Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 26, No. 1, (August 2023) pp. 47–57
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2023.26.1.47

@FMI

On an algebraic structure induced by a fuzzy bi-partially ordered space

Ju-mok Oh

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 26, No. 1, August 2023

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 26, No. 1, (August 2023) pp. 47–57
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2023.26.1.47

@FMI

On an algebraic structure induced by a fuzzy bi-partially ordered space

Ju-mok Oh

Received 17 May 2023; Revised 3 June 2023; Accepted 6 July 2023

Abstract. We introduce an algebraic structure induced by a fuzzy bi-partial order space on a complete residuated lattice. We establish that the two families of f-stable and g-stable fuzzy sets are complete lattices and that they are isomorphic. Additionally, we demonstrate that the composition map gf, when the two maps f and g are restricted to suitable Alexandrov topologies, can be regarded as an interior operator, while the map fg can be viewed as a closure operator.

2020 AMS Classification: 03E72, 54A40, 54B10

Keywords: Complete residuated lattices, Fuzzy bi-partial ordered space, Alexandrov topologies, f-stable fuzzy sets, g-stable fuzzy sets.

Corresponding Author: Ju-mok Oh (jumok@gwnu.ac.kr)

1. Introduction

Stone [1] established the isomorphism between a distributive lattice L and the lattice of compact open subsets of $P(L)$, where $P(L)$ represents the collection of all prime ideals of L. This result, known as Stone's representation, provided a profound insight into the structure of distributive lattices. Building upon Stone's representation, Urquhart [2] introduced topological representations for lattices by incorporating a doubly ordered structure $(\leq_1, \leq_2)$ along with two maps l and r defined by

$$\begin{aligned} l(A) &= \{x \in X \mid (\forall y \in X)(x \leq_1 y \Rightarrow y \notin A)\}, \\ r(A) &= \{x \in X \mid (\forall y \in X)(x \leq_2 y \Rightarrow y \notin A)\}, \end{aligned}$$

where $A \subseteq X$. Urquhart demonstrated that the dual space of a bounded lattice can be viewed as a doubly ordered topological space.

The exploration of representation theorems has been a significant area of research, encompassing various algebraic structures, algebras, and logical relational systems [3, 4, 5, 6]. These theorems have provided valuable insights into the connections between algebraic structures and topological spaces. Moreover, they have served as a foundation for the development of relational semantics for logic, which forms an essential component of the theoretical underpinnings of computer science.

This paper delves into the exploration of an algebraic structure that arises from the utilization of two maps $f : L^X \to L^X$ and $g : L^X \to L^X$ defined by

$$f(A)(x) \quad = \bigwedge_{y\in X}\left[e^2_X(x,y) \to A(y)\right] \text{ and } g(B)(x) = \bigvee_{y\in X}\left[B(y)\odot e^1_X(y,x)\right],$$

within the context of a fuzzy bi-partially ordered space. The underlying space, denoted as $\left(X, e^1_X, e^2_X\right)$, is based on a complete residuated lattice $(L, \vee, \wedge, \odot, \to, \bot, \top)$. By shifting our focus from the doubly ordered space framework, characterized by the presence of two maps l and r, to this alternative framework, we aim to uncover new insights and implications.

In Theorem 3.6, we demonstrate that the families of f-stable and g-stable fuzzy sets form complete lattices and establish their isomorphism.

To further explore the properties and behavior of f and g, we narrow our focus to their restrictions within the framework of Alexandrov topologies. Specifically, we consider the restricted maps $f : \tau_{e^1_X} \to \tau_{e^2_X}$ and $g : \tau_{e^2_X} \to \tau_{e^1_X}$, where $\tau_{e^i_X}$ represents the Alexandrov topology associated with the fuzzy partial ordering e^i_X. This restriction allows us to analyze the influence of f and g within the context of interior and closure operators. In Theorem 3.4, we establish that $gf : \tau_{e^1_X} \to \tau_{e^1_X}$ operates as an interior operator, while $fg : \tau_{e^2_X} \to \tau_{e^2_X}$ operates as a closure operator.

2. Preliminaries

Definition 2.1 ([7, 8, 9, 10])**.** An algebra $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is called a *complete residuated lattice* if it satisfies the following conditions:
(L1) $(L, \leq, \vee, \wedge, \bot, \top)$ is a complete lattice with the greatest element $\top$ and the least element $\bot$,
(L2) $(L, \odot, \top)$ is a commutative monoid with identity $\top$,
(L3) the residuation property, i.e., $x \odot y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in L$.

In this paper, we always assume that $(L, \wedge, \vee, \odot, \to, \bot, \top)$ is a complete residuated lattice.

Lemma 2.2 ([7, 8, 9, 10])**.** *Let $x,\ y,\ z,\ w \in L$ and let $\{x_i\}_{i\in\Gamma}, \{y_i\}_{i\in\Gamma} \subseteq L$. Then the followings hold.*
(1) $\top \to x = x$, $\bot \odot x = \bot$.
(2) *If* $y \leq z$, *then* $x \odot y \leq x \odot z$, $x \to y \leq x \to z$ *and* $z \to x \leq y \to x$.
(3) $x \leq y$ *if and only if* $x \to y = \top$.
(4) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$.
(5) $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.
(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$.
(7) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ *and* $y \leq x \to x \odot y$.
(8) $(x \to y) \odot (z \to w) \leq (x \odot z) \to (y \odot w)$ *and* $x \to y \leq (x \odot z) \to (y \odot z)$.
(9) $(x \to y) \odot (y \to z) \leq x \to z$ *and* $(x \to y) \odot z \leq x \to (y \odot z)$.

(10) $\bigwedge_{i\in\Gamma}(x_i \to y_i) \leq \bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i$ *and* $\bigwedge_{i\in\Gamma}(x_i \to y_i) \leq \bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i$.
(11) $x \to y \leq (y \to z) \to (x \to z)$ *and* $x \to y \leq (z \to x) \to (z \to y)$.

