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Abstract. In this paper, we define fuzzy rare set, fuzzy dense set and fuzzy rarely continuous functions in double fuzzy topological spaces. We will explore several interesting properties and characterizations of these newly defined notions.

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Keywords: Double fuzzy topological spaces, Rare set, Dense set, Rarely continuous functions.

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1. Introduction

The fuzzy concept has overrun almost all branches of mathematics since the definition of the concept by Zadeh [1]. Fuzzy sets have applications in many fields such as information theory [2] and control theory [3]. The theory of fuzzy topological spaces was defined and developed by Chang [2] and since then various notions in general topology have been generalized to Chang's fuzzy topological spaces. Šostak [4] and Kubiak [5] introduced the fuzzy topology as an extension of Chang's fuzzy topology. It has been developed in many directions. Šostak [6] also published a survey article of the developed areas of fuzzy topological spaces. The topologists used to call Chang's fuzzy topology by "topology"and Kubiak-Šostak's fuzzy topology by "L-fuzzy topology", where L is any appropriate lattice. In [3], Atanassov introduced the idea of intuitionistic fuzzy sets. Then Çoker [7, 8], introduced the concept of intuitionistic fuzzy topological spaces. On the other hand, as a generalization of fuzzy topological spaces Samanta and Mondal [9], introduced the concept of intuitionistic gradation of openness. Also in 2020, Kim et al. [10], introduced and investigated the octahedron sets in order to reduce the loss of information in solving the problem of uncertainty. The research has been also done in other useful research avenues (See for example, [11, 12, 13, 14, 15, 16, 17, 18, 19]). In 2005, the term intuitionistic is ended by Garcia and Rodabaugh [20]. They proved that the term intuitionistic is unsuitable in mathematics and applications and they replaced it by double. Many other topologists (See [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]) studied various notions in double fuzzy topological space. In July 2020, two theses have been written on double fuzzy topological spaces (See [33, 34]). The aim of this paper is to introduce and study the concept of double fuzzy rarely continuous function as a generalization of double fuzzy weakly continuous functions in the context of double fuzzy topological spaces. These types of functions have, among others applications in the theory of double fuzzy multifunctions which are useful in the theory of economical analysis. We present some interesting, important and basic properties and characterizations double fuzzy rarely continuous functions. In more details, we focus on the following:

1. Introduction of some new notions such as double fuzzy rare sets, double fuzzy dense set and study some of their basic properties.
2. Introduction of the notion of doubly fuzzy rarely continuous function.

2. Preliminaries

Throughout this paper, X is a non-empty set, I the unit interval $[0, 1]$, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. The family of all fuzzy sets on X is denoted by I^X. By $\overline{0}$ and $\overline{1}$, we denote the smallest and the greatest fuzzy sets on X. For a fuzzy set $\lambda \in I^X$, $\overline{1} - \lambda$ denotes its complement. Given a function $f : I^X \longrightarrow I^Y$ and its inverse $f^{-1} : I^Y \longrightarrow I^X$ is defined by $f(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ and $f^{-1}(\mu)(x) = \mu(f(x))$, for each $\lambda \in I^X, \mu \in I^Y$ and $x \in X$ respectively. All other notations are standard notations of fuzzy set theory.

Definition 2.1 ([8, 9])**.** A *double fuzzy topology* on X is a pair of maps $\tau, \tau^\star : I^X \to I$, which satisfies the following properties: for any $\lambda,\ \lambda_1,\ \lambda_2 \in I^X$ and each $(\lambda_i)_{i\in\Gamma} \subset I^X$,

(i) $\tau(\lambda) \leq \overline{1} - \tau^\star(\lambda)$,

(ii) $\tau(\lambda_1 \wedge \lambda_2) \geq \tau(\lambda_1) \wedge \tau(\lambda_2)$ and $\tau^\star(\lambda_1 \wedge \lambda_2) \leq \tau^\star(\lambda_1) \vee \tau^\star(\lambda_2)$,

(iii) $\tau(\bigvee\limits_{i\in\Gamma} \lambda_i) \geq \bigwedge\limits_{i\in\Gamma} \tau(\lambda_i)$ and $\tau^\star(\bigvee\limits_{i\in\Gamma} \lambda_i) \leq \bigvee_{i\in\Gamma} \tau^\star(\lambda_i)$. The triplet $(X, \tau, \tau^\star)$ is called a *double fuzzy topological space*.

Definition 2.2 ([8, 9])**.** Let $\lambda \in I^X$. Then λ is said to be:

(i) (r, s)-*fuzzy open* in X, if $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq s$,

(ii) (r, s)-*fuzzy closed* in X, if $\overline{1} - \lambda$ is an (r, s)-fuzzy open set in X.

