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Recent advances in rough statistical convergence for difference sequences in neutrosophic normed spaces

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Abstract. This paper expands upon existing theories regarding the convergence of sequences in neutrosophic normed spaces (NNSs) by introducing the concept of rough statistical convergence for difference sequences in NNSs. The study investigates novel notions, namely rough convergence and rough statistical convergence, specifically in the context of difference sequences in NNSs. Furthermore, the paper analyzes various properties and characteristics of a mathematical construct denoted as $St_{(\Theta,\Psi,\Omega)} - \mathrm{LIM}^{r}_{\Delta t_w}$, which is referred to as the r-statistical limit set of the difference sequence (Δt_w). The examination of these features aims to deepen the understanding of the behavior and attributes associated with the r-statistical limit set in the context of rough statistical convergence in NNSs.

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1. Introduction

Fast [1] was the original proponent of statistical convergence for a sequence of real numbers. Subsequently, the notion of statistical convergence became associated with summability theory and has been extensively studied by researchers such as Fridy [2] and others in various fields [3, 4, 5, 6, 7, 8]).

The pioneering work on the Theory of Fuzzy Sets (FSs) was introduced by Zadeh [9], which had a profound impact across various scientific fields. This theory has been widely employed to handle imprecise, vague and inexact data. However, FSs alone may not always be sufficient to address the challenge of incomplete knowledge regarding membership degrees. To overcome this limitation, Atanassov [10] extended the Theory of FSs by developing the theory of Interval-valued Fuzzy Sets (IFSs). Another line of research explored by Kramosil and Michalek [11] is the study of Fuzzy Metric Spaces (FMSs), which combine concepts from fuzzy logic and probabilistic metric spaces. Kaleva and Seikkala [12] further investigated FMSs, examining the distance between two points as a non-negative fuzzy number. George and Veeramani [13] contributed to FMSs by providing certain qualifications. They also established fundamental properties and proved significant theorems related to FMSs. FMSs have found practical applications in decision-making, fixed point theory and medical imaging, among others. Building upon FMSs, Park [14] generalized the concept and introduced the notion of Interval-valued Fuzzy Metric Spaces (IFMSs). Park [14] incorporated the use of t-norm and t-conorm, as proposed by George and Veeramani [13], while describing IFMS and investigating its fundamental properties. Saadati and Park [15] delved into the exploration of Intuitionistic Fuzzy Normed Spaces (IFNSs) and initially examined their properties. For further background on IFNSs and related topics, readers are encouraged to refer to the studies [16, 17, 18, 19].

Smarandache [20] conducted a scientific exploration into the concept of "Neutrosophic sets" (NSs), which is a broader extension of Fuzzy Sets (FSs), Interval-valued Fuzzy Sets (IFSs) and other similar frameworks. The purpose of this investigation was to address the complexities arising from uncertainty when dealing with practical problems in our daily lives in a more precise manner. Decision-makers often encounter hesitations when making choices and a binary approach (such as a simple yes or no) may not always be sufficient. Furthermore, certain real-life events, such as sports competitions and voting procedures, can yield outcomes that have three possible components. Taking all these factors into account, Smarandache applied the principles of IFSs theory by introducing a new component called the indeterminacy membership function. Consequently, an element in an NSs consists of a triplet comprising a truth-membership function (T), an indeterminacy-membership function (I) and a falsity-membership function (F). A Neutrosophic set is defined as a set where each component of the universe possesses degrees of T, F and I. These three functions are independent of one another within the context of NSs. Consequently, the term "neutrosophy" indicates an impartial knowledge of thought, while "neutral" denotes the fundamental distinction between neutral, fuzzy, intuitive fuzzy sets and logic.

In Interval-valued Fuzzy Sets (IFSs), uncertainty is based on the degree of belongingness. However, in NSs, uncertainty is considered independently from the values of T and F. Since there are no restrictions on the degrees of T, F, and I, NSs are actually more general than IFSs.

Kirişci and Şimşek [21] introduced a novel concept called *Neutrosophic metric space* (NMS) utilizing continuous t-norms and continuous t-conorms. This new framework was thoroughly investigated, highlighting its notable characteristics.

The research conducted by Kirişci and Şimşek [22] delved into the study of Neutrosophic Normed Space (NNS) and statistical convergence in NNS. Their work explored the applications of Neutrosophic Set Theory and Neutrosophic Logic in various domains such as decision-making, robotics and summability theory. In the study [23], the researchers explored the concept of neutrosophic norm in a soft linear space, introducing a novel framework known as the neutrosophic soft normed linear space (NSNLS).

In Phu's work [24], the author extensively explored the properties of the set $LIM^r x$, demonstrating its boundedness, closedness and convexity. This concept holds significant importance and Phu thoroughly investigated its fundamental characteristics. It is noteworthy that the concept of rough convergence arises organically in the field of numerical analysis and finds intriguing applications therein.

Building upon the groundwork laid in [24], Phu delved deeper into the concept of rough convergence within the framework of infinite-dimensional normed spaces in [25]. By extending the scope of investigation to such spaces, the author expanded our understanding of rough convergence and its implications in this broader context. Following Phu's definition [24], Aytar [26] investigated the concept of rough statistical convergence.