Definition 2.3 ([7, 8, 9, 10])**.** Let X be a set. Then a map $e_X : X \times X \to L$ is said to be:
(E1) *reflexive* if $e_X(x,x) = \top$ for all $x \in X$,
(E2) *transitive* if $e_X(x,y) \odot e_X(y,z) \leq e_X(x,z)$ for all $x,y,z \in X$,
(E3) *antisymmetric* if $e_X(x,y) = e_X(y,x) = \top$, then $x = y$.
If e_X satisfies (E1),(E2) and (E3), e_X is called a *fuzzy partial order*. Let e^1_X and e^2_X be fuzzy partial orders on X. Then $\left(X, e^1_X, e^2_X\right)$ is called a *fuzzy bi-partially ordered space*.

Define $e_{L^X} : L^X \times L^X \to L$ by $e_{L^X}(A,B) = \bigwedge_{x\in X}[A(x) \to B(x)]$. Then e_{L^X} is a fuzzy partial order by Lemma 2.2 (9).
Let $\tau \subseteq L^X$. Define a $e_\tau : \tau \times \tau \to L$ by $e_\tau(A,B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then e_τ is a fuzzy partial order.
Let $\alpha \in L$ and $A \in L^X$. Define three maps $(\alpha_X \to A)$, $(\alpha_X \odot A)$, $\alpha_X : X \to L$ by $(\alpha_X \to A)(x) = \alpha \to A(x)$, $(\alpha_X \odot A)(x) = \alpha \odot A(x)$ and $\alpha_X(x) = \alpha$.

Definition 2.4 ([7, 8, 9, 10])**.** Let $\tau \subseteq L^X$. Then τ is called an *Alexandrov topology* on X, if it satisfies the following conditions:
(A1) $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau$ for all $\{A_i\}_{i\in I} \subseteq \tau$,
(A2) $\alpha_X \to A, \alpha_X \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.
The pair (X,τ) is called an *Alexandrov topological space* on X.

Theorem 2.5. *Let (X, e_X) be a fuzzy partially ordered space. Let*

$$\tau_{e_X} = \left\{A \in L^X \mid A(x) \odot e_X(x,y) \leq A(y) \text{ for all } x,y \in X\right\}.$$

Then the followings hold.
(1) *τ_{e_X} is an Alexandrov topology.*
(2) *For each $A \in L^X$, $A \in \tau_{e_X}$ if and only if $\bigvee_{x\in X}[A(x) \odot e_X(x,y)] = A(y)$ for all $y \in X$ if and only if $\bigwedge_{y\in X}[e_X(x,y) \to A(y)] = A(x)$ for all $x \in X$.*

Proof. (1) (A1) Let $\{A_i\}_{i\in\Gamma} \subseteq \tau_{e_X}$. Then

$$\begin{aligned}\left(\bigwedge_{i\in\Gamma} A_i\right)(x) \odot e_X(x,y) &\leq \bigwedge_{i\in\Gamma}[A_i(x) \odot e_X(x,y)]\\ &\leq \bigwedge_{i\in\Gamma} A_i(y) \quad [\text{Since } A_i \in \tau_{e_X}]\\ &= \left(\bigwedge_{i\in\Gamma} A_i\right)(y)\end{aligned}$$

and

$$\begin{aligned}\left(\bigvee_{i\in\Gamma} A_i\right)(x) \odot e_X(x,y) &= \bigvee_{i\in\Gamma}[A_i(x) \odot e_X(x,y)] \quad [\text{By Lemma 2.2 (6)}]\\ &\leq \bigvee_{i\in\Gamma} A_i(y) \quad [\text{Since } A_i \in \tau_{e_X}]\\ &= \left(\bigvee_{i\in\Gamma} A_i\right)(y).\end{aligned}$$

Thus $\bigwedge_{i\in\Gamma} A_i, \bigvee_{i\in\Gamma} \in \tau_{e_X}$.
(A2) Let $A \in \tau_{e_X}$ and $\alpha \in L$. Then

$$\begin{aligned}[\alpha_X \to A](x) \odot e_X(x,y) &= \alpha \to [A(x) \odot e_X(x,y)] \quad [\text{By Lemma 2.2 (9)}]\\ &\leq \alpha \to A(y) \quad [\text{Since } A \in \tau_{e_X}]\\ &= [\alpha_X \to A](y)\end{aligned}$$

and

$$\begin{aligned}[\alpha_X \odot A](x) \odot e_X(x,y) &= \alpha \odot [A(x) \odot e_X(x,y)] \\ &\le \alpha \odot A(y) \quad [\text{Since } A \in \tau_{e_X}] \\ &= [\alpha_X \odot A](y).\end{aligned}$$

Thus $\alpha_X \to A, \alpha_X \odot A \in \tau_{e_X}$. So τ_{e_X} is an Alexandrov topology.
(2) Suppose $A \in \tau_{e_X}$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $x,\ y \in X$. Thus $\bigvee_{x\in X} [A(x) \odot e_X(x,y)] \le A(y)$ for all $y \in X$. On the other hand, we have

$$\bigvee_{x\in X} [A(x) \odot e_X(x,y)] \ge A(y) \odot e_X(y,y) = A(y).$$

So $\bigvee_{x\in X} [A(x) \odot e_X(x,y)] = A(y)$ for all $y \in X$.

Conversely, suppose $\bigvee_{x\in X} [A(x) \odot e_X(x,y)] = A(y)$ for all $y \in X$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $y \in X$. Thus $A \in \tau_{e_X}$.

Now suppose $A \in \tau_{e_X}$. Then $A(x) \odot e_X(x,y) \le A(y)$ for all $x, y \in X$. By residuation, $A(x) \le e_X(x,y) \to A(y)$ for all $x, y \in X$. Thus we have

$$A(x) \le \bigwedge_{y\in X} [e_X(x,y) \to A(y)] \text{ for all } x \in X.$$

On the other hand, $\bigwedge_{y\in X} [e_X(x,y) \to A(y)] \le e_X(x,x) \to A(x) = A(x)$. So $\bigwedge_{y\in X} [e_X(x,y) \to A(y)] = A(x)$ for all $x \in X$.