Definition 2.3 ([35, 36])**.** Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. Then the *double fuzzy closure operator* and the *double fuzzy interior operator* of $\lambda \in I^X$, denoted by $C_{\tau,\tau^\star}(\lambda, r, s)$ and $I_{\tau,\tau^\star}(\lambda, r, s)$, are fuzzy sets in X respectively defined by:

$$C_{\tau,\tau^\star}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \tau(\overline{1} - \mu) \geq r, \tau^\star(\overline{1} - \mu) \leq s\},$$

$$I_{\tau,\tau^\star}(\lambda, r, s) = \bigvee\{\mu \in I^X \mid \mu \leq \lambda, \tau(\mu) \geq r, \tau^\star(\mu) \leq s\},$$

where $r \in I_0$ and $s \in I_1$ such that $r + s \leq \overline{1}$.

3. Double fuzzy rare sets

Definition 3.1. A fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is said to be:
(i) an (r, s)-*fuzzy rare set* in X, if $I_{\tau,\tau^\star}(\lambda, r, s) = \overline{0}$,
(ii) an (r, s)-*fuzzy dense set* in X, if $C_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$.

Theorem 3.2. *An (r, s)-fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is an (r, s)-fuzzy rare set if and only if $\overline{1} - \lambda$ is an (r, s)-fuzzy dense set.*

Proof. Suppose λ is an (r, s)-fuzzy rare set in X. Then $I_{\tau,\tau^\star}(\lambda, r, s) = \overline{0}$ or $\overline{1} - I_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$. Thus $C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) = \overline{1}$.

Conversely, suppose $\overline{1}-\lambda$ is an (r, s)-fuzzy dense set in X. Then $C_{\tau,\tau^\star}(\overline{1}-\lambda, r, s) = \overline{1}$, that is, $\overline{1} - I_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$. Thus $I_{\tau,\tau^\star}(\lambda, r, s) = \overline{0}$. □

Theorem 3.3. *A fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is both an (r, s)-fuzzy open set and an (r, s)-fuzzy rare set if and only if it is the $\overline{0}$ set.*

Proof. λ is a fuzzy set which is an (r, s)-fuzzy open set and an (r, s)-fuzzy rare set if and only if $I_{\tau,\tau^\star}(\lambda, r, s) = \lambda = \overline{0}$. □

Corollary 3.4. *A fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is both an (r, s)-fuzzy closed set and (r, s)-fuzzy dense set if and only if it is $\overline{1}$ set.*

Remark 3.5. A fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ can be both an (r, s)-fuzzy rare set and an (r, s)-fuzzy dense set. This follows from the following example.

Example 3.6. Let $X = \{a, b, c\}$, $\mu_1 = \left(\frac{a}{0.1}, \frac{b}{0.2}, \frac{c}{0.6}\right)$ and $\mu_2 = \left(\frac{a}{0.1}, \frac{b}{0.1}, \frac{c}{0.6}\right)$. Define fuzzy topologies $\tau, \tau^\star, \sigma, \sigma^\star : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 0 & \textit{otherwise} \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 1 & \textit{otherwise.} \end{cases}$$

Then $C_{\tau,\tau^\star}(\mu_2, \frac{1}{2}, \frac{1}{2}) = \overline{1}$, $I_{\tau,\tau^\star}(\mu_2, \frac{1}{2}, \frac{1}{2}) = \overline{0}$. Thus μ_2 is both $\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy rare set and an $\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy dense set.

Theorem 3.7. *A fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is both an (r, s)-fuzzy rare set and an (r, s)-fuzzy dense set, if and only if there exists neither an (r, s)-fuzzy open set contained in λ nor an (r, s)-fuzzy closed set containing λ, except for $\overline{0}$ and $\overline{1}$ respectively.*

Proof. Suppose λ is both an (r, s)-fuzzy rare set and an (r, s)-fuzzy dense set. Then by definition, $C_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$ and $I_{\tau,\tau^\star}(\lambda, r, s) = \overline{0}$. This implies that λ contains neither (r, s)-fuzzy open set except $\overline{0}$ nor contained in (r, s)-fuzzy closed set except $\overline{1}$. The proof of the converse is similar. □

Remark 3.8. If an (r, s)-fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is both an (r, s)-fuzzy rare set and an (r, s)-fuzzy dense set, then λ is neither an

(r, s)-fuzzy open set nor an (r, s)-fuzzy closed set. The converse need not be true as it can be seen from the following example.