Kizmaz [27] introduced the notion of a difference sequence, denoted as $\Delta y = (\Delta y_i) = (y_i - y_{i+1})$, where $(y_i)_{i\in\mathbb{N}}$ represents a real sequence. This concept serves as a foundational framework for the present study. Subsequently, Et [28], Et and Çolak [29], Et and Nuray [30], Bektaş et al. [31], Et and Esi [32], Gümüş and Nuray [33] and numerous others have examined various aspects and properties of difference sequences. Demir and Gümüş [34] presented the concept of rough convergence for difference sequences in a finite dimensional normed space. In another study, they investigated the rough statistical convergence of difference sequences [35]. Furthermore, Antala et al. [36] proposed the concept of rough statistical convergence in the context of intuitionistic fuzzy normed spaces (IFNSs). In recent years, significant advancements have been made in the field of rough convergence, motivating our investigation into the concept of rough statistical convergence in intuitionistic fuzzy normed spaces (IFNSs). IFNS is a well-established area of research, serving as a valuable framework for modeling imprecision in real-life scenarios. In addition, studies have been conducted in the field of rough convergence in NNS in recent years.

This study focuses on investigating the concept of rough statistical convergence for difference sequences in NNS. To establish the necessary background, we begin by reviewing relevant literature on difference sequences.

2. Preliminary

Triangular norms (t-norms), initially introduced by Menger [37], provide a generalization of the probability distribution with regards to the triangle inequality in metric spaces. These t-norms play a crucial role in fuzzy operations, particularly in terms of intersections. On the other hand, triangular conorms (t-conorms) serve as the dual operations of t-norms and are fundamental for fuzzy unions. The utilization of t-norms and t-conorms hold significant importance in various fuzzy operations. We recall the concepts of rough convergence, rough statistical convergence, neutrosophic norm are as follows.

Definition 2.1 ([24])**.** Consider a sequence $w = (w_k)$ of real numbers and a non-negative real number r. We say that the sequence (w_k) is *rough convergent to* $w_0 \in \mathbb{R}$, denoted by $w_k \xrightarrow{r} w_0$, if it satisfies the following condition: for every $\varepsilon > 0$, there exists a natural number $k_\varepsilon \in \mathbb{N}$ such that $|w_k - w_0| < r + \varepsilon$ for all $k \geq k_\varepsilon$.

Definition 2.2 ([26])**.** Let $w=(w_k)$ be a sequence of real numbers and r be a non-negative real number. A sequence (w_k) is said to be *rough statistically convergent to* $w_0 \in \mathbb{R}$, denoted by $w_k \overset{r-st}{\to} w_0$, if

$$\forall \varepsilon > 0,\ d\left(\{k \in \mathbb{N} : |w_k - w_0| \geq r + \varepsilon\}\right) = 0.$$

Definition 2.3 ([26])**.** Let $w=(w_k)$ be a sequence of real numbers and r be a non-negative real number. Then the set

$$st - LIM^r w = \left\{ w_0 \in \mathbb{R} : w_k \overset{r}{\to} w_0 \right\}$$

is known as the rough statistical limit set of $w=(w_k)$.

Definition 2.4 ([37])**.** Consider an operation $\circ : [0,1] \times [0,1] \to [0,1]$. Then $\circ$ is called a *continuous triangular norm* (TN), if it satisfies the following conditions: for any $p,\ q,\ r,\ s \in [0,1]$,

(i) $p \circ 1 = p$,
(ii) if $p \leq r$ and $q \leq s$, then $p \circ q \leq r \circ s$,
(iii) $\circ$ is continuous,
(iv) $\circ$ is associative and commutative.

Definition 2.5 ([37])**.** Consider an operation $\bullet : [0,1] \times [0,1] \to [0,1]$. Then $\bullet$ is called a *continuous triangular conorm* (TC), if it satisfies the following conditions: for any $p,\ q,\ r,\ s \in [0,1]$,

(i) $p \bullet 0 = p$,
(ii) if $p \leq r$ and $q \leq s$, then $p \bullet q \leq r \bullet s$,
(iii) $\bullet$ is continuous,
(iv) $\bullet$ is associative and commutative.

Definition 2.6 ([20])**.** Let X be a space of points (objects). A *Neutrosophic Set* (NS) $\mathcal{N}$ on X is characterized by a truth-membership function Θ, an indeterminacy membership function Ψ, and a falsity-membership function Ω, where $\Theta(u), \Psi(u)$ and $\Omega(u)$ and real standard and non-standard subset of $]^-0, 1^+[$ i.e., $\Theta, \Psi, \Omega : X \to]^-0, 1^+[$. Thus the NS $\mathcal{N}$ over X is defined as:

$$\mathcal{N} = \langle u, \Theta(u), \Psi(u), \Omega(u) : u \in X \rangle.$$

We will simply write neutrosophic set $\mathcal{N}$ as $\langle \Theta, \Psi, \Omega \rangle$. On the same of $\Theta(u), \Psi(u)$ and $\Omega(u)$ there is no restriction and so $^-0 \leq \sup \Theta(u) + \sup \Psi(u) + \sup \Omega(u) \leq 3^+$ for each $u \in X$. Here $1^+ = 1 + \epsilon$, where 1 is its standard part and ϵ its non-standard part. Also, $^-0 = 0 - \epsilon$, where 0 is its standard part and ϵ its non-standard part.
From philosophical point of view, a NS takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But to practice in real scientific and engineering areas, it is difficult to use NS with value from real standard or nonstandard subset of $]^-0, 1^+[$. Hence, we consider the NS which takes the value from the subset of $[0,1]$.