Conversely, suppose $\bigwedge_{y\in X} [e_X(x,y) \to A(y)] = A(x)$ for all $x \in X$. Then $A(x) \le e_X(x,y) \to A(y)$ for all $x, y \in X$. By residuation, $e_X(x,y) \odot A(x) \le A(y)$ for all $x, y \in X$. Thus $A \in \tau_{e_X}$. □

3. Fuzzy concept lattices and fuzzy bi-partially ordered spaces

Definition 3.1. Let $\left(X, e^1_X, e^2_X\right)$ be a fuzzy bi-partially ordered space. Define $f : L^X \to L^X$ and $g : L^X \to L^X$ by: for each $x \in X$,

$$f(A)(x) = \bigwedge_{y\in X} \left[e^2_X(x,y) \to A(y)\right] \text{ and } g(B)(x) = \bigvee_{y\in X} \left[B(y) \odot e^1_X(y,x)\right].$$

A fuzzy set $A \in L^X$ is called *g-stable* (resp. *f-stable*), if $g(f(A)) = A$ (resp. $f(g(A)) = A$).

The family of all g-stable (resp. f-stable) fuzzy sets will be denoted by $G\left(L^X\right)$ (resp. $F\left(L^X\right)$).

Theorem 3.2. *Let (X, e^1_X, e^2_X) be a fuzzy bi-partially ordered space. Let f and g be the maps defined in Definition 3.1. Then the following hold.*

(1) *Let $A, B \in L^X$. Then*

$$e_{L^X}(A,B) \le e_{L^X}(f(A), f(B)) \text{ and } e_{L^X}(A,B) \le e_{L^X}(g(A), g(B)).$$

In particular, if $A \le B$, then $f(A) \le f(B)$ and $g(A) \le g(B)$.

(2) *Let $A \in L^X$. Then $g(A) \in \tau_{e^1_X}$, $f(A) \in \tau_{e^2_X}$, $f(A) \le A$ and $g(A) \ge A$.*

(3) *If $A \in \tau_{e^1_X}$, then $gf(A) \le A$. If $A \in \tau_{e^2_X}$, then $fg(A) \ge A$. Moreover, if $A \in \tau_{e^1_X} \cap \tau_{e^2_X}$, then $gf(A) = A$ and $fg(A) = A$.*

(4) *$G\left(L^X\right) = \left\{gf(A) \mid A \in L^X\right\}$ and $F\left(L^X\right) = \left\{fg(A) \mid A \in L^X\right\}$.*

(5) *If $A \in \tau_{e^1_X}$, then $f(A) \in F\left(L^X\right)$. Similarly, if $A \in \tau_{e^2_X}$, then $g(A) \in G\left(L^X\right)$.*

(6) *Let $\{A_i\}_{i\in\Gamma} \subseteq G\left(L^X\right)$, $A \in G\left(L^X\right)$ and $\alpha \in L$. Then*

$$\bigvee_{i\in\Gamma} A_i, \alpha_X \odot A \in G\left(L^X\right).$$

(7) *Let* $\{A_i\}_{i\in\Gamma} \subseteq F\left(L^X\right)$, $A \in F\left(L^X\right)$ *and* $\alpha \in L$. *Then*

$$\bigwedge_{i\in\Gamma} A_i, \alpha_X \to A \in F\left(L^X\right).$$

Proof. (1) Let $A,\ B \in L^X$. Then we have

$e_{L^X}(f(A), f(B))$
$= \bigwedge_{x\in X}\left[\bigwedge_{y\in X}\left[e^2_X(x,y) \to A(y)\right] \to \bigwedge_{y\in X}\left[e^2_X(x,y) \to B(y)\right]\right]$
$\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\left[\left[e^2_X(x,y) \to A(y)\right] \to \left[e^2_X(x,y) \to B(y)\right]\right]$ [By Lemma 2.2 (10)]
$\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\left[A(y) \to B(y)\right]$ [By Lemma 2.2 (11)]
$= e_{L^X}(A, B)$
and
$e_{L^X}(g(A), g(B))$
$= \bigwedge_{x\in X}\left[\bigvee_{y\in X}\left[A(y) \odot e^1_X(y,x)\right] \to \bigvee_{y\in X}\left[B(y) \odot e^1_X(y,x)\right]\right]$
$\geq \bigwedge_{x\in X}\bigwedge_{y\in X}\left[\left[A(y) \odot e^1_X(y,x)\right] \to \left[B(y) \odot e^1_X(y,x)\right]\right]$ [By Lemma 2.2 (11)]
$\geq \bigwedge_{y\in X}\bigwedge_{y\in X}\left[A(y) \to B(y)\right]$ [By Lemma 2.2 (8)]
$= e_{L^X}(A, B)$.

Now suppose $A \leq B$. Then by Lemma 2.2 (3), $\top = e_{L^X}(A, B)$. Thus

$$\top = e_{L^X}(f(A), f(B)) \text{ and } \top = e_{L^X}(g(A), g(B)).$$

So $f(A) \leq f(B)$ and $g(A) \leq g(B)$.

(2) Let $A \in L^X$ and let $x,\ y,\ z \in X$. Then we get

$$\begin{aligned} f(A)(x) \odot e^2_X(x,y) \odot e^2_X(y,z) &\leq f(A)(x) \odot e^2_X(x,z) \\ &= \textstyle\bigwedge_{y\in X}\left[e^2_X(x,y) \to A(y)\right] \odot e^2_X(x,z) \\ &\leq \left[e^2_X(x,z) \to A(z)\right] \odot e^2_X(x,z) \\ &\leq A(z). \end{aligned}$$

Thus by residuation, we have

$$f(A)(x) \odot e^2_X(x,y) \leq \bigwedge_{z\in X}\left[e^2_X(y,z) \to A(z)\right] = f(A)(y).$$

So $f(A) \in \tau_{e^2_X}$.

Also, we have

$$\begin{aligned} g(A)(x) \odot e^1_X(x,z) &= \textstyle\bigvee_{y\in X}\left[(A(y) \odot e^1_X(y,x)\right] \odot e^1_X(x,z) \\ &\leq \textstyle\bigvee_{y\in X}\left[A(y) \odot e^1_X(y,z)\right] \text{ [By Lemma 2.2 (6) and (E2)]} \\ &= g(A)(z). \end{aligned}$$

Then $g(A) \in \tau_{e^1_X}$.

On the other hand, we get

$$\begin{aligned} f(A)(x) &= \textstyle\bigwedge_{y\in X}\left[e^2_X(x,y) \to A(y)\right] \leq \left[e^2_X(x,x) \to A(x)\right] = A(x), \\ g(A)(x) &= \textstyle\bigvee_{y\in X}\left[A(y) \odot e^1_X(y,x)\right] \geq \left[A(x) \odot e^1_X(x,x)\right] = A(x). \end{aligned}$$

Then we have $f(A) \leq A$ and $A \leq g(A)$.

