorem 2.1. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-contraction with respect to $\xi \in \mathcal{L}_\gamma$. If ϑ is non-decreasing, then T has a unique fixed point and for every initial point $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to the fixed point.*

Proof. Firstly, we show uniqueness of fixed point whenever it exists.

Assume that w and u are fixed points of T.

If $u \neq w$, then $d(w, u) > 0$ and $\frac{1}{2}d(w, Tw) = 0 < d(w, u)$. By Lemma 1.6, $\varpi(m(w, u), d(w, u)) < 0$. We infer that $M(w, u) = d(w, u)$ and $m(w, u) = 0$. Thus it follows from (2.1) that

$$\begin{aligned} 1 \leq & \xi(\vartheta(d(Tw, Tu)), \vartheta(M(w, u) + Lm(w, u))) \\ = & \xi(\vartheta(d(Tw, Tu)), \vartheta(d(w, u))) \\ = & \xi(\vartheta(d(w, u)), \vartheta(d(w, u))) \\ < & \frac{\gamma(\vartheta(d(w, u)))}{\gamma(\vartheta(d(w, u)))} = 1 \end{aligned}$$

which is a contradiction. So $w = u$ and fixed point of T is unique.

Secondly, we prove existence of fixed point.

Let $x_0 \in X$ be a point. Define a sequence $\{x_n\} \subset X$ by $x_n = Tx_{n-1} = T^n x_0 \ \forall n = 1, 2, 3 \cdots$.

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of T and the proof is finished.

Assume that

$$x_{n-1} \neq x_n \ \forall n = 1, 2, 3 \cdots . \tag{2.2}$$

We infer that

$$\frac{1}{2}d(x_{n-1}, Tx_{n-1}) = \frac{1}{2}d(x_{n-1}, x_n) < d(x_{n-1}, x_n). \tag{2.3}$$

By applying Lemma 1.6 with (2.3), we obtain that

$$\varpi(m(x_{n-1}, x_n), d(x_{n-1}, x_n)) < 0.$$

We have that

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \tag{2.4}$$

and

$$m(x_{n-1}, x_n) = \min\{d(x_{n-1}, x_n), d(x_n, x_n)\} = 0.$$

It follows from (2.1), (2.2), (2.3) and (2.4) that $\forall n = 1, 2, 3, \cdots$,

$$\begin{aligned} 1 \leq & \xi(\vartheta(d(Tx_{n-1}, Tx_n)), \vartheta(M(x_{n-1}, x_n) + Lm(x_{n-1}, x_n))) \\ = & \xi(\vartheta(d(x_n, x_{n+1})), \vartheta(M(x_{n-1}, x_n))) \\ < & \frac{\gamma(\vartheta(M(x_{n-1}, x_n)))}{\gamma(\vartheta(d(x_n, x_{n+1})))} \end{aligned} \tag{2.5}$$

which implies

$$\gamma(\vartheta(d(x_n, x_{n+1}))) < \gamma(\vartheta(M(x_{n-1}, x_n))) \ \forall n = 1, 2, 3, \cdots .$$

Consequently, we obtain that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)\ \forall n = 1, 2, 3, \cdots .$$

Then $\{d(x_{n-1}, x_n)\}$ is a decreasing sequence. Thus there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = l.$$

We now show that $l = 0$.

Assume that $l > 0$ and let $s_n = \vartheta(d(x_{n-1}, x_n))$ and $t_n = \vartheta(d(x_n, x_{n+1}))\ \forall n = 1, 2, 3, \cdots$. Then $t_n < s_n\ \forall n = 1, 2, 3, \cdots$. By Lemma 1.5, we have that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = \lim_{n\to l^+} \vartheta(t) > 1.$$

It follows from ($\xi 3$) that

$$1 \leq \lim_{n\to\infty} \sup \xi(t_n, s_n) < 1$$

which yields a contradiction. Thus we get

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0. \tag{2.6}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence.