Definition 2.7 ([39])**.** Let X be a linear space over $\mathbb{R}$ ($\mathbb{R}$ denotes the set of all real numbers) and let $*$ be a continuous t-norm. A fuzzy subset N on $X \times \mathbb{R}$ is called a *fuzzy norm* on X, if for $u,\ v \in X$ and $c \in F$,

(N1) $\forall \lambda \in \mathbb{R}$ with $\lambda \leq 0$, $N(u, \lambda) = 0$,
(N2) $\forall \lambda \in \mathbb{R}$ with $\lambda > 0$, $N(u, \lambda) = 1$ iff $u = 0$,

(N3) $\lambda \in \mathbb{R}$, $\lambda > 0$,

$$N(cu, \lambda) = N\left(u, \frac{\lambda}{|c|}\right), \text{ if } c \neq 0,$$

(N4) $\forall \lambda, \mu \in \mathbb{R}$, $\forall u,\ v \in X$,

$$N(u+v, \lambda+\mu) \geq N(u, \lambda) * N(v, \mu),$$

(N5) $\lim_{u\to\infty} N(u, \lambda) = 1$.

The triplet $(X, N, *)$ will be referred to as a *fuzzy normed linear space.*

Definition 2.8 ([38])**.** Let F be a vector space and let $\mathcal{N} = \langle \Theta, \Psi, \Omega \rangle$ be a neutrosophic set on $F \times \mathbb{R}$. Then $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ is called a *neutrosophic normed space* (NNS), if it satisfies the following conditions: for all $u,\ v \in F$ and $\lambda,\ \mu > 0$ and for all $\sigma \neq 0$,

(i) $0 \leq \Theta(u, \lambda) \leq 1$, $0 \leq \Psi(u, \lambda) \leq 1$, $0 \leq \Omega(u, \lambda) \leq 1$,
(ii) $\Theta(u, \lambda) + \Psi(u, \lambda) + \Omega(u, \lambda) \leq 3$,
(iii) $\Theta(u, \lambda) = 1$ iff $u = 0$,
(iv) $\Theta(\sigma u, \lambda) = \Theta\left(u, \frac{\lambda}{|\sigma|}\right)$,
(v) $\Theta(u, \mu) \circ \Theta(v, \lambda) \leq \Theta(u+v, \mu+\lambda)$,
(vi) $\Theta(u, .)$ is non-decreasing continuous function,
(vii) $\lim_{\lambda\to\infty} \Theta(u, \lambda) = 1$,
(viii) $\Psi(u, \lambda) = 0$ iff $u = 0$,
(ix) $\Psi(\sigma u, \lambda) = \Psi\left(u, \frac{\lambda}{|\sigma|}\right)$,
(x) $\Psi(u, \mu) \bullet \Psi(v, \lambda) \geq \Psi(u+v, \mu+\lambda)$,
(xi) $\Psi(u, .)$ is non-decreasing continuous function,
(xii) $\lim_{\lambda\to\infty} \Psi(u, \lambda) = 0$,
(xiii) $\Omega(u, \lambda) = 0$ iff $u = 0$,
(xiv) $\Omega(\sigma u, \lambda) = \Omega\left(u, \frac{\lambda}{|\sigma|}\right)$,
(xv) $\Omega(u, \mu) \bullet \Omega(v, \lambda) \geq \Omega(u+v, \mu+\lambda)$,
(xvi) $\Omega(u, .)$ is non-decreasing continuous function,
(xvii) $\lim_{\lambda\to\infty} \Omega(u, \lambda) = 0$,
(xviii) if $\lambda \leq 0$, then $\Theta(u, \lambda) = 0, \Psi(u, \lambda) = 1$ and $\Omega(u, \lambda) = 1$.

In this case, $\mathcal{N} = \langle \Theta, \Psi, \Omega \rangle$ is called a *neutrosophic norm* (NN) on F.

Definition 2.9 ([22])**.** Let $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ be an NNS, $\sigma \in (0, 1)$ and $\lambda > 0$ and let (x_k) be a sequence in $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$. Then (x_k) is said to be *Cauchy*, if there is a $N \in \mathbb{N}$ such that $\Theta(x_k - x_m, \lambda) > 1 - \sigma$, $\Psi(x_k - x_m, \lambda) < \sigma$, $\Omega(x_k - x_m, \lambda) < \sigma$ for $k,\ m \geq N$.