(3) Suppose $A \in \tau_{e_X^1}$ and let $x \in X$. Then

$$\begin{aligned} gf(A)(x) &= \bigvee_{y\in X} \left[f(A)(y) \odot e_X^1(y,x)\right] \\ &= \bigvee_{y\in X} \left[\bigwedge_{w\in X} \left[e_X^2(y,w) \to A(w)\right] \odot e_X^1(y,x)\right] \\ &\le \bigvee_{y\in X} \left[\left[e_X^2(y,y) \to A(y)\right] \odot e_X^1(y,x)\right] \\ &= \bigvee_{y\in X} \left[A(y) \odot e_X^1(y,x)\right] \\ &= A(x) \text{ [By Theorem 2.5 (2)].} \end{aligned}$$

Thus $gf(A) \le A$.

Suppose $A \in \tau_{e_X^2}$ and let $x \in X$. Then

$$\begin{aligned} fg(A)(x) &= \bigwedge_{y\in X} \left[e_X^2(x,y) \to \bigvee_{w\in X} \left[A(w) \odot e_X^1(w,y)\right]\right] \\ &\ge \bigwedge_{y\in X} \left[e_X^2(x,y) \to \left[A(y) \odot e_X^1(y,y)\right]\right] \text{ [By Lemma 2.2 (2)]} \\ &= \bigwedge_{y\in X} \left[e_X^2(x,y) \to A(y)\right] \\ &= A(x) \text{ [By Theorem 2.5 (2)].} \end{aligned}$$

Thus $A \le fg(A)$.

Suppose $A \in \tau_{e_X^1} \cap \tau_{e_X^2}$ and let $x \in X$. Then

$$\begin{aligned} gf(A)(x) &= \bigvee_{y\in X} \left[\bigwedge_{w\in X} \left[e_X^2(y,w) \to A(w)\right] \odot e_X^1(y,x)\right] \\ &= \bigvee_{y\in X} \left[A(y) \odot e_X^1(y,x)\right] \text{ [Since } A \in \tau_{e_X^2}] \\ &= A(x) \text{ [Since } A \in \tau_{e_X^1}] \end{aligned}$$

and

$$\begin{aligned} fg(A)(x) &= \bigwedge_{y\in X} \left[e_X^2(x,y) \to \bigvee_{w\in X} \left[A(w) \odot e_X^1(w,y)\right]\right] \\ &= \bigwedge_{y\in X} \left[e_X^2(x,y) \to A(y)\right] \text{ [Since } A \in \tau_{e_X^1}] \\ &= A(x) \text{ [Since } A \in \tau_{e_X^2}]. \end{aligned}$$

Thus $gf(A) = A = fg(A)$.

(4) Let $G_1\left(L^X\right) = \left\{gf(A) \mid A \in L^X\right\}$. We prove that $G\left(L^X\right) = G_1\left(L^X\right)$.

Claim 1: $G\left(L^X\right) \subseteq G_1\left(L^X\right)$. Let $A \in G\left(L^X\right)$. Then clearly, $A = gf(A) \in G_1\left(L^X\right)$.

Claim 2: $G_1\left(L^X\right) \subseteq G\left(L^X\right)$. Let $gf(A) \in G_1\left(L^X\right)$, where $A \in L^X$. Since $f(A) \in \tau_{e_X^2}$ by (2), $f(A) \le fgf(A)$ by (3). Since g is increasing by (1), $gf(A) \le gfgf(A)$. On the other hand, since $gf(A) \in \tau_{e_X^1}$ by (2), $gfgf(A) \le gf(A)$ by (3). Then $gfgf(A) = gf(A)$ and $gf(A) \in G\left(L^X\right)$.

Now let $F_1\left(L^X\right) = \left\{fg(A) \mid A \in L^X\right\}$. We show that $F\left(L^X\right) = F_1\left(L^X\right)$.

Claim 1: $F\left(L^X\right) \subseteq F_1\left(L^X\right)$. Let $A \in F\left(L^X\right)$. Then $A = fg(A) \in F_1\left(L^X\right)$.

Claim 2: $F_1\left(L^X\right) \subseteq F\left(L^X\right)$. Let $fg(A) \in F_1\left(L^X\right)$, where $A \in L^X$. Since $g(A) \in \tau_{e_X^1}$ by (2), $gfg(A) \le g(A)$ by (3). Since f is increasing by (1), $fgfg(A) \le fg(A)$. On the other hand, since $fg(A) \in \tau_{e_X^2}$ by (2), $fgfg(A) \ge fg(A)$ by 3. Then $fgfg(A) = fg(A)$ and $fg(A) \in F\left(L^X\right)$.

(5) Suppose $A \in \tau_{e_X^1}$. Then by (3), $gf(A) \le A$. Since f is increasing by (1), $fgf(A) \le f(A)$. On the other hand, since $f(A) \in \tau_{e_X^2}$ by (2), $fgf(A) \ge f(A)$ by (3). Thus $fgf(A) = f(A)$ and $f(A) \in F\left(L^X\right)$.

Now suppose $A \in \tau_{e_X^2}$. Then by (3), $A \le fg(A)$. Since g is increasing by (1), $g(A) \le gfg(A)$. On the other hand, since $g(A) \in \tau_{e_X^1}$ by (2), $gfg(A) \le g(A)$ by 3. Thus $gfg(A) = g(A)$ and $g(A) \in G\left(L^X\right)$.

(6) Let $\{A_i\}_{i\in\Gamma} \subseteq G\left(L^X\right)$. Then by (2), $A_i = gf(A_i) \in \tau_{e^1_X}$ for all $i \in \Gamma$. Thus $\bigvee_{i\in\Gamma} A_i \in \tau_{e^1_X}$. By (3), $gf\left(\bigvee_{i\in\Gamma} A_i\right) \leq \bigvee_{i\in\Gamma} A_i$. On the other hand, since gf is increasing by (1), $A_i = gf(A_i) \leq gf\left(\bigvee_{i\in\Gamma} A_i\right)$ for all $i \in \Gamma$. So $\bigvee_{i\in\Gamma} A_i \leq gf\left(\bigvee_{i\in\Gamma} A_i\right)$. Hence $gf\left(\bigvee_{i\in\Gamma} A_i\right) = \bigvee_{i\in\Gamma} A_i$ and $\bigvee_{i\in\Gamma} A_i \in G\left(L^X\right)$.