Example 3.9. Let $X = \{a, b, c\}$, $\mu_1 = \left(\frac{a}{0.2}, \frac{b}{0.0}, \frac{c}{0.0}\right)$ and $\mu_2 = \left(\frac{a}{0.0}, \frac{b}{0.5}, \frac{c}{0.0}\right)$. Define fuzzy topologies $\tau, \tau^\star : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 0 & otherwise \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 1 & otherwise \end{cases}$$

Then λ is neither$\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy open set nor $\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy closed set. But $I_{\tau,\tau^\star}(\mu_2, \frac{1}{2}, \frac{1}{2}) = \overline{0}$; $C_{\tau,\tau^\star}(\mu_2, \frac{1}{2}, \frac{1}{2}) = \left(\frac{a}{0.8}, \frac{b}{1.0}, \frac{c}{1.0}\right) \neq \overline{1}$. Thus μ_2 is not an $\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy dense set.

Theorem 3.10. *In a double fuzzy topological space* $(X, \tau, \tau^\star)$,

(1) $\overline{1}$ *is an* (r, s)*-fuzzy dense set, but it is not an* (r, s)*-fuzzy rare set,*

(2) $\overline{0}$ *is an* (r, s)*-fuzzy rare set,but it is not an* (r, s)*-fuzzy dense set,*

(3) *arbitrary intersection* (resp. *union*) *of* (r, s)*-fuzzy rare* (resp. *fuzzy dense*) *set is* (r, s)*-fuzzy rare* (resp. (r, s)*-fuzzy dense*) *set.*

Proof. (1) and (2) are obvious.

(3) Let $\lambda = \bigwedge_{\alpha\in\Delta} \mu_\alpha$ be an arbitrary intersection of (r, s)-fuzzy rare sets, that is $I_{\tau,\tau^\star}(\mu_\alpha, r, s) = \overline{0}$ for each $\alpha \in \Delta$. Then $\bigwedge_{\alpha\in\Delta} \mu_\alpha = \overline{0}$. We have $\overline{0} = \bigwedge_{\alpha\in\Delta} \mu_\alpha \geq I_{\tau,\tau^\star}(\bigwedge_{\alpha\in\Delta} \mu_\alpha, r, s)$. This implies that $\overline{0} \geq I_{\tau,\tau^\star}(\mu, r, s)$ or $I_{\tau,\tau^\star}(\mu, r, s) = \overline{0}$. Similarly, it can be shown that arbitrary union of (r, s)-fuzzy dense sets is (r, s)-fuzzy dense set. □

Example 3.11. Finite union of (r, s)-fuzzy rare sets need not be (r, s)-fuzzy rare set. This follows from the following example

Example 3.12. Let $X = \{a, b, c\}$, $\mu_1 = \left(\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.0}\right)$, $\mu_2 = \left(\frac{a}{0.2}, \frac{b}{0.0}, \frac{c}{0.0}\right)$ and $\mu_3 = \left(\frac{a}{0.0}, \frac{b}{0.2}, \frac{c}{0.0}\right)$. Define fuzzy topologies $\tau, \tau^\star : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 0 & otherwise \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 1 & otherwise \end{cases}$$

It is clear that μ_2 and μ_3 are $\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy rare sets but $\mu_2 \vee \mu_2$ is not an $\left(\frac{1}{2}, \frac{1}{2}\right)$-fuzzy rare set.

Theorem 3.13. *A fuzzy set* λ *of a double fuzzy topological space* $(X, \tau, \tau^\star)$ *is an* (r, s)*-fuzzy dense (resp.* (r, s)*-fuzzy rare) set if and only if for every* (r, s)*-fuzzy open* (resp. (r, s)*-fuzzy closed*) *set* μ *satisfying* $\lambda \leq \mu$ (resp. $\mu \leq \lambda$)*, we have* $C_{\tau,\tau^\star}(\lambda, r, s) \geq \lambda$ (resp. $I_{\tau,\tau^\star}(\lambda, r, s) \leq \mu$).

Proof. First assume that λ is an (r, s)-fuzzy dense set and take an (r, s)-fuzzy open set μ with $\lambda \leq \mu$. Then $C_{\tau,\tau^\star}(\lambda, r, s) = \overline{1} \geq \mu$.

Conversely, suppose the necessary conditions hold and take $\mu = \overline{1}$. Then μ is an (r,s)-fuzzy open set and $\lambda \leq \mu$. Thus $C_{\tau,\tau^\star}(\lambda,r,s) \geq \mu = \overline{1}$; that is, $C_{\tau,\tau^\star}(\lambda,r,s) = \overline{1}$. So λ is an (r,s)-fuzzy dense set. The other part can be proved similarly. □

Remark 3.14. A fuzzy set λ of a double fuzzy topological space $(X,\tau,\tau^\star)$ is an (r,s)-fuzzy rare set, if there exists no (r,s)-fuzzy open set other than $\overline{0}$ contained in λ.