On the contrary, assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which

$$m(k) > n(k) > k,\ d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1}, x_{n(k)}) < \epsilon. \tag{2.7}$$

From (2.7), we have

$$\begin{aligned}\epsilon &\leq d(x_{m(k)}, x_{n(k)}) \\ &\leq d(x_{n(k)}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \epsilon + d(x_{m(k)-2}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}).\end{aligned} \tag{2.8}$$

Letting $k \to \infty$ in (2.8), we obtain

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.9}$$

On the other hand, we obtain

$$d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{m(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{n(k)+1}, x_{n(k)}) + d(x_{n(k)}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1}).$$

Then we get

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \tag{2.10}$$

It follows from (2.6) that there exists $N \in \mathbb{N}$ such that

$$d(x_{n(k)}, x_{n(k)+1}) < \frac{1}{4}\epsilon,\ \forall k > N. \tag{2.11}$$

We infer that $\forall k > N$

$$\frac{1}{2} d(x_{n(k)}, Tx_{n(k)}) = \frac{1}{2} d(x_{n(k)}, x_{n(k)+1}) < \frac{1}{8}\epsilon < d(x_{n(k)}, x_{m(k)})$$

and thus

$$\varpi(m(x_{n(k)}, x_{m(k)}), d(x_{m(k)}, x_{n(k)})) < 0. \tag{2.12}$$

We deduce that

$$M(x_{n(k)}, x_{m(k)}) = \max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1})\}$$

and

$$m(x_{n(k)}, x_{m(k)}) = \min\{d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{n(k)+1})\}. \tag{2.13}$$

From (2.7), we infer that

$$\begin{aligned}
\epsilon &\le d(x_{m(k)+1}, x_{n(k)})\\
&\le d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)+1})\\
&< \frac{1}{4}\epsilon + d(x_{n(k)+1}, x_{m(k)}) + \frac{1}{4}\epsilon\\
&= \frac{1}{2}\epsilon + d(x_{n(k)+1}, x_{m(k)}),\ \forall k > N
\end{aligned}$$

which implies

$$\frac{1}{2}\epsilon < d(x_{n(k)+1}, x_{m(k)}),\ \forall k > N.$$

So we get

$$d(x_{n(k)}, x_{n(k)+1}) < \frac{1}{4}\epsilon < \frac{1}{2}\epsilon < d(x_{n(k)+1}, x_{m(k)})\ \forall k > N.$$

From (2.13) we have

$$m(x_{n(k)}, x_{m(k)}) = d(x_{n(k)}, x_{n(k)+1})\ \forall k > N.$$

It follows from (2.1), (2.12) and (2.13) that

$$\begin{aligned}
1 &\le \xi(\vartheta(d(Tx_{n(k)}, Tx_{m(k)})), \vartheta(M(x_{n(k)}, x_{m(k)}) + Lm(x_{n(k)}, x_{m(k)})))\\
&= \xi(\vartheta(d(x_{n(k)+1}, x_{m(k)+1})), \vartheta(M(x_{n(k)}, x_{m(k)}) + Ld(x_{n(k)}, x_{n(k)+1})))\\
&< \frac{\gamma(\vartheta(M(x_{n(k)}, x_{m(k)}) + Ld(x_{n(k)}, x_{n(k)+1})))}{\gamma(\vartheta(d(x_{n(k)+1}, x_{m(k)+1})))}\ \forall k > N
\end{aligned}$$

which implies

$$\gamma(\vartheta(d(x_{n(k)+1}, x_{m(k)+1}))) < \gamma(\vartheta(M(x_{n(k)}, x_{m(k)}) + Ld(x_{n(k)}, x_{(k)+1})))\ \forall k > N.$$

Hence we infer that

$$\vartheta(d(x_{n(k)+1}, x_{m(k)+1})) < \vartheta(M(x_{n(k)}, x_{m(k)}) + Ld(x_{n(k)}, x_{(k)+1}))\ \forall k > N.$$

Let for each $k > N$,

$$t_k = \vartheta(d(x_{n(k)+1}, x_{m(k)+1})) \text{ and } s_k = \vartheta(M(x_{n(k)}, x_{m(k)}) + Ld(x_{n(k)}, x_{n(k)+1})).$$

Then $t_k < s_k\ \forall k > N$. From (2.6), (2.9) and (2.10), we obtain

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \lim_{k\to\infty}[M(x_{n(k)}, x_{m(k)}) + Ld(x_{n(k)}, x_{n(k)+1})] = \epsilon.$$

By Lemma 1.5, we obtain that

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} s_k = \lim_{t\to\epsilon^+} \vartheta(t) > 1.$$

From ($\xi3$), we have

$$1 \leq \lim_{k\to\infty} \sup \xi(t_k, s_k) < 1$$

which leads a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0. \tag{2.14}$$