Let $A \in G\left(L^X\right)$ and $\alpha \in L$. Then by (2), $A = gf(A) \in \tau_{e^1_X}$. Thus $\alpha_X \odot A \in \tau_{e^1_X}$. By (3), $gf(\alpha_X \odot A) \leq \alpha_X \odot A$. On the other hand,

$$\begin{aligned} f(\alpha_X \odot A)(x) &= \bigwedge_{y\in X}\left[e^2_X(x,y) \to [\alpha_X \odot A](y)\right] \\ &\geq \bigwedge_{y\in X}\left[\left[e^2_X(x,y) \to A(y)\right] \odot \alpha\right] \text{ [By Lemma 2.2 (9)]} \\ &\geq f(A)(x) \odot \alpha. \end{aligned}$$

So we get

$$\begin{aligned} gf(\alpha_X \odot A)(x) &= \bigvee_{y\in X}\left[f(\alpha_X \odot A)(y) \odot e^1_X(y,x)\right] \\ &\geq \bigvee_{y\in X}\left[f(A)(y) \odot \alpha \odot e^1_X(y,x)\right] \\ &= \bigvee_{y\in X}\left[f(A)(y) \odot e^1_X(y,x)\right] \odot \alpha \\ &= gf(A)(x) \odot \alpha \\ &= (\alpha_X \odot A)(x). \end{aligned}$$

Hence $gf(\alpha_X \odot A) = \alpha_X \odot A$ and $\alpha_X \odot A \in G\left(L^X\right)$.

(7) Let $\{A_i\}_{i\in\Gamma} \subseteq F\left(L^X\right)$. By (2), $A_i = fg(A_i) \in \tau_{e^2_X}$ for all $i \in \Gamma$. Then $\bigwedge_{i\in\Gamma} A_i \in \tau_{e^2_X}$. Thus by (3), $\bigwedge_{i\in\Gamma} A_i \leq fg\left(\bigwedge_{i\in\Gamma} A_i\right)$. On the other hand, since fg is increasing by (1), $fg\left(\bigwedge_{i\in\Gamma} A_i\right) \leq fg(A_i) = A_i$ for all $i \in \Gamma$. So $fg\left(\bigwedge_{i\in\Gamma} A_i\right) \leq \bigwedge_{i\in\Gamma} A_i$. Hence $fg\left(\bigwedge_{i\in\Gamma} A_i\right) = \bigwedge_{i\in\Gamma} A_i$ and $\bigwedge_{i\in\Gamma} A_i \in F\left(L^X\right)$.

Let $A \in F\left(L^X\right)$ and $\alpha \in L$. By (2), $A = fg(A) \in \tau_{e^2_X}$. Then $\alpha_X \to A \in \tau_{e^2_X}$. Thus by (3), $\alpha_X \to A \leq fg(\alpha_X \to A)$. On the other hand, we have

$$\begin{aligned} g(\alpha_X \to A)(x) &= \bigvee_{y\in X}\left[[\alpha_X \to A](y) \odot e^1_X(y,x)\right] \\ &\leq \bigvee_{y\in X}\left[\alpha \to \left[A(y) \odot e^1_X(y,x)\right]\right] \text{ [By Lemma 2.2 (9)]} \\ &\leq \alpha \to g(A)(x). \text{[By Lemma 2.2 (2)]} \end{aligned}$$

So we get

$$\begin{aligned} fg(\alpha_X \to A)(x) &= \bigwedge_{y\in X}\left[e^2_X(x,y) \to g(\alpha_X \to A)(y)\right] \\ &\leq \bigwedge_{y\in X}\left[e^2_X(x,y) \to [\alpha \to g(A)(y)]\right] \\ &= \bigwedge_{y\in X}\left[\alpha \to \left[e^2_X(x,y) \to g(A)(y)\right]\right] \text{ [By Lemma 2.2 (7)]} \\ &= \alpha \to fg(A)(x) \text{[By Lemma 2.2 (4)]} \\ &= \alpha \to A(x) \\ &= (\alpha_X \to A)(x). \end{aligned}$$

Hence $fg(\alpha_X \to A) = \alpha_X \to A$ and $\alpha_X \to A \in F\left(L^X\right)$. □

Definition 3.3. Let $\tau_X \subseteq L^X$ be an Alexandrov topology on X.

(i) A map $I : \tau_X \to \tau_X$ is called an *interior operator*, if
- (I1) $I(A) \leq A$ for all $A \in \tau_X$,
- (I2) $I(A) \leq I(I(A))$ for all $A \in \tau_X$,
- (I3) $e_{\tau_X}(A, B) \leq e_{\tau_X}(I(A), I(B))$ for all $A,\ B \in \tau_X$.

(2) A map $C : \tau_X \to \tau_X$ is called a *closure operator*, if
- (C1) $A \leq C(A)$ for all $A \in \tau_X$,
- (C2) $C(C(A)) \leq C(A)$ for all $A \in \tau_X$,

(C3) $e_{\tau_X}(A, B) \leq e_{\tau_X}(C(A), C(B))$ for all $A,\ B \in \tau_X$.

Theorem 3.4. *Let (X, e^1_X, e^2_X) be a fuzzy bi-partially ordered space. Let $f : \tau_{e^1_X} \to \tau_{e^2_X}$ and $g : \tau_{e^2_X} \to \tau_{e^1_X}$ be the restrictions of f and d defined in Definition 3.1. Then the followings hold:*

(1) *$gf : \tau_{e^1_X} \to \tau_{e^1_X}$ is an interior operator such that $gf(A) = \bigvee \left\{C \in G\left(L^X\right) \mid C \leq A\right\}$,*

(2) *$fg : \tau_{e^2_X} \to \tau_{e^2_X}$ is a closure operator such that $fg(A) = \bigwedge \left\{D \in F\left(L^X\right) \mid A \leq D\right\}$.*

Proof. By Theorem 3.2(2), f and g are well-defined.

(1) (I1) Let $A \in \tau_{e^1_X}$. Then by Theorem 3.2 (3), $gf(A) \leq A$.