Theorem 3.15. *The union* (resp. *intersection*) *of* (r,s)*-fuzzy dense* (resp (r,s)*-fuzzy rare*) *sets and* (r,s)*-fuzzy closed* (resp. (r,s)*-fuzzy open*) *sets is fuzzy dense* (resp. (r,s)*-fuzzy rare*) *set.*

Proof. (1) Let λ be an (r,s)-fuzzy dense set and μ an (r,s)-fuzzy closed set. If ν is an (r,s)-fuzzy open set with $\lambda \vee \mu \leq \nu$, then $\lambda \leq \nu$. Thus $C_{\tau,\tau^\star}(\lambda,r,s) \geq \nu$. Now $C_{\tau,\tau^\star,r,s}(\lambda \vee \mu) \geq C_{\tau,\tau^\star}(\lambda,r,s) \vee C_{\tau,\tau^\star}(\mu,r,s) \geq \mu \vee \nu \geq \nu$. So the union of an (r,s)-fuzzy dense set and an (r,s)-fuzzy closed set is an (r,s)-fuzzy dense set.

(2) Let λ be an (r,s)-fuzzy rare set and μ an (r,s)-fuzzy open set. If ν is an (r,s)-fuzzy closed set with $\nu \leq \lambda \vee \mu$, then $\nu \leq \lambda$. Thus $I_{\tau,\tau^\star}(\lambda,r,s) \leq \nu$. Now $I_{\tau,\tau^\star}(\lambda \wedge \mu,r,s) = I_{\tau,\tau^\star}(\lambda,r,s) \wedge I_{\tau,\tau^\star}(\mu,r,s) \leq \mu \wedge \nu \leq \nu$. So the intersection of an (r,s)-fuzzy rare set and an (r,s)-fuzzy open set is an (r,s)-fuzzy rare set. □

Theorem 3.16. $C_{\tau,\tau^\star}(\lambda,r,s)$ (resp. $I_{\tau,\tau^\star}(\lambda,r,s)$) *is an* (r,s)*-fuzzy dense* (resp. (r,s)*-fuzzy rare*) *set, whenever* λ *is an* (r,s)*-fuzzy dense* (resp. (r,s)*-fuzzy rare*) *set.*

Proof. Let λ be an (r,s)-fuzzy dense set. Then $C_{\tau,\tau^\star}(\lambda,r,s) = \overline{1}$. This implies that $C_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r,s),r,s) = C_{\tau,\tau^\star}(1,r,s) = \overline{1}$. Thus $C_{\tau,\tau^\star}(\lambda,r,s)$ is an (r,s)-fuzzy dense set. The proof of the second case is by the same token. □

Definition 3.17. A fuzzy set λ of a double fuzzy topological space $(X,\tau,\tau^\star)$ is said to be an (r,s)*-fuzzy closed rare* (resp. (r,s)*-fuzzy open dense*) *set* in X, if the (r,s)-fuzzy set λ is both an (r,s)-fuzzy closed set and an (r,s)-fuzzy rare set (resp. an (r,s)-fuzzy open set and an (r,s)-fuzzy dense set) in X.

Theorem 3.18. *A fuzzy set* λ *of a double fuzzy topological space* $(X,\tau,\tau^\star)$ *is an* (r,s)*-fuzzy closed rare set, if and only if* λ *is an* (r,s)*-fuzzy closed set which does not contain any* (r,s)*-fuzzy open set other than* $\overline{0}$*.*

Proof. Let λ be an (r,s)-fuzzy closed rare set in X. Then we have

$$I_{\tau,\tau^\star}(\lambda,r,s) = \overline{0} \text{ and } C_{\tau,\tau^\star}(\lambda,r,s) = \lambda.$$

This shows that λ is an (r,s)-fuzzy closed set which does not contain any (r,s)-fuzzy open set other than $\overline{0}$. □

Theorem 3.19. *A fuzzy set* λ *of a double fuzzy topological space* $(X,\tau,\tau^\star)$ *is an* (r,s)*-fuzzy open dense set, if and only if* λ *is an* (r,s)*-fuzzy closed set which does not contain any* (r,s)*-fuzzy open set other than* $\overline{1}$*.*

Proof. Let λ be an (r,s)-fuzzy open dense set in X. Then we get

$$I_{\tau,\tau^\star}(\lambda,r,s) = \lambda \text{ and } C_{\tau,\tau^\star}(\lambda,r,s) = \overline{1}.$$

This shows that λ is an (r,s)-fuzzy open set, which does not contain any (r,s)-fuzzy closed set other than $\overline{1}$. □

Definition 3.20. Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. Then we define

$$Fr_{\tau,\tau^\star}(\lambda, r, s) = C_{\tau,\tau^\star}(\lambda, r, s) \wedge C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s),$$

where $r \in I_0$ and $s \in I_1$ such that $r + s \leq \overline{1}$.