From (2.6) and (2.14), we may assume that

$$d(x_n, Tx_n) = d(x_n, x_{n+1}) \leq d(x_n, z) \ \forall n = 1, 2, 3, \cdots .$$

which implies

$$\frac{1}{2} d(x_n, Tx_n) < d(x_n, z) \ \forall n = 1, 2, 3, \cdots .$$

Applying Lemma 1.6, we have that

$$\varpi(m(x_n, z), d(x_n, z)) < 0. \tag{2.15}$$

We deduce that

$$\begin{aligned} M(x_n, z) &= \max\{d(x_n, z), d(x_n, x_{n+1}), d(z, Tz)\} \\ &= \max\{d(x_n, z), d(z, Tz)\} \end{aligned} \tag{2.16}$$

and

$$m(x_n, z) = \min\{d(x_n, x_{n+1}), d(z, x_{n+1})\} = d(x_n, x_{n+1}). \tag{2.17}$$

It follows from (2.1) that

$$\begin{aligned} 1 \leq& \xi(\vartheta(d(Tx_n, Tz)), \vartheta(M(x_n, z) + Lm(x_n, z)))) \\ =& \xi(\vartheta(d(Tx_n, Tz)), \vartheta(M(x_n, z) + Ld(x_n, x_{n+1}))) \\ <& \frac{\gamma(\vartheta(M(x_n, z) + Ld(x_n, x_{n+1})))}{\gamma(\vartheta(d(Tx_n, Tz)))} \ \forall n = 1, 2, 3, \cdots \end{aligned}$$

which implies

$$\gamma(\vartheta(d(Tx_n, Tz))) < \gamma(\vartheta(M(x_n, z) + Ld(x_n, x_{n+1}))) \ \forall n = 1, 2, 3, \cdots .$$

Thus we have

$$\vartheta(d(Tx_n, Tz)) < \vartheta(M(x_n, z) + Ld(x_n, x_{n+1})) \ \forall n = 1, 2, 3, \cdots . \tag{2.18}$$

Assume that $M(x_n, z) = d(z, Tz)$.

If $d(z, Tz) = 0$, then T has a fixed point, and the proof is finished.

Let $d(z, Tz) > 0$. Then

$$\vartheta(d(Tx_n, Tz)) < \vartheta(d(z, Tz) + Ld(x_n, x_{n+1})) \ \forall n = 1, 2, 3, \cdots$$

which implies

$$d(x_{n+1}, Tz) < d(z, Tz) + Ld(x_n, x_{n+1}) \ \forall n = 1, 2, 3, \cdots .$$

Thus we obtain that

$$\begin{aligned} d(z, Tz) \leq& d(z, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tz) \\ \leq& d(z, x_n) + d(x_n, x_{n+1}) + d(z, Tz) + Ld(x_n, x_{n+1}). \end{aligned}$$

Letting $n \to \infty$ in above inequality, we obtain

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = d(z, Tz).$$

We infer that

$$\lim_{n\to\infty} \{d(z, Tz) + Ld(x_n, x_{n+1})\} = d(z, Tz).$$

Let $t_n = \vartheta(d(x_{n+1}, Tz))$ and $s_n = \vartheta(d(z, Tz) + Ld(x_n, x_{n+1}))$ $\forall n = 1, 2, 3, \cdots$. Then $t_n < s_n$ $\forall n = 1, 2, 3, \cdots$. By applying Lemma 1.5, we deduce that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \lim_{n\to d(z,Tz)^+} \vartheta(t) > 1.$$

It follows from ($\xi 3$) that

$$1 \le \limsup \xi(t_n, s_n) < 1$$

which is a contradiction. Thus the case does not occur.

If $M(x_n, z) = d(x_n, z)$, then from (2.18) we obtain that

$$d(Tx_n, Tz) < d(x_n, z) + Ld(x_n, x_{n+1}),\ \forall n = 1, 2, 3, \cdots.$$

Thus

$$\lim_{n\to\infty} d(x_{n+1}, Tz) = 0. \tag{2.19}$$

Applying Lemma 1.4 with (2.14) and (2.19), we have $z = Tz$. □

We give an example to illustrate Theorem 2.1.