(I2) Let $A \in \tau_{e^1_X}$. Then by Theorem 3.2 (5), $f(A) \in F\left(L^X\right)$. Thus $fgf(A) = f(A)$. So $gfgf(A) = gf(A)$.

(I3) Let $A,\ B \in \tau_{e^1_X}$. Then we have

$$e_{\tau_{e^1_X}}(fg(A), gf(B)) \geq e_{\tau_{e^2_X}}(f(A), f(B)) \geq e_{\tau_{e^1_X}}(A, B). \text{ [By Theorem 3.2 (1)]}$$

Thus gf is an interior operator.

Let $\mathcal{H}(A) = \bigvee \left\{C \in G\left(L^X\right) \mid C \leq A\right\}$, where $A \in \tau_{e^1_X}$. Since $gf(A) \leq A$ by Theorem 3.2 (3) and $gf(A) \in G\left(L^X\right)$ by Theorem 3.2 (4), we have $gf(A) \leq \mathcal{H}(A)$. Conversely, $\mathcal{H}(A) \leq A$ by definition and $\mathcal{H}(A) \in G\left(L^X\right)$ by Theorem 3.2 (6). Since gf is increasing by Theorem 3.2 (1), $\mathcal{H}(A) = gf\left(\mathcal{H}(A)\right) \leq gf(A)$. Then $gf(A) = \mathcal{H}(A)$.

(2) (C1) Let $A \in \tau_{e^2_X}$. Then $A \leq fg(A)$ by Theorem 3.2 (3).

(C2) Let $A \in \tau_{e^2_X}$. Then $g(A) \in G\left(L^X\right)$ by Theorem 3.2 (5). Thus $gfg(A) = g(A)$ and $fgfg(A) = fg(A)$.

(C3) Let $A, B \in \tau_{e^2_X}$. Then we get

$$e_{\tau_{e^2_X}}(fg(A), fg(B)) \geq e_{\tau_{e^1_X}}(g(A), g(B)) \geq e_{\tau_{e^2_X}}(A, B). \text{ [By Theorem 3.2 (1)]}$$

Thus fg is a closure operator.

Let $\mathcal{J}(A) = \bigwedge \left\{D \in F\left(L^X\right) \mid A \leq D\right\}$, where $A \in \tau_{e^2_X}$. Since $A \leq fg(A)$ by Theorem 3.2 (3) and $fg(A) \in F\left(L^X\right)$ by Theorem 3.2 (4), we have $\mathcal{J}(A) \leq fg(A)$. Conversely, $A \leq \mathcal{J}(A)$ by definition and $\mathcal{J}(A) \in F\left(L^X\right)$ by Theorem 3.2 (7). Since fg is increasing by Theorem 3.2 (1), $\mathcal{J}(A) = fg\left(\mathcal{J}(A)\right) \geq fg(A)$. Then $fg(A) = \mathcal{J}(A)$. □

Definition 3.5. Let $(L_1, \leq, \wedge, \vee)$ and $(L_2, \leq, \wedge, \vee)$ be complete lattices. L_1 and L_2 are *isomorphic*, if there exists a bijective map $h : L_1 \to L_2$ such that

$$h\left(\bigvee_{i\in\Gamma} x_i\right) = \bigvee_{i\in\Gamma} h\left(x_i\right) \text{ and } h\left(\bigwedge_{i\in\Gamma} x_i\right) = \bigwedge_{i\in\Gamma} h\left(x_i\right) \text{ for all } \{x_i\}_{i\in\Gamma} \subseteq L_1.$$

Theorem 3.6. *Let $\left(X, e^1_X, e^2_X\right)$ be a fuzzy bi-partially ordered space. Let f and g be the maps defined in Definition 3.1. Then the followings hold.*

(1) *$\left(G\left(L^X\right), \sqcap, \bigvee, \perp_X, \top_X\right)$ is a complete lattice with*

$$\sqcap_{i\in\Gamma} A_i = g\left(\bigwedge_{i\in\Gamma} f\left(A_i\right)\right) \text{ and } \bigvee_{i\in\Gamma} A_i \text{ for all } \{A_i\}_{i\in\Gamma} \subseteq G\left(L^X\right).$$

(2) *$\left(F\left(L^X\right), \bigwedge, \sqcup, \perp_X, \top_X\right)$ is a complete lattice with*

$$\bigwedge_{i\in\Gamma} B_i \text{ and } \sqcup_{i\in\Gamma} B_i = f\left(\bigvee_{i\in\Gamma} g\left(B_i\right)\right) \text{ for all } \{B_i\}_{i\in\Gamma} \subseteq F\left(L^X\right).$$

(3) *$G\left(L^X\right)$ and $F\left(L^X\right)$ are isomorphic.*

Proof. (1) Let $\{A_i\}_{i\in\Gamma} \subseteq G\left(L^X\right)$. Then by Theorem 3.2 (6), $\bigvee_{i\in\Gamma} A_i \in G\left(L^X\right)$. We show $\sqcap_{i\in\Gamma} A_i \in G\left(L^X\right)$. Since $g\left(\bigwedge_{i\in\Gamma} f\left(A_i\right)\right) \in \tau_{e_X^1}$ by Theorem 3.2 (2), $gfg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) \leq g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)$ by Theorem 3.2 (3). On the other hand, since $f(A_i) \in \tau_{e_X^2}$ by Theorem 3.2 (2), we have $\bigwedge_{i\in\Gamma} f(A_i) \in \tau_{e_X^2}$. Thus by Theorem 3.2 (3), $fg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) \geq \bigwedge_{i\in\Gamma} f(A_i)$. Since g is increasing by Theorem 3.2 (1), we have $gfg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) \geq g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)$. So $gfg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) = g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)$ and $\sqcap_{i\in\Gamma} A_i \in G\left(L^X\right)$.

We show that $\sqcap_{i\in\Gamma} A_i$ is the infimum of $\{A_i\}_{i\in\Gamma}$ in $G\left(L^X\right)$. Since $\bigwedge_{i\in\Gamma} f(A_i) \leq f(A_i)$ for all $i \in \Gamma$ and g is increasing by Theorem 3.2 (1), we have $g\left(\bigwedge_{i\in\Gamma} f(A_i)\right) \leq gf(A_i) = A_i$ for all $i \in \Gamma$. Hence $g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)$ is a lower bound of $\{A_i\}_{i\in\Gamma}$ in $G\left(L^X\right)$.