Definition 2.10 ([22])**.** A sequence (x_m) is said to be *statistically convergent to* $\xi \in F$ w.r.t neutrosophic norm $\langle \Theta, \Psi, \Omega \rangle$, provided that for each $\lambda > 0$ and $\sigma > 0$, the set

$$P_\sigma := \{m \leq n : \Theta(x_m - \xi, \lambda) \leq 1 - \sigma \text{ or } \Psi(x_m - \xi, \lambda) \geq \sigma \text{ or } \Omega(x_m - \xi, \lambda) \geq \sigma\}$$

has natural density zero. That is $\delta(P_\sigma) = 0$ or

$$\lim_{n\to\infty} \frac{1}{n} \left|\{m \leq n : \Theta(x_m - \xi, \lambda) \leq 1 - \sigma \text{ or } \Psi(x_m - \xi, \lambda) \geq \sigma \text{ or } \Omega(x_m - \xi, \lambda) \geq \sigma\}\right| = 0.$$

It is demonstrated by $S_{\mathcal{N}}$-$\lim x_m = \xi$.

Definition 2.11. Let $(X, \|.\|)$ be a normed linear space and (Δw_k) be a difference sequence in X. Then $(\Delta w') = (\Delta w_{k_n})$ is called a *non-thin subsequence* of (Δw_k), if $\delta(\{k_n : n \in \mathbb{N}\}) \neq 0$.

3. Main results

In this section, our focus is on exploring the notions of rough convergence and rough statistical convergence for difference sequences in an NNS. Throughout the article, we will denote the NNS as $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ with neutrosophic norm $\langle \Theta, \Psi, \Omega \rangle$. The investigation is carried out in the following manner.

Definition 3.1. A sequence $\Delta t = (\Delta t_w)$ in $\mathcal{X}$ is said to be *rough convergent to* $\eta \in \mathcal{X}$ w.r.t the norm $\langle \Theta, \Psi, \Omega \rangle$ for some non-negative real number r, provided that there exist $w_0 \in \mathbb{N}$ for each $\xi > 0$ and $\sigma \in (0, 1)$ such that

$$\Theta(\Delta t_w - \eta; r + \xi) > 1 - \sigma, \ \Psi(\Delta t_w - \eta; r + \xi) < \sigma, \ \Omega(\Delta t_w - \eta; r + \xi) < \sigma$$

for all $w \geq w_0$. It is indicated by $r_{\langle \Theta, \Psi, \Omega \rangle} - \lim_{w \to \infty} \Delta t_w = \eta$ or $\Delta t_w \overset{r_{\langle \Theta, \Psi, \Omega \rangle}}{\rightarrow} \eta$.

Definition 3.2. A sequence $\Delta t = (\Delta t_w)$ in $\mathcal{X}$ is said to be *rough statistically convergent to* $\eta \in \mathcal{X}$ w.r.t the norm $\langle \Theta, \Psi, \Omega \rangle$ for some $r \geq 0$, provided that for each $\xi > 0$ and $\sigma \in (0, 1)$,

$$\delta(\{w \in \mathbb{N} : \Theta(\Delta t_w - \eta; r + \xi) \leq 1 - \sigma \text{ or } \Psi(\Delta t_w - \eta; r + \xi) \geq \sigma \\ \text{ or } \Omega(\Delta t_w - \eta; r + \xi) \geq \sigma\}) = 0.$$

It is denoted by $r - St_{\langle \Theta, \Psi, \Omega \rangle} - \lim_{w \to \infty} \Delta t_w = \eta$ or $\Delta t_w \overset{r - St_{\langle \Theta, \Psi, \Omega \rangle}}{\rightarrow} \eta$.

In the special case where $r = 0$, the concept of rough statistical convergence w.r.t $\langle \Theta, \Psi, \Omega \rangle$ is equivalent to statistical convergence w.r.t the norm $\langle \Theta, \Psi, \Omega \rangle$ in an NNS.

The $r - St_{\langle \Theta, \Psi, \Omega \rangle}$-limit of a difference sequence may be not unique in an NNS. Therefore, we establish $r - St_{\langle \Theta, \Psi, \Omega \rangle}$-limit set of the sequence $\Delta t = (\Delta t_w)$ as

$$St_{\langle \Theta, \Psi, \Omega \rangle} - \text{LIM}^r_{\Delta t_w} = \left\{ \eta : \Delta t_w \overset{r - St_{\langle \Theta, \Psi, \Omega \rangle}}{\rightarrow} \eta \right\}.$$

In addition, a sequence $\Delta t = (\Delta t_w)$ is $r_{\langle \Theta, \Psi, \Omega \rangle}$-convergent when $\text{LIM}^{r_{\langle \Theta, \Psi, \Omega \rangle}}_{\Delta t_w} \neq \varnothing$, where

$$\text{LIM}^{r_{\langle \Theta, \Psi, \Omega \rangle}}_{\Delta t_w} = \left\{ \eta^* \in \mathcal{X} : \Delta t_w \overset{r_{\langle \Theta, \Psi, \Omega \rangle}}{\rightarrow} \eta^* \right\}.$$

For unbounded sequence $\text{LIM}^{r_{\langle \Theta, \Psi, \Omega \rangle}}_{\Delta t_w}$ is always empty. Whereas this sequence might be rough statistically convergent, i.e., $St_{\langle \Theta, \Psi, \Omega \rangle} - \text{LIM}^r_{\Delta t_w} \neq \varnothing$. The subsequent example describes this situation.