Example 2.2. Let $X = \{1 - \frac{1}{n} : n = 1, 2, 3, \cdots\} \cup \{1, 2\}$, and let $d : X \times X \to [0, \infty)$ be a map defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x \in \{1,2\} \text{ and } y = 1 - \frac{1}{n}, n = 1,2,3,\cdots \\ \frac{1}{n} & \text{if } x = 1 - \frac{1}{n}, n = 1,2,3,\cdots \text{ and } y \in \{1,2\} \\ 1 & \text{otehwise.} \end{cases}$$

Then (X, d) is a complete Branciari distance space and it is not a metric space. In fact, we have that

$$d(\frac{3}{4}, 2) + d(2, \frac{2}{3}) < d(\frac{3}{4}, \frac{2}{3}).$$

Define a map $T : X \to X$ by

$$Tx = \begin{cases} 1 & \text{if } x = 1, 2 \\ 1 - \frac{1}{n+1} & \text{if } x = 1 - \frac{1}{n}. \end{cases}$$

Let $\vartheta(t) = e^t \forall t > 0$ and $\gamma(t) = 1 + \ln(t)$ $\forall t \ge 1$.

We now show that T is a generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-contraction with respect to ξ_b, where $\xi_b(t, s) = \frac{\gamma(s)^k}{\gamma(t)}$ $\forall t, s \ge 1$, $k = \frac{1}{2}$ and $L = 2$.

We have that

$d(Tx, Ty) > 0 \Leftrightarrow (x = 1, y = 2), (x = 1, y = 1 - \frac{1}{n}), (x = 2, y = 1 - \frac{1}{n})$, or $(x = 1 - \frac{1}{n}, y = 1 - \frac{1}{m}, n \ne m)$.

We consider the following four cases.

Case 1: $x = 1$ and $y = 2$. We infer that

$$m(1,2) = 0, M(1,2) = 1 \text{ and } d(1,2) = 1.$$

Then

$$\varpi(m(1,2), d(1,2)) = \varpi(0,1) < 0.$$

Thus we obtain that

$$\begin{aligned}&\xi_b(\vartheta(d(T1,T2)), \vartheta(M(1,2) + Lm(1,2)))\\ =&\xi_b(\vartheta(0), \vartheta(1)) = \xi_b(e^0, e^1) = \frac{\gamma(e)^k}{\gamma(e^0)} = \sqrt{2} > 1.\end{aligned}$$

Case 2: $x = 1$ and $y = 1 - \frac{1}{n}, n = 1, 2, 3, \cdots$. We have that

$$m(1, 1-\frac{1}{n}) = 0, M(1, 1-\frac{1}{n}) = 1 \text{ and } d(1, 1-\frac{1}{n}) = \frac{1}{n}$$

and

$$\varpi(m(1, 1-\frac{1}{n}), d(1, 1-\frac{1}{n})) = \varpi(0, \frac{1}{n}) < 0.$$

Then we obtain that

$$\begin{aligned}&\xi_b(\vartheta(d(T1, T1-\frac{1}{n})), \vartheta(M(1, 1-\frac{1}{n}) + Lm(1, 1-\frac{1}{n})))\\ =&\xi_b(\vartheta(\frac{1}{n+1}), \vartheta(e^1)) = \xi_b(e^{\frac{1}{n+1}}, e^1)\\ =&\frac{[\gamma(e)]^k}{\gamma(e^{\frac{1}{n+1}})} = \frac{\sqrt{2}}{1+\frac{1}{n+1}} > 1, n = 1, 2, 3, \cdots.\end{aligned}$$

Case 3: $x = 2$ and $y = 1 - \frac{1}{n}, n = 1, 2, 3, \cdots$. We obtain that

$$m(2, 1-\frac{1}{n}) = \frac{1}{n}, M(2, 1-\frac{1}{n}) = 1 \text{ and } d(2, 1-\frac{1}{n}) = 1$$

and

$$\varpi(m(2, 1-\frac{1}{n}), d(2, 1-\frac{1}{n})) = \varpi(\frac{1}{n}, 1) < 0.$$

Then we have that

$$\begin{aligned}&\xi_b(\vartheta(d(T2, T1-\frac{1}{n})), \vartheta(M(2, 1-\frac{1}{n}) + Lm(2, 1-\frac{1}{n})))\\ =&\xi_b(\vartheta(\frac{1}{n+1}), \vartheta(1+\frac{2}{n})) = \xi_b(e^{\frac{1}{n+1}}, e^{1+\frac{2}{n}})\\ =&\frac{[\gamma(e^{1+2/n})]^k}{\gamma(e^{\frac{1}{n+1}})} = \frac{[2+2/n]^{1/2}}{1+\frac{1}{n+1}} > 1, n = 1, 2, 3, \cdots.\end{aligned}$$