Suppose $B \leq A_i$ for all $i \in \Gamma$, where $B \in G\left(L^X\right)$. Then by Theorem 3.2 (1), $f(B) \leq f(A_i)$ for all $i \in \Gamma$. Thus $f(B) \leq \bigwedge_{i\in\Gamma} f(A_i)$. Since g is increasing by Theorem 3.2 (1), we have $B = gf(B) \leq g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)$. So $\sqcap_{i\in\Gamma} A_i$ the infimum of $\{A_i\}_{i\in\Gamma}$ in $G\left(L^X\right)$.

(2) Let $\{B_i\}_{i\in\Gamma} \subseteq F\left(L^X\right)$. Then $\bigwedge_{i\in\Gamma} B_i \in F\left(L^X\right)$ by Theorem 3.2 (7). We show $\sqcup_{i\in\Gamma} B_i \in F\left(L^X\right)$. Since $f\left(\bigvee_{i\in\Gamma} g(B_i)\right) \in \tau_{e_X^2}$ by Theorem 3.2 (2), $fgf\left(\bigvee_{i\in\Gamma} g(B_i)\right) \geq f\left(\bigvee_{i\in\Gamma} g(B_i)\right)$ by Theorem 3.2(3). On the other hand, since $g(B_i) \in \tau_{e_X^1}$ by Theorem 3.2 (2), we have $\bigvee_{i\in\Gamma} g(B_i) \in \tau_{e_X^1}$. Thus $gf\left(\bigvee_{i\in\Gamma} g(B_i)\right) \leq \bigvee_{i\in\Gamma} g(B_i)$ by Theorem 3.2 (3). Since f is increasing by Theorem 3.2 (1),

$$fgf\left(\bigvee_{i\in\Gamma} g(B_i)\right) \leq f\left(\bigvee_{i\in\Gamma} g(B_i)\right).$$

So we have $fgf\left(\bigvee_{i\in\Gamma} g(B_i)\right) = f\left(\bigvee_{i\in\Gamma} g(B_i)\right)$ and $\sqcup_{i\in\Gamma} B_i \in F\left(L^X\right)$.

We now show that $\sqcup_{i\in\Gamma} B_i$ is the supremum of $\{B_i\}_{i\in\Gamma}$ in $F\left(L^X\right)$. Since $g(B_i) \leq \bigvee_{i\in\Gamma} g(B_i)$ for all $i \in \Gamma$ and f is increasing by Theorem 3.2 (1), we have $B_i = fg(B_i) \leq f\left(\bigvee_{i\in\Gamma} g(B_i)\right)$ for all $i \in \Gamma$. Then $f\left(\bigvee_{i\in\Gamma} g(B_i)\right)$ is an upper bound of $\{B_i\}_{i\in\Gamma}$ in $F\left(L^X\right)$.

Suppose $B_i \leq B$ for all $i \in \Gamma$, where $B \in F\left(L^X\right)$. Then by Theorem 3.2 (1), $g(B_i) \leq g(B)$ for all $i \in \Gamma$. Thus $\bigvee_{i\in\Gamma} g(B_i) \leq g(B)$. Since f is increasing by Theorem 3.2 (1), we have $f\left(\bigvee_{i\in\Gamma} g(B_i)\right) \leq fg(B) = B$. So $\sqcup_{i\in\Gamma} B_i$ is the supremum of $\{B_i\}_{i\in\Gamma}$ in $F\left(L^X\right)$.

(3) Define $f_1 : G\left(L^X\right) \to F\left(L^X\right)$ by

$$f_1(A)(x) = \bigwedge_{y\in X}\left[e_X^2(x,y) \to A(y)\right].$$

Let $A \in G\left(L^X\right)$. Then $fgf_1(A) = fgf(A) = f(A) = f_1(A)$. Thus $f_1(A) \in F\left(L^X\right)$. So f_1 is well-defined.

Suppose $f_1(A) = f_1(B)$, where $A, B \in G\left(L^X\right)$. Then $A = gf(A) = gf_1(A) = gf_1(B) = gf(B) = B$. Thus f_1 is injective.

Let $C \in F\left(L^X\right)$. Then $C = fg(C) = f_1\left(g(C)\right)$ and $gf\left(g(C)\right) = g(C)$. Thus f_1 is surjective.

Let $\{A_i\}_{i\in\Gamma} \subseteq G\left(L^X\right)$. Then

$$f_1\left(\bigvee_{i\in\Gamma} A_i\right) = f\left(\bigvee_{i\in\Gamma} gf(A_i)\right) = \sqcup_{i\in\Gamma} f_1(A_i).$$

We show $\bigwedge_{i\in\Gamma} f(A_i) \in F\left(L^X\right)$. Since $f(A_i) \in \tau_{e^2_X}$ by Theorem 3.2 (2), we have $\bigwedge_{i\in\Gamma} f(A_i) \in \tau_{e^2_X}$. Then $\bigwedge_{i\in\Gamma} f(A_i) \le fg\left(\bigwedge_{i\in\Gamma} f(A_i)\right)$ by Theorem 3.2 (3). On the other hand, since $\bigwedge_{i\in\Gamma} f(A_i) \le f(A_i)$ for all $i \in \Gamma$ and fg is increasing by Theorem 3.2 (1), we have $fg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) \le fgf(A_i) = f(A_i)$ for all $i \in \Gamma$. Thus $fg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) \le \bigwedge_{i\in\Gamma} f(A_i)$. So we get

$$fg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) = \bigwedge_{i\in\Gamma} f(A_i) \text{ and } \bigwedge_{i\in\Gamma} f(A_i) \in F\left(L^X\right).$$

Now, note that

$$f_1\left(\sqcap_{i\in\Gamma} A_i\right) = f_1\left(g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)\right) = fg\left(\bigwedge_{i\in\Gamma} f(A_i)\right) = \bigwedge_{i\in\Gamma} f_1(A_i).$$

Hence $G\left(L^X\right)$ and $F\left(L^X\right)$ are isomorphic. □

Example 3.7. Let $(X, \le, \wedge, \vee, \bot, \top)$ be a bounded lattice. Let $([0,1], \odot, \to, 0, 1)$ be the complete residuated lattice where

$$x \odot y := (x + y - 1) \vee 0 \text{ and } x \to y := (1 - x + y) \wedge 1.$$