Example 3.3. Let $(\mathcal{X}, \|.\|)$ be any real normed space. Give the operations $\circ$, $\bullet$ as t-norm $u \circ v = uv$, t-conorm $u \bullet v = u + v - uv$ for all $u, v \in [0, 1]$. For $m > \|\Delta t_w\|$,

$$\Theta(\Delta t_w, m) = \frac{m}{m + \|\Delta t_w\|}, \ \Psi(\Delta t_w, m) = \frac{\|\Delta t_w\|}{m + \|\Delta t_w\|}, \ \Omega(\Delta t_w, m) = \frac{\|\Delta t_w\|}{m + \|\Delta t_w\|}$$

for all $m > 0$ and all $\Delta t = (\Delta t_w) \in \mathcal{X}$. Then four-tuple $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ is an NNS. Let

$$\Delta t_w = \begin{cases} (-1)^w & \text{if } w \neq p^2 \\ w & \text{if not,} \end{cases}$$

i.e.,

$$(\Delta t_w) = (-1, 2, 3, 1, 5, 6, 7, 8, -1, ...).$$

Then we obtain

$$\delta(\{w \in \mathbb{N} : \Theta(\Delta t_w - \eta; r + \xi) \leq 1 - \sigma \text{ or } \Psi(\Delta t_w - \eta; r + \xi) \geq \sigma \text{ or } \Omega(\Delta t_w - \eta; r + \xi) \geq \sigma\}) = 0$$

for every $\xi > 0$ and $\sigma \in (0, 1)$. Also, we get

$$St_{\langle\Theta,\Psi,\Omega\rangle} - \text{LIM}^r_{\Delta t_w} = \begin{cases} \varnothing & \text{if } r < 1 \\ [1 - r, r - 1] & \text{otherwise} \end{cases}$$

and $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w} = \varnothing$ for all $r \geq 0$. Thus the sequence is divergent in ordinary sense since it is unbounded. Moreover, the sequence is not rough convergent in an NNS $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ for any r.

In this section, we direct our attention towards the examination of rough statistically bounded difference sequences in an NNS. The investigation proceeds as follows:

Definition 3.4. A sequence $\Delta t = (\Delta t_w)$ in $\mathcal{X}$ is said to be *rough statistically bounded* w.r.t. the norm $\langle\Theta, \Psi, \Omega\rangle$ for some $r \geq 0$, provided that for every $\xi > 0$ and $\sigma \in (0, 1)$, there is a real number $T > 0$ such that

$$\delta(\{w \in \mathbb{N} : \Theta(\Delta t_w; T) \leq 1 - \sigma \text{ or } \Psi(\Delta t_w; T) \geq \sigma \text{ or } \Omega(\Delta t_w; T) \geq \sigma\}) = 0.$$

Based on the aforementioned definitions, our investigation has yielded noteworthy results concerning the rough statistical convergence of difference sequences in an NNS.

Theorem 3.5. *Consider $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ as an NNS. A sequence $\Delta t = (\Delta t_w)$ in $\mathcal{X}$ is statistically bounded iff $St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w} \neq \varnothing$ for some $r \geq 0$.*

Proof. Necessary part: Consider the sequence $\Delta t = (\Delta t_w)$ which is statistically bounded in $\mathcal{X}$. Then for all $\sigma \in (0, 1)$ and some $r \geq 0$, there is a real number $T > 0$ such that

$$\delta(\{w \in \mathbb{N} : \Theta(\Delta t_w; T) \leq 1 - \sigma \text{ or } \Psi(\Delta t_w; T) \geq \sigma \text{ or } \Omega(\Delta t_w; T) \geq \sigma\}) = 0.$$

Let $P = \{w \in \mathbb{N} : \Theta(\Delta t_w; T) \leq 1 - \sigma \text{ or } \Psi(\Delta t_w; T) \geq \sigma \text{ or } \Omega(\Delta t_w; T) \geq \sigma\}$.
For $w \in P^c$, we have $\Theta(\Delta t_w; T) > 1 - \sigma$, $\Psi(\Delta t_w; T) < \sigma$, $\Psi(\Omega t_w; T) < \sigma$.
Also, we get

$$\begin{aligned} \Theta(\Delta t_w; r + T) &\geq \min\{\Theta(0; r), \Theta(\Delta t_w; T)\} \\ &= \min\{1, \Theta(\Delta t_w; T)\} \\ &> 1 - \sigma, \\ \Psi(\Delta t_w; r + T) &\leq \max\{\Psi(0; r), \Psi(\Delta t_w; T)\} \\ &= \max\{0, \Psi(\Delta t_w; T)\} \\ &< \sigma \end{aligned}$$

and

$$\begin{aligned} \Omega(\Delta t_w; r + T) &\leq \max\{\Omega(0; r), \Omega(\Delta t_w; T)\} \\ &= \max\{0, \Omega(\Delta t_w; T)\} \\ &< \sigma. \end{aligned}$$

Then $0 \in St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$. As a result, $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w} \neq \varnothing$ for some $r \geq 0$.