Theorem 3.21. *If a fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is an (r, s)-fuzzy dense set, then $Fr_{\tau,\tau^\star}(\lambda, r, s) = \overline{1} - I_{\tau,\tau^\star}(\lambda, r, s)$.*

Proof. Suppose λ is an (r, s)-fuzzy dense set in X, i.e., $C_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$. Then

$$\begin{aligned} Fr_{\tau,\tau^\star}(\lambda, r, s) &= C_{\tau,\tau^\star}(\lambda, r, s) \wedge C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) \\ &= \overline{1} \wedge C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) \\ &= C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) \\ &= \overline{1} - I_{\tau,\tau^\star}(\lambda, r, s). \end{aligned}$$

□

Theorem 3.22. *If a fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is an (r, s)-fuzzy rare set, then $Fr_{\tau,\tau^\star}(\lambda, r, s) = C_{\tau,\tau^\star}(\lambda, r, s)$.*

Proof. Suppose λ is an (r, s)-fuzzy rare set in X. Then $I_{\tau,\tau^\star}(\lambda, r, s) = \overline{0}$. Thus we have

$$\begin{aligned} Fr_{\tau,\tau^\star}(\lambda, r, s) &= C_{\tau,\tau^\star}(\lambda, r, s) \wedge C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) \\ &= C_{\tau,\tau^\star}(\lambda, r, s) \wedge (\overline{1} - I_{\tau,\tau^\star}(\lambda, r, s)) \\ &= C_{\tau,\tau^\star}(\lambda, r, s) \wedge (\overline{1} - \overline{0}) \\ &= C_{\tau,\tau^\star}(\lambda, r, s). \end{aligned}$$

□

Theorem 3.23. *A fuzzy set λ of a double fuzzy topological space $(X, \tau, \tau^\star)$ is both (r, s)-fuzzy dense and (r, s)-fuzzy rare if and only if $Fr_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$.*

Proof. Suppose $Fr_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}$. Then $C_{\tau,\tau^\star}(\lambda, r, s) \wedge C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) = \overline{1}$, i.e,

$$C_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}, \tag{3.1}$$

$$C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) = \overline{1}. \tag{3.2}$$

By (3.1), λ is (r, s)-fuzzy dense set in X. From (3.2), we have

$$C_{\tau,\tau^\star}(\overline{1} - \lambda, r, s) = \overline{1} - I_{\tau,\tau^\star}(\lambda, r, s) = \overline{1}, \text{ i.e, } I_{\tau,\tau^\star}(\lambda, r, s) = \overline{0}.$$

Thus λ is (r, s)-fuzzy rare set. The converse follows from Theorem 3.21. □

Theorem 3.24. *In a double fuzzy topological space $(X, \tau, \tau^\star)$, we have the following*

(1) $\overline{0}$ *is an (r, s)-fuzzy closed rare set in X,*

(2) *arbitrary intersections of (r, s)-fuzzy closed rare sets in X are (r, s)-fuzzy closed rare set in X.*

Proof. (1) Obvious.

(2) Let $(\lambda_i)_{i \in \Gamma}$ be a collection of (r, s)-fuzzy closed rare sets in X, i.e,

$$C_{\tau,\tau^\star}(\lambda_i, r, s) = \lambda_i \text{ and } I_{\tau,\tau^\star}(\lambda_i, r, s) = \overline{0} \text{ for each } \in \Gamma.$$

Then $C_{\tau,\tau^\star}(\bigwedge_{i \in I} \lambda_i, r, s) = \bigwedge_{i \in I} C_{\tau,\tau^\star}(\lambda_i, r, s) = \bigwedge_{i \in I} \lambda_i$. This proves that arbitrary intersection of (r, s)-fuzzy closed sets is (r, s)-fuzzy closed set. Since each λ_i is (r, s)-rare set in X, $I_{\tau,\tau^\star}(\lambda_i, r, s) = \overline{0}$. Thus $I_{\tau,\tau^\star}(\bigwedge_{i \in I} \lambda_i, r, s) = \bigwedge_{i \in I} I_{\tau,\tau^\star}(\lambda_i, r, s) = \overline{0}$. □

4. Double fuzzy rarely continuous functions

Definition 4.1. A function $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ is said to be:

(i) *double weakly continuous*, if for each $\mu \in I^Y$ with $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, $f^{-1}(\mu) \leq I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\mu, r, s)), r, s)$,

(ii) *double rarely continuous*, if for each $\mu \in I^Y$ with $\sigma(\mu) \geq r$ and $\sigma^\star(\mu) \leq s$, there exists an (r, s)-fuzzy rare set $\gamma \in I^Y$ with $\mu + C_{\sigma,\sigma^\star}(\gamma, r, s) \geq \overline{1}$ and $\rho \in I^X$, where $\tau(\rho) \geq r$ and $\tau^\star(\rho) \leq s$ such that $f(\rho) \leq \mu \vee \gamma$,

(iii) *double fuzzy open*, if for each $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq s$, $\sigma(f(\lambda)) \geq r$ and $\sigma^\star(f(\lambda)) \leq s$.