Case 4: $x = 1 - \frac{1}{n}$ and $y = 1 - \frac{1}{m}, n \neq m$. We infer that

$$m(1-\frac{1}{n}, 1-\frac{1}{m}) = 1, M(1-\frac{1}{n}, 1-\frac{1}{m}) = 1 \text{ and } d(1-\frac{1}{n}, 1-\frac{1}{m}) = 1$$

and

$$\varpi(m(1-\frac{1}{n}, 1-\frac{1}{m}), d(1-\frac{1}{n}, 1-\frac{1}{m})) = \varpi(1,1) < 0.$$

Then we have that

$$\begin{aligned}&\xi_b(\vartheta(d(T1-\frac{1}{n},T1-\frac{1}{m})),\vartheta(M(1-\frac{1}{n},1-\frac{1}{m})+Lm(1-\frac{1}{n},1-\frac{1}{m})))\\ =&\xi_b(\vartheta(1),\vartheta(3))=\xi_b(e,e^3)\\ =&\frac{[\gamma(e^3)]^k}{\gamma(e)}=\frac{\sqrt{1+3\ln e}}{1+\ln e}=\frac{\sqrt{4}}{2}=1, n=1,2,3,\cdots.\end{aligned}$$

Thus T is a generalized Suzuki-Berinde type $\mathcal{L}_\gamma$-contraction with respect to ξ_b. So the assumptions of Theorem 2.1 are satisfied and T has a fixed point $z=1$.

Note that the Banach contraction condition is not satisfied. In fact, if for $x=\frac{1}{2}, y=\frac{3}{4}$,

$$d(T\frac{1}{2},T\frac{3}{4})\leq kd(\frac{1}{2},\frac{3}{4})\ k\in(0,1)$$

then

$$d(\frac{2}{3},\frac{4}{5})\leq kd(\frac{1}{2},\frac{3}{4}).$$

Thus $k\geq 1$.

Also, note that the ϑ-contraction condition [16] does not hold.

Let $\vartheta(t)=e^t, \forall t>0$. Then ($\vartheta 1$),($\vartheta 2$) and ($\vartheta 4$) are satisfied.

Let $x=\frac{1}{2}, y=\frac{3}{4}$. If

$$\vartheta(d(T\frac{1}{2},T\frac{3}{4})\leq[\vartheta(d(\frac{1}{2},\frac{3}{4}))]^k\ k\in(0,1),$$

then

$$\vartheta(d(\frac{2}{3},\frac{4}{5}))\leq\vartheta(d(\frac{1}{2},\frac{3}{4}))^k.$$

Thus $e\leq e^k$. So $k\geq 1$. Hence T is not a ϑ-contraction map.

Remark 2.3. Theorem 2.1 is a generalization of Theorem 1 of [13]. By taking $L=0$ and $M(x,y)=d(x,y)$ in Theorem 2.1, we have Theorem 1 of [13]. Also, Theorem 2.1 is a generalization of Theorem 2 of [13] without continuity of ϑ. In fact, let $M(x,y)=d(x,y)$ in Theorem 2.1. Then Theorem 2.1 reduces to Theorem 2 of [13].

Corollary 2.4. *Let (X,d) be a complete Branciari distance space, and let $T:X\to X$ be a mapping such that for all $x,y\in X$ with $d(Tx,Ty)>0$*

$$\varpi(m(x,y),d(x,y))<0 \text{ implies } \xi(\vartheta(d(Tx,Ty)),\vartheta(M(x,y)+Ln(x,y)))\geq 1,$$

where $\xi\in\mathcal{L}_\gamma$ is non-decreasing with respect to the second coordinate, $L\geq 0$ and $n(x,y)=\min\{d(x,Tx),d(x,Ty),d(y,Tx)\}$. If ϑ is non-decreasing, then T has a unique fixed point.

Corollary 2.5. *Let (X,d) be a complete Branciari distance space, and let $T:X\to X$ be a mapping such that for all $x,y\in X$ with $d(Tx,Ty)>0$*

$$\varpi(m(x,y),d(x,y))<0 \text{ implies } \xi(\vartheta(d(Tx,Ty)),\vartheta(M(x,y)+Lp(x,y)))\geq 1,$$

where $\xi\in\mathcal{L}_\gamma$ is non-decreasing with respect to the second coordinate, $L\geq 0$ and $p(x,y)=\min\{d(x,Ty),d(y,Tx),\frac{1}{2}[d(x,Tx)+d(y,Ty)]\}$. If ϑ is non-decreasing, then T has a unique fixed point.