Sufficient part: Assume that $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w} \neq \varnothing$ for some $r \geq 0$. Then there is a $\eta \in \mathcal{X}$ such that $\eta \in St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$. For each $\xi > 0$ and $\sigma \in (0,1)$, we obtain

$$\delta\left(\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \eta; r+\xi\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma \text{ or } \Omega\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma\}\right) = 0.$$

Thus almost all Δt_w's are included in some ball with center η which gives that sequence $\Delta t = (\Delta t_w)$ is statistically bounded in $\mathcal{X}$. □

If $\Delta t' = \left(\Delta t_{w_j}\right)$ is a subsequence of $\Delta t = (\Delta t_w)$ in an NNS $\mathcal{X}$, then $\text{LIM}^r_{\Delta t_w} \subset \text{LIM}^r_{\Delta t_{w_j}}$. However, it should be noted that this observation does not hold true in the case of statistical convergence. To illustrate this, we provide the following example.

Example 3.6. Let $(\mathcal{X}, \|.\|)$ be any real normed space. Give the operations $\circ$, $\bullet$ as t-norm $u \circ v = uv$, t-conorm $u \bullet v = u + v - uv$ for all $u, v \in [0,1]$. For $m > \|\Delta t_w\|$,

$$\Theta\left(\Delta t_w, m\right) = \tfrac{m}{m+\|\Delta t_w\|},\ \Psi\left(\Delta t_w, m\right) = \tfrac{\|\Delta t_w\|}{m+\|\Delta t_w\|},\ \Omega\left(\Delta t_w, m\right) = \tfrac{\|\Delta t_w\|}{m+\|\Delta t_w\|},$$

for all $m > 0$ and all $\Delta t = (\Delta t_w) \in \mathcal{X}$. Then the four-tuple $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ is an NNS. Let

$$\Delta t_w = \begin{cases} w & \text{when } w \neq p^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w} = [-r, r]$ and $\left(\Delta t_{w_j}\right) = \{1, 4, 9, ...\}$ is a subsequence of (Δt_w). Also, we obtain $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_{w_j}} = \varnothing$.

However, this holds true for non-thin subsequences of the rough statistically convergent difference sequence in an NNS, as demonstrated by the following result.

Theorem 3.7. *If $\Delta t' = \left(\Delta t_{w_j}\right)$ is a nonthin subsequence of $\Delta t = (\Delta t_w)$ in an NNS, then*

$$St_{\langle\Theta,\Psi,\Omega\rangle} - LIM^r_{\Delta t_w} \subset St_{\langle\Theta,\Psi,\Omega\rangle} - LIM^r_{\Delta t_{w_j}}.$$

Proof. It is obvious and so we are omitting it. □

Theorem 3.8. *Consider $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ as an NNS. Take t-norm as $t(u,v) = \min(u,v)$ and t-conorm as $s(u,v) = \max(u,v)$. The set $St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}$ of a difference sequence $\Delta t = (\Delta t_w)$ in $\mathcal{X}$ is a closed set.*

Proof. We have nothing to prove as $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w} = \varnothing$. Let $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w} \neq \varnothing$ for some $r \geq 0$ and consider $\Delta p = (\Delta p_w)$ be a convergent difference sequence in $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$ w.r.t. the norm $\langle\Theta, \Psi, \Omega\rangle$ to $p_0 \in \mathcal{X}$. Then for each $\xi > 0$ and $\sigma \in (0,1)$, there is a $w_1 \in \mathbb{N}$ such that

$$\Theta\left(\Delta p_w - p_0; \frac{\xi}{2}\right) > 1-\sigma,\ \Psi\left(\Delta p_w - p_0; \frac{\xi}{2}\right) < \sigma,\ \Omega\left(\Delta p_w - p_0; \frac{\xi}{2}\right) < \sigma$$

for all $w \geq w_1$.
Let us consider $\Delta p_s \in St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$ with $s > w_1$ such that

$$\delta\left(\left\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_s; r+\tfrac{\xi}{2}\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta t_w - \Delta p_s; r+\tfrac{\xi}{2}\right) \geq \sigma \text{ or } \Omega\left(\Delta t_w - \Delta p_s; r+\tfrac{\xi}{2}\right) \geq \sigma\right\}\right) = 0. \tag{3.1}$$

For

$$j \in \Big\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) > 1 - \sigma,\ \Psi\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) < \sigma, \\ \Omega\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) < \sigma\Big\},$$

we obtain

$$\Theta\left(\Delta t_j - \Delta p_s; r + \frac{\xi}{2}\right) > 1-\sigma,\ \Psi\left(\Delta t_j - \Delta p_s; r + \frac{\xi}{2}\right) < \sigma,\ \Omega\left(\Delta t_j - \Delta p_s; r + \frac{\xi}{2}\right) < \sigma.$$