Remark 4.2. It is clear that every double weakly continuous function is double rarely continuous function. The following examples show that the converse statement may not be true.

Example 4.3. Let $X = \{a, b\}$, $\mu_1 = \left(\frac{a}{0.5}, \frac{b}{0.1}\right)$, $\mu_2 = \left(\frac{a}{0.8}, \frac{b}{0.7}\right)$, $\mu_3 = \left(\frac{a}{0.1}, \frac{b}{0.2}\right)$ and $\mu_4 = \left(\frac{a}{1.0}, \frac{b}{0.1}\right)$. Define fuzzy topologies $\tau, \tau^\star, \sigma, \sigma^\star : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 0 & otherwise \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{2} & \text{if } \lambda = \mu_1 \\ 1 & otherwise \end{cases}$$

$$\sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{1}{10} & \text{if } \lambda = \mu_2 \\ \frac{1}{11} & \text{if } \lambda = \mu_3 \\ 0 & otherwise \end{cases} \qquad \sigma^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1} \\ \frac{9}{10} & \text{if } \lambda = \mu_2 \\ \frac{10}{11} & \text{if } \lambda = \mu_3 \\ 1 & otherwise. \end{cases}$$

Let $r = \frac{1}{10}$ and $s = \frac{9}{10}$. Then the identity function $f : (X, \tau, \tau^\star) \to (X, \sigma, \sigma^\star)$ is $\left(\frac{1}{10}, \frac{9}{10}\right)$-fuzzy rarely continuous but not $\left(\frac{1}{10}, \frac{9}{10}\right)$-fuzzy weakly continuous, since $f^{-1}(\mu_4) \not\leq I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\mu_4, \frac{1}{10}, \frac{9}{10})), \frac{1}{10}, \frac{9}{10})$.

Proposition 4.4. *If $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ is both double fuzzy open and double fuzzy continuous, then it is double weakly continuous.*

Proof. Let $\lambda \in I^X$ such that $\tau(\lambda) \geq r$ and $\tau^*(\lambda) \leq s$. Since f is double fuzzy open, $\sigma(f(\lambda)) \geq r$ and $\sigma^\star(f(\lambda)) \leq s$. Also, since f is double fuzzy continuous, $f^{-1}(f(\lambda)) \in I^X$, where $\tau(f^{-1}(f(\lambda))) \geq r$ and $\tau^\star(f^{-1}(f(\lambda))) \leq s$. Note that

$$I_{\tau,\tau^\star}(f^{-1}(f(\lambda)), r, s) \leq I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(f(\lambda), r, s)), r, s).$$

Then $f^{-1}(f(\lambda)) \leq f^{-1}(C_{\sigma,\sigma^\star}(f(\lambda)))$. Since $\tau(f^{-1}(f(\lambda))) \geq r$ and $\tau^\star(f^{-1}(f(\lambda))) \leq s$, $f^{-1}(f(\lambda)) \leq I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(f(\lambda), r, s)), r, s)$. Thus f is double weakly continuous. □

Definition 4.5. Let $(X, \tau, \tau^\star)$ be a double fuzzy topological space. An (r, s)-*fuzzy open cover* of $(X, \tau, \tau^\star)$ is the collection $\{\lambda_i \in I^X; \tau(\lambda_i) \geq r, \tau^\star(\lambda_i) \leq s, i \in J\}$ such that $\bigvee_{i \in J} \lambda_i = \overline{1}$

Definition 4.6. A double fuzzy topological space $(X,\tau,\tau^\star)$ is said to be a *fuzzy compact space*, if every (r,s)-fuzzy open cover of $(X,\tau,\tau^\star)$ has a finite subcover.

Definition 4.7. A double fuzzy topological space $(X,\tau,\tau^\star)$ is said to be *double rarely fuzzy almost compact*, if for every (r,s)-fuzzy open cover $\{\lambda_i \in I^X; \tau(\lambda_i) \geq r, \tau^\star(\lambda_i) \leq s\}$ of $(X,\tau,\tau^\star)$, there exists a finite subset J_0 of J such that $\bigvee_{i\in J} \lambda_i \vee \rho_i = \overline{1}$, where $\rho_i \in I^X$ are (r,s)-fuzzy rare sets.