Remark 2.6. Corollary 2.5 is a generalization of Theorem 15 of [14]. By Taking $M(x, y) = d(x, y), \gamma(t) = t,\ \forall t \geq 1$ and applying Lemma 1.6 in Corollary 2.5, we have Theorem 15 of [14] without condition (ϑ_2) and continuity of T.

Corollary 2.7. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$\varpi(m(x, y), d(x, y)) < 0 \textit{ implies } \xi(\vartheta(d(Tx, Ty)), \vartheta(M(x, y) + Lq(x, y))) \geq 1,$$

where $\xi \in \mathcal{L}_\gamma$ is non-decreasing with respect to the second coordinate, $L \geq 0$ and $q(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$. If ϑ is non-decreasing, then T has a unique fixed point.

By taking $L = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.8. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$\varpi(m(x, y), d(x, y)) < 0 \textit{ implies } \xi(\vartheta(d(Tx, Ty)), \vartheta(M(x, y))) \geq 1,$$

where $\xi \in \mathcal{L}_\gamma$. If ϑ is non-decreasing, then T has a unique fixed point.

From Theorem 2.1, we have the following result.

Corollary 2.9. *Let (X, d) be a complete Branciari distance space, and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$*

$$\xi(\vartheta(d(Tx, Ty)), \vartheta(d(x, y)) + Lm(x, y)) \geq 1,$$

where $\xi \in \mathcal{L}_\gamma$ is non-decreasing with respect to the second coordinate and $L \geq 0$.
If ϑ is non-decreasing, then T has a unique fixed point.

Remark 2.10. Corollary 2.9 is a generalization of Theorem 2.1 of [17]. In fact, if $L = 0$ and $\gamma(t) = t,\ \forall t \geq 1$, then Corollary 2.9 reduces Theorem 2.1 of [17].

3. Consequence

Applying simulation functions given in Examples 1.2 and 1.3, we have some fixed point results.

By taking $\xi = \xi_b$ in Theorem 2.1, we obtain the following result.

Corollary 3.1. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x, y), d(x, y)) < 0 \textit{ implies } \gamma(\vartheta(d(Tx, Ty))) \leq [\gamma(\vartheta(M(x, y) + Lm(x, y)))]^k,$$

where $k \in (0, 1)$ and $L \geq 0$. If ϑ is non-decreasing, then T has a unique fixed point.

Corollary 3.2. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x, y), d(x, y)) < 0 \textit{ implies } \gamma(\vartheta(d(Tx, Ty))) \leq [\gamma(\vartheta(M(x, y) + Ln(x, y)))]^k,$$

where $k \in (0, 1)$ and $L \geq 0$. If ϑ is non-decreasing, then T has a unique fixed point.

Corollary 3.3. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies } \gamma(\vartheta(d(Tx,Ty))) \leq [\gamma(\vartheta(M(x,y)) + Lq(x,y)))]^k,$$

where $k \in (0,1)$ and $L \geq 0$. If ϑ is non-decreasing, then T has a unique fixed point.

Corollary 3.4. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies } \gamma(\vartheta(d(Tx,Ty))) \leq [\gamma(\vartheta(M(x,y)))]^k,$$

where $k \in (0,1)$. If ϑ is non-decreasing, then T has a unique fixed point.

Corollary 3.5. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\vartheta(d(Tx,Ty)) \leq [\vartheta(d(x,y))]^k,$$

where $k \in (0,1)$. If ϑ is non-decreasing, then T has a unique fixed point.

Remark 3.6. (1) Corollary 3.2 is an extention and generalization of Theorem 3.2 of [16] to Branciari distance space without the condition ($\vartheta 2$).

(2) Corollary 3.5 is a generalization of Theorem 2.1 of [15] without the conditions ($\vartheta 2$) and ($\vartheta 3$) and is an extension of Theorem 2.2 of [16] to Branciari distance space without the condition ($\vartheta 2$).

(3) Corollary 3.5 is an answer to open question of [25].

By taking $\xi = \xi_w$ in Theorem 2.1, we obtain the following result.

Corollary 3.7. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies } \gamma(\vartheta(d(Tx,Ty))) \leq \frac{\gamma(\vartheta(M(x,y) + Lm(x,y)))}{\phi(\gamma(\vartheta(M(x,y) + Lm(x,y))))},$$

where $L \geq 0$ and $\phi : [1,\infty) \to [1,\infty)$ is non-decreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$. If ϑ is non-decreasing, then T has a unique fixed point.