Then we obtain

$$\begin{aligned} \Theta\left(\Delta t_j - p_0; r + \xi\right) &\geq \min\Big\{\Theta\left(\Delta t_j - \Delta p_s; r + \tfrac{\xi}{2}\right), \Theta\left(\Delta p_s - p_0; r + \tfrac{\xi}{2}\right)\Big\} \\ &> 1 - \sigma, \\ \Psi\left(\Delta t_j - p_0; r + \xi\right) &\leq \max\Big\{\Psi\left(\Delta t_j - \Delta p_s; r + \tfrac{\xi}{2}\right), \Psi\left(\Delta p_s - p_0; r + \tfrac{\xi}{2}\right)\Big\} \\ &< \sigma \end{aligned}$$

and

$$\begin{aligned} \Omega\left(\Delta t_j - p_0; r + \xi\right) &\leq \max\Big\{\Omega\left(\Delta t_j - \Delta p_s; r + \tfrac{\xi}{2}\right), \Omega\left(\Delta p_s - p_0; r + \tfrac{\xi}{2}\right)\Big\} \\ &< \sigma. \end{aligned}$$

Thus we have

$$j \in \{w \in \mathbb{N} : \Theta\left(\Delta t_w - p_0; r + \xi\right) > 1 - \sigma,\ \Psi\left(\Delta t_w - p_0; r + \xi\right) < \sigma, \\ \Omega\left(\Delta t_w - p_0; r + \xi\right) < \sigma\}.$$

Now, we get

$$\begin{aligned} &\Big\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) > 1 - \sigma, \Psi\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) < \sigma, \\ &\Omega\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) < \sigma\Big\} \\ &\subseteq \{w \in \mathbb{N} : \Theta\left(\Delta t_w - p_0; r + \xi\right) > 1 - \sigma, \Psi\left(\Delta t_w - p_0; r + \xi\right) < \sigma, \\ &\Psi\left(\Delta t_w - p_0; r + \xi\right) < \sigma\} \end{aligned}$$

So we have

$$\begin{aligned} &\delta\left(\Big\{w \in \mathbb{N} : \Theta\left(\Delta t_w - p_0; r + \tfrac{\xi}{2}\right) \leq 1 - \sigma \text{ or } \Psi\left(\Delta t_w - p_0; r + \tfrac{\xi}{2}\right) \geq \sigma \right. \\ &\quad \left. \text{or } \Omega\left(\Delta t_w - p_0; r + \tfrac{\xi}{2}\right) \geq \sigma\Big\}\right) \\ &\leq \delta\left(\Big\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) \leq 1 - \sigma \text{ or } \Psi\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) \geq \sigma \right. \\ &\quad \left. \text{or } \Omega\left(\Delta t_w - \Delta p_s; r + \tfrac{\xi}{2}\right) \geq \sigma\Big\}\right). \end{aligned}$$

Utilizing (3.1), we obtain

$$\begin{aligned} &\delta\left(\Big\{w \in \mathbb{N} : \Theta\left(\Delta t_w - p_0; r + \tfrac{\xi}{2}\right) \leq 1 - \sigma, \Psi\left(\Delta t_w - p_0; r + \tfrac{\xi}{2}\right) \geq \sigma, \right. \\ &\left. \Omega\left(\Delta t_w - p_0; r + \tfrac{\xi}{2}\right) \geq \sigma\Big\}\right) = 0. \end{aligned}$$

As a result, $p_0 \in St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$. □

Now, we examine the convexity of the set $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$.

Theorem 3.9. *Let $\Delta t = (\Delta t_w)$ be a sequence in an NNS. Then rough statistical limit set $St_{\langle\Theta,\Psi,\Omega\rangle}-LIM_{\Delta t_w}$ w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ is convex for some $r \geq 0$.*

Proof. Take $\eta_1, \eta_2 \in St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$. For the convexity of the set $St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$, we have to demonstrate that

$$[(1-\alpha)\eta_1 + \alpha\eta_2] \in St_{(\Theta,\Psi)} - LIM^r_{\Delta t_w} \text{ for some } \alpha \in (0,1).$$

For each $\xi > 0$ and $\sigma \in (0,1)$, we consider

$$K_1 = \Big\{ w \in \mathbb{N} : \Theta\left(\Delta t_w - \eta_1; \tfrac{r+\xi}{2(1-\alpha)}\right) \le 1-\sigma \text{ or } \Psi\left(\Delta t_w - \eta_1; \tfrac{r+\xi}{2(1-\alpha)}\right) \ge \sigma \text{ or } \Omega\left(\Delta t_w - \eta_1; \tfrac{r+\xi}{2(1-\alpha)}\right) \ge \sigma \Big\},$$

$$K_2 = \Big\{ w \in \mathbb{N} : \Theta\left(\Delta t_w - \eta_2; \tfrac{r+\xi}{2\alpha}\right) \le 1-\sigma \text{ or } \Psi\left(\Delta t_w - \eta_2; \tfrac{r+\xi}{2\alpha}\right) \ge \sigma \text{ or } \Omega\left(\Delta t_w - \eta_2; \tfrac{r+\xi}{2\alpha}\right) \ge \sigma \Big\}.$$