Define two maps $e^1 : X \times X \to [0,1]$ and $e^2 : X \times X \to [0,1]$ by

$$e^1(x,y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \ne y \end{cases} \quad \text{and } e^2(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ 0, & \text{if } x \not\le y. \end{cases}$$

Then one can see that e^1 and e^2 are fuzzy partial orders and

$$\begin{aligned} \tau_{e^1} &= [0,1]^X, \\ \tau_{e^2} &= \{A \in L^X \mid \text{ if } x \le y, \text{ then } A(x) \le A(y)\}. \end{aligned}$$

Consider the fuzzy bi-partially ordered space $\left(X, e^1, e^2\right)$. Let f and g be maps defined in Definition 3.1. Then

$$\begin{aligned} f(A)(x) &= \bigwedge_{y\in X}\left[e^2(x,y) \to A(y)\right] = \bigwedge_{\substack{x\le y \\ y\in X}} A(y), \\ g(A)(x) &= \bigvee_{y\in X}\left[A(y) \odot e^1(y,x)\right] = A(x), \end{aligned}$$

where $A \in [0,1]^X$.

Let $A \in \tau_{e^1} = [0,1]^X$ and $B \in \tau_{e^2}$. Then

$$\begin{aligned} fg(A)(x) &= \bigvee_{y\in X}\left[f(A)(y) \odot e^1(y,x)\right] = f(A)(x) = \bigwedge_{\substack{x\le y \\ y\in X}} A(y), \\ fg(B)(x) &= \bigwedge_{y\in X}\left[e^2(x,y) \to g(B)(y)\right] = \bigwedge_{y\in X}\left[e^2(x,y) \to B(y)\right] = \bigwedge_{\substack{x\le y \\ y\in X}} B(y) \\ &= B(x). \quad \text{[Since } B \text{ is increasing]} \end{aligned}$$

Thus by Theorem 3.4, $gf : \tau_{e^1} \to \tau_{e^1}$ by $gf(A)(x) = \bigwedge_{\substack{x\le y \\ y\in X}} A(y)$ is an interior operator and $fg : \tau_{e^2} \to \tau_{e^2}$ by $fg(B)(x) = B(x)$ is a closure operator.

Moreover,

$$G\left([0,1]^X\right) = \left\{A \in [0,1]^X \mid \text{if } x \le y, \text{ then } A(x) \le A(y)\right\} = F\left([0,1]^X\right).$$

Let $\{A_i\}_{i\in\Gamma} \subseteq G\left([0,1]^X\right) = F\left([0,1]^X\right)$. Then

$$\begin{aligned} \left(\sqcap_{i\in\Gamma} A_i\right)(x) &= g\left(\bigwedge_{i\in\Gamma} f(A_i)\right)(x) = \bigwedge_{i\in\Gamma} f(A_i)(x) = \bigwedge_{i\in\Gamma} \bigwedge_{\substack{x\le y \\ y\in X}} A_i(y) \\ &= \bigwedge_{i\in\Gamma} A_i(x) \quad \text{[Since } A_i \text{ is increasing]} \end{aligned}$$

and

$$\begin{aligned}(\sqcup_{i\in\Gamma} A_i)(x) &= f\left(\bigvee_{i\in\Gamma} g(A_i)\right)(x) = \bigwedge_{\substack{x\le y\\ y\in X}} \bigvee_{i\in\Gamma} g(A_i)(y) = \bigwedge_{\substack{x\le y\\ y\in X}} \bigvee_{i\in\Gamma} A_i(y)\\ &= \bigvee_{i\in\Gamma} A_i(x). \quad \text{[Since } A_i \text{ is increasing]}\end{aligned}$$

So by Theorem 3.6, we conclude that

$$\left(G\left([0,1]^X\right) = F\left([0,1]^X\right), \sqcap = \bigwedge, \sqcup = \bigvee, 0_X, 1_X\right)$$

is a complete lattice.

4. Conclusion

In this paper, we are interested in the algebraic structures induced by bi-partial orders based on complete residuated lattices. We have shown that the two families of f-stable and g-stable fuzzy sets are complete lattices and they are isomorphic. Moreover, we have demonstrated that the composition map gf, when the two maps f and g are restricted to suitable Alexandrov topologies, can be regarded as an interior operator, while the map fg can be viewed as a closure operator.

In the future, by using bi-partial orders on complete residuated lattices, we might investigate various fuzzy concept lattices, information systems and decision rules on complete residuated lattices.

References

[1] M. H. Stone, Topological representation of distibutive lattices and Brouwerian logics, Časopis Pěst. Mat. Fys. 67 (1937) 1–25.
[2] A. Urquhart, A topological representation theorem for lattices, Algebra Universalis 8 (1978) 45–58.
[3] I. Düntsch and Edwin Mares, Alasdair Urquhart on Nonclassical and Algebraic Logic and Complexity of Proofs, Springer Verlag 2021.
[4] I. Düntsch, E. Orłowska and A. M. Radzikowska, Lattice-based relation algebras and their rpresentabilty, Lecture Notes in Computer Science 2929, Springer-Verlag, 234-258 2003.
[5] I. Düntsch, E. Orłowska and A. M. Radzikowska, Lattice-based relation algebras II, Lecture Notes in Artificial Intelligence 4342, Springer-Verlag, 267-289 2006.
[6] E. Orłowska and A. M. Radzikowska, Representation theorems for some fuzzy logics based on residuated non-distributive lattices, Fuzzy Sets and Systems 159 (2008) 1247–1259.
[7] R. Bělohlávek, Fuzzy Relational Systems, Kluwer Academic Publishers, New York 2002.
[8] P. Hájek, Metamathematices of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht 1998.
[9] U. Höhle and E. P. Klement, Non-classical logic and their applications to fuzzy subsets, Kluwer Academic Publishers, Boston 1995.
[10] U. Höhle and S .E. Rodabaugh, Mathematics of Fuzzy Sets, Logic, Topology and Measure Theory, The Handbooks of Fuzzy Sets Series, Kluwer Academic Publishers, Dordrecht 1999.

Ju-mok Oh (jumokoh@gwnu.ac.kr)
Department of Mathematics and Physics, University of Gangneung-Wonju, Gangneung, Gangwondo, 25457, Korea