Corollary 3.8. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies } \vartheta(d(Tx,Ty)) \leq \frac{\vartheta(d(x,y))}{\phi(\vartheta(d(x,y)))},$$

where $L \geq 0$ and $\phi : [1,\infty) \to [1,\infty)$ is non-decreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$. If ϑ is non-decreasing, then T has a unique fixed point.

Remark 3.9. Corollary 3.8 is a generalization of Corollary 8 of [17].

By taking $\xi = \xi_n$ in Theorem 2.1, we obtain the following result.

Corollary 3.10. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies}$$
$$[\gamma(\vartheta(d(Tx,Ty)))]^\lambda \leq \gamma(\vartheta(M(x,y) + Lm(x,y))),$$

where $\lambda > 1$. If ϑ is non-decreasing, then T has a unique fixed point.

By taking $\xi = \xi_r$ in Theorem 2.1 with $\gamma(t) = t, \forall t \geq 1$, we obtain the following result.

Corollary 3.11. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies}$$

$$\vartheta(d(Tx,Ty)) \leq \vartheta(d(x,y))\phi(\vartheta(d(x,y))),$$

where ϑ is non-decreasing and $\phi : [1, \infty) \to [1, \vartheta(1))$ is a function such that

$$\lim_{t \to s^+} \sup \phi(t) < \vartheta(1) \forall t > 1.$$

Then T has a unique fixed point.

Corollary 3.12. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \text{ implies}$$

$$d(Tx,Ty) \leq d(x,y)\varphi(d(x,y)), \tag{3.1}$$

where $\varphi : [0, \infty) \to [0, 1)$ is a function such that

$$\lim_{t \to s^+} \sup \varphi(t) < 1 \ \forall s > 0.$$

Then T has a unique fixed point.

Proof. Let $\vartheta(t) = e^t \ \forall t > 0$ and let $\varphi(t) = \ln(\phi(\vartheta(t))) \ \forall t \geq 0$ where $\phi : [1, \infty) \to [1, \theta(1))$ is a function. Then we have that

$$\begin{aligned}\lim_{t \to s^+} \sup \varphi(t) &= \lim_{t \to s^+} \sup \ln(\phi(\vartheta(t))) \\ &= \ln(\lim_{t \to s^+} \sup \phi(\vartheta(t))) \\ &< \ln(\vartheta(1))\end{aligned}$$

which implies

$$\lim_{t \to s^+} \sup \phi(\vartheta(t)) < \vartheta(1), \forall t > 0.$$

Thus

$$\lim_{t \to s^+} \sup \phi(t) < \vartheta(1), \forall t > 1.$$

It follows from (3.1) that for all $x, y \in X$ with $d(Tx, Ty) > 0$ and $\varpi(m(x,y), d(x,y)) < 0$,

$$\begin{aligned}\vartheta(d(Tx,Ty)) &\leq \vartheta(d(x,y)\varphi(d(x,y))) \\ &= \vartheta(\ln(\phi(\vartheta(d(x,y)))d(x,y)) \\ &= e^{\ln(\phi(\vartheta(d(x,y)))d(x,y)} \\ &\leq \phi(\vartheta(d(x,y)))\vartheta(d(x,y)).\end{aligned}$$

So by Corollary 3.11, T has a unique fixed point. □

Taking $\xi = \xi_g$ in Theorem 2.1 with $\gamma(t) = t, \forall t \geq 1$, we have the following result.

Corollary 3.13. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x, y), d(x, y)) < 0 \text{ implies}$$

$$\vartheta(d(Tx, Ty)) \leq \vartheta(d(x, y))\alpha(\vartheta(d(x, y))),$$

where ϑ is non-decreasing and $\alpha : [1, \infty) \to [1, \vartheta(1))$ is a function such that

$$\lim_{n\to\infty} \alpha(t_n) = \vartheta(1) \Longleftrightarrow \lim_{n\to\infty} t_n = 1.$$

Then T has a unique fixed point.

Corollary 3.14. *Let (X, d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx, Ty) > 0$,*

$$\varpi(m(x, y), d(x, y)) < 0 \text{ implies}$$

$$d(Tx, Ty) \leq d(x, y)\beta(d(x, y)), \tag{3.2}$$

where $\beta : [0, \infty) \to [0, 1)$ is a function such that

$$\lim_{n\to\infty} \beta(t_n) = 1 \Longleftrightarrow \lim_{n\to\infty} t_n = 0, \forall t_n > 0. \tag{3.3}$$

Then T has a unique fixed point.