Proposition 4.8. *Let $f : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ be double rarely continuous. If $(X,\tau,\tau^\star)$ is fuzzy compact, then $(Y,\sigma,\sigma^\star)$ is rarely fuzzy almost compact.*

Proof. Let $\{\lambda_i \in I^Y : i \in J\}$ be an (r,s)-fuzzy open cover of $(Y,\sigma,\sigma^\star)$. Then $\overline{1} = \bigvee_{i\in J} \lambda_i$. Since f is double rarely continuous, there exists an (r,s)-fuzzy rare set $\rho_i \in I^Y$ such that $\lambda_i + C_{\sigma,\sigma^\star}(\rho_i,r,s) \geq \overline{1}$ and an (r,s)-fuzzy open set $\mu_i \in I^X$ such that $f(\mu_i) \leq \lambda_i \vee \rho_i$. Since $(X,\tau,\tau^\star)$ is fuzzy compact, every (r,s)-fuzzy open cover of $(X,\tau,\tau^\star)$ has a finite subcover. Thus $\overline{1} \leq \bigvee_{i\in J_0} \mu_i$. So we have

$$\overline{1} = f(\overline{1}) = f(\bigvee_{i\in J_0} \mu_i) = \bigvee_{i\in J_0} f(\mu_i) \leq \bigvee_{i\in J_0} \lambda_i \vee \rho_i.$$

Hence $(Y,\sigma,\sigma^\star)$ is rarely fuzzy almost compact. □

Proposition 4.9. *If $f : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ is double fuzzy open and one-to-one, then f preserves (r,s)-fuzzy rare sets.*

Proof. The proof is trivial. □

Proposition 4.10. *If $f : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ is double rarely continuous, double fuzzy open and $g : (Y,\sigma,\sigma^\star) \to (Z,\eta,\eta^\star)$ is double fuzzy open and one-to-one, then $g \circ f : (X,\tau,\tau^\star) \to (Z,\eta,\eta^\star)$ is double rarely continuous.*

Proof. Let $\lambda \in I^X$ with $\tau(\lambda) \geq r$ and $\tau^\star(\lambda) \leq s$. Since f is double fuzzy open, $f(\lambda) \in I^Y$ with $\sigma(f(\lambda)) \geq r$ and $\sigma^\star(f(\lambda)) \leq s$. Since f is double rarely continuous, there exist an (r,s)-fuzzy rare set $\rho \in I^Y$ with $f(\lambda) + C_{\tau,\tau^\star}(\rho,r,s) \geq \overline{1}$ and an (r,s)-fuzzy open set $\mu \in I^X$ such that $f(\mu) \leq f(\lambda) \vee \rho$. Then $g(\rho) \in I^Z$ is also an (r,s)-fuzzy rare set. By the injectivity of g and for $\rho \in I^Y$ such that $\rho < \gamma$ for all $\gamma \in I^X$ with $\sigma(\gamma) \geq r$, $\sigma^\star(\gamma) \leq s$, it follows that $C_{\eta,\eta^\star}(g(\rho),r,s) + (g \circ f)(\lambda) \geq \overline{1}$. Thus $(g \circ f)(\mu) = g(f(\mu)) \leq g(f(\lambda) \vee \rho) \leq g(f(\lambda)) \vee g(\rho) \leq (g \circ f)(\lambda) \vee g(\rho)$. □

Theorem 4.11. *If $f : (X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ is a double fuzzy open onto function and $g : (Y,\sigma,\sigma^\star) \to (Z,\eta,\eta^\star)$ is any function such that $g \circ f : (X,\tau,\tau^\star) \to (Z,\eta,\eta^\star)$ is double rarely continuous, then g is double rarely continuous.*

Proof. Let $\lambda \in I^X$, $\mu \in I^Y$ and $f(\lambda) = \mu$. Suppose $(g \circ f)(\lambda) = \gamma \in I^Z$ with $\eta(\gamma) \geq r$ and $\eta^\star(\gamma) \leq s$. Since $g \circ f$ is double rarely continuous, there exist an (r,s)-fuzzy rare set $\rho \in I^Z$ with $\gamma + C_{\eta,\eta^\star}(\rho,r,s) \geq r$ and an (r,s)-fuzzy open set $\theta \in I^X$ such that $(g \circ f)(\theta) \leq \gamma \vee \rho$. Since f is double fuzzy open, $f(\theta) \in I^Y$ is an (r,s)-fuzzy open set. Then there exist an (r,s)-fuzzy rare set $\rho \in I^Z$ with $\gamma + C_{\eta,\eta^\star}(\rho,r,s) \geq \overline{1}$ and an (r,s)-fuzzy open set $f(\theta) \in I^Y$ such that $g(f(\theta))\gamma \vee \rho$. Thus g is double rarely continuous. □

Definition 4.12. A double fuzzy topological space $(X, \tau, \tau^\star)$ is called a *fuzzy weak rare space*, if for every (r, s)-fuzzy open set $\lambda \in I^X$, there exists an (r, s)-fuzzy rare set $\rho \in I^X$ with $\lambda + C_{\tau,\tau^\star}(\rho, r, s) \geq \overline{1}$ such that $\tau(\lambda \vee \rho) \geq r$ and $\tau^\star(\lambda \vee \rho) \leq s$.