Proof. Let $\vartheta(t) = e^t\ \forall t > 0$ and let $\beta(t) = \ln(\alpha(\vartheta(t))), \forall t \geq 0$, where $\alpha : [1, \infty) \to [1, \vartheta(1))$ is a function. Let $\{s_n\} \subset [1, \infty)$ be a sequence and let $\{t_n = \ln s_n\} \subset [0, \infty)$ be a sequence. Then from (3.3), we infer that

$$\begin{aligned} &\lim_{n\to\infty} \beta(t_n) = 1 \\ \Longleftrightarrow &\ln(\lim_{n\to\infty} \alpha(\vartheta(t_n)) = \lim_{n\to\infty} \ln(\alpha(\vartheta(t_n))) = \ln(\vartheta(1)) \\ \Longleftrightarrow &\lim_{n\to\infty} \alpha(\vartheta(t_n)) = \vartheta(1) \\ \Longleftrightarrow &\lim_{n\to\infty} \alpha(s_n) = \vartheta(1). \end{aligned} \tag{3.4}$$

Also, we have that

$$\begin{aligned} &\lim_{n\to\infty} t_n = 0 \\ \Longleftrightarrow &\lim_{n\to\infty} s_n = \lim_{n\to\infty} \vartheta(t_n) = \vartheta(0) = 1. \end{aligned} \tag{3.5}$$

It follows from (3.5) that

$$\lim_{n\to\infty} \alpha(s_n) = \vartheta(1) \Longleftrightarrow \lim_{n\to\infty} s_n = 1\ \forall t_n > 1.$$

Thus from (3.2), we obtain that

$$\begin{aligned} &\vartheta(d(Tx, Ty) = e^{d(Tx,Ty)} \\ \leq &e^{d(x,y)\beta(d(x,y))} \\ = &e^{\ln(\alpha(\vartheta((d(x,y)))d(x,y)} \\ = &\alpha(\vartheta(d(x, y)))d(x, y) \\ \leq &\alpha(\vartheta((d(x, y)))\vartheta(d(x, y)). \end{aligned}$$

So by Corollary 3.13, T has a unique fixed point. □

Remark 3.15. Corollary 3.14 is a generalization and extention of Theorem 2.1 of [26] to Branciari distance.

By taking $\vartheta(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\alpha})\ \forall t > 0$, where $\alpha \in (0,1)$ in Corollary 3.8, we have the following result.

Corollary 3.16. *Let (X,d) be a complete Branciari distance space and let $T : X \to X$ be a mapping such that for all $x, y \in X$ with $d(Tx,Ty) > 0$,*

$$\varpi(m(x,y), d(x,y)) < 0 \textit{ implies}$$

$$2 - \frac{2}{\pi}\arctan(\frac{1}{d(Tx,Ty)^\alpha}) \leq \frac{2 - \frac{2}{\pi}\arctan(\frac{1}{[M(x,y)+Lm(x,y)]^\alpha})}{\phi(2 - \frac{2}{\pi}\arctan(\frac{1}{[M(x,y)+Lm(x,y)]^\alpha}))},$$

where $\alpha \in (0,1)$ and $\phi : [1,\infty) \to [1,\infty)$ is nondecreasing and lower semicontinuous such that $\phi^{-1}(\{1\}) = 1$. Then T has a unique fixed point.

4. Conclusion

One can unify and merge some existing fixed point theorems by using $\mathcal{L}_\gamma$-simulation functions in Branciari distance spaces. One can obtain some concequence of the main result by applying $\mathcal{L}_\gamma$-simulation functions given in Example 1.2 and Example 1.3. Further, one can derive all the results of the paper in the setting of metric spaces.

Suggestion

We suggest that the main theorem can be extended and generalized to fuzzy mappings defined on abstract distance spaces by using $\mathcal{L}$-simulation functions and $\mathcal{L}_\gamma$-simulation functions. Also, we suggest that the fuzzy $\mathcal{L}$-simulation function and fuzzy $\mathcal{L}_\gamma$-simulation function can be extended in a similar way to the one in which the $\mathcal{Z}$-simulation function is extended to the $\mathcal{FZ}$-simulation function. The main theorem can be extended and generalized to fuzzy metric spaces using certain extended simulation functions, and the existing fixed point theorem can be interpreted.

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