Proposition 4.13. *If $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ is a bijective double fuzzy continuous function and $(Y, \sigma, \sigma^\star)$ a fuzzy weak rare space, then $I_{\tau,\tau^\star}(f^{-1}(\lambda), r, s) \leq f^{-1}(\lambda \vee \rho)$ for $\rho, \lambda \in I^X$.*

Proof. Since $(Y, \sigma, \sigma^\star)$ is fuzzy weak rare space, for every (r, s)-fuzzy open set $\lambda \in I^X$, there exists an (r, s)-fuzzy rare set $\rho \in I^Y$ with $\lambda + C_{\sigma,\sigma^\star}(\rho, r, s) \geq \overline{1}$ such that $\sigma(\lambda \vee \mu) \geq r$ and $\sigma^\star(\lambda \vee \mu) \leq s$. Since f is double fuzzy continuous, $f^{-1}(\lambda \vee \rho)$ is (r, s)-fuzzy open. Clearly, $\lambda \leq \lambda \vee \rho$ and then $f^{-1}(\lambda) \leq f^{-1}(\lambda \vee \rho)$. Thus $I_{\tau,\tau^\star}(f^{-1}(\lambda), r, s) \leq f^{-1}(\lambda \vee \rho)$. □

Definition 4.14. A double fuzzy topological space $(X, \tau, \tau^\star)$ is called a *fuzzy rarely T_2 space*, if for each pair $\lambda, \mu \in I^X$ with $\lambda \neq \mu$ there exist (r, s)-fuzzy open sets $\rho_1, \rho_2 \in I^X$ with $\rho \neq \rho_2$ and a fuzzy rare set $\gamma \in I^X$ with $\rho_1 + C_{\tau,\tau^\star}(\gamma, r, s) \geq \overline{1}$ and $\rho_2 + C_{\tau,\tau^\star}(\gamma, r, s) \geq \overline{1}$ such that $\lambda \leq \rho_1 \vee \gamma$ and $\mu \leq \rho_2 \vee \gamma$.

Proposition 4.15. *If $f : (X, \tau, \tau^\star) \to (Y, \sigma, \sigma^\star)$ is a bijective double fuzzy open function and $(X, \tau, \tau^\star)$ a fuzzy rarely T_2 space, then $(Y, \sigma, \sigma^\star)$ is also a fuzzy rarely T_2 space.*

Proof. Let $\lambda, \mu \in I^X$ with $\lambda \neq \mu$. Since f is injective, $f(\lambda) \neq f(\mu)$. Since $(X, \tau, \tau^\star)$ is fuzzy rarely T_2 space, there exist (r, s)-fuzzy open sets $\rho_1, \rho_2 \in I^X$ with $\rho_1 \neq \rho_2$ and an (r, s)-fuzzy rare set $\gamma \in I^X$ with $\rho_1 + C_{\tau,\tau^\star}(\gamma, r, s) \geq \overline{1}$ and $\rho_2 + C_{\tau,\tau^\star}(\gamma, r, s) \geq \overline{1}$ such that $\lambda \leq \rho_1 \vee \gamma$ and $\mu \leq \rho_2 \vee \gamma$. By the fact that f is double fuzzy open, $f(\rho_1), f(\rho_2) \in I^Y$ are (r, s)-fuzzy open sets with $f(\rho_1) \neq f(\rho_2)$. Since f is double fuzzy open and one-to-one, $f(\gamma)$ is also an (r, s)-fuzzy rare set with $f(\rho_1) + C_{\tau,\tau^\star}(\gamma, r, s) \geq \overline{1}$ and $f(\rho_2) + C_{\tau,\tau^\star}(\gamma, r, s) \geq \overline{1}$ such that $f(\lambda) \leq f(\rho_i \vee \gamma)$ and $f(\mu) \leq f(\rho_2 \vee \gamma)$. Then $(Y, \sigma, \sigma^\star)$ is a fuzzy rarely T_2 space. □

5. Conclusion

In the course of our research, we defined new terminology with respect to the theory of double fuzzy rarely continuous functions, such as fuzzy rare set, fuzzy dense set and fuzzy rarely continuous functions for which we presented some fundamental properties and characterizations. In our future work we will investigate the case of double fuzzy multifunction for such a functions. This type of multifunction is a generalization of double fuzzy weakly continuous multifunctions (See [37]).

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