Since $\eta_1,\ \eta_2 \in St_{\langle\Theta,\Psi,\Omega\rangle}-\text{LIM}^r_{\Delta t_w}$, we get $\delta(K_1) = \delta(K_2) = 0$. For $w \in K_1^c \cap K_2^c$, we get

$$\begin{aligned}
&\Theta\left(\Delta t_w - [(1-\alpha)\eta_1 + \alpha\eta_2]; r+\xi\right) \\
&= \Theta\left((1-\alpha)(\Delta t_w - \eta_1) + \alpha(\Delta t_w - \eta_2); r+\xi\right) \\
&\ge \min\Big\{\Theta\left((1-\alpha)(\Delta t_w - \eta_1); \tfrac{r+\xi}{2}\right), \Theta\left(\alpha(\Delta t_w - \eta_2); \tfrac{r+\xi}{2}\right)\Big\} \\
&= \min\Big\{\Theta\left(\Delta t_w - \eta_1; \tfrac{r+\xi}{2(1-\alpha)}\right), \Theta\left(\Delta t_w - \eta_2, \tfrac{r+\xi}{2\alpha}\right)\Big\} \\
&> 1-\sigma, \\
&\Psi\left(\Delta t_w - [(1-\alpha)\eta_1 + \alpha\eta_2]; r+\xi\right) \\
&= \Psi\left((1-\alpha)(\Delta t_w - \eta_1) + \alpha(\Delta t_w - \eta_2); r+\xi\right) \\
&\le \max\Big\{\Psi\left((1-\alpha)(\Delta t_w - \eta_1); \tfrac{r+\xi}{2}\right), \Psi\left(\alpha(\Delta t_w - \eta_2); \tfrac{r+\xi}{2}\right)\Big\} \\
&= \max\Big\{\Psi\left(\Delta t_w - \eta_1; \tfrac{r+\xi}{2(1-\alpha)}\right), \Psi\left(\Delta t_w - \eta_2, \tfrac{r+\xi}{2\alpha}\right)\Big\} \\
&< \sigma
\end{aligned}$$

and

$$\begin{aligned}
&\Omega\left(\Delta t_w - [(1-\alpha)\eta_1 + \alpha\eta_2]; r+\xi\right) \\
&= \Omega\left((1-\alpha)(\Delta t_w - \eta_1) + \alpha(\Delta t_w - \eta_2); r+\xi\right) \\
&\le \max\Big\{\Omega\left((1-\alpha)(\Delta t_w - \eta_1); \tfrac{r+\xi}{2}\right), \Omega\left(\alpha(\Delta t_w - \eta_2); \tfrac{r+\xi}{2}\right)\Big\} \\
&= \max\Big\{\Omega\left(\Delta t_w - \eta_1; \tfrac{r+\xi}{2(1-\alpha)}\right), \Omega\left(\Delta t_w - \eta_2, \tfrac{r+\xi}{2\alpha}\right)\Big\} \\
&< \sigma.
\end{aligned}$$

Then we have

$\delta\left(\{w \in \mathbb{N} : \Theta\left(\Delta t_w - [(1-\alpha)\eta_1 + \alpha\eta_2]\,; r+\xi\right) \leq 1-\sigma \quad \text{or}\right.$
$\Psi\left(\Delta t_w - [(1-\alpha)\eta_1 + \alpha\eta_2]\,; r+\xi\right) \geq \sigma \text{ or } \Omega\left(\Delta t_w - [(1-\alpha)\eta_1 + \alpha\eta_2]\,; r+\xi\right) \geq \sigma\})$
$= 0.$

As a result, $[(1-\alpha)\eta_1 + \alpha\eta_2] \in St_{\langle\Theta,\Psi,\Omega\rangle}\!-\!\mathrm{LIM}^r_{\Delta t_w}$, i.e., $St_{\langle\Theta,\Psi,\Omega\rangle}\!-\!\mathrm{LIM}^r_{\Delta t_w}$ is a convex set. □

Theorem 3.10. *A sequence $\Delta t = (\Delta t_w)$ in an NNS $\mathcal{X}$ is r-statistically convergent to $\eta \in \mathcal{X}$ w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ for some $r \geq 0$, if there is a sequence $\Delta p = (\Delta p_w)$ in $\mathcal{X}$ which is statistically convergent to $\eta \in \mathcal{X}$ w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ and for each $\sigma \in (0,1)$, we get $\Theta\left(\Delta t_w - \Delta p_w; r\right) > 1-\sigma$, $\Psi\left(\Delta t_w - \Delta p_w; r\right) < \sigma$ and $\Omega\left(\Delta t_w - \Delta p_w; r\right) < \sigma$ for all $w \in \mathbb{N}$.*

Proof. Let $\xi > 0$ and $\sigma \in (0,1)$. Consider $\Delta p_w \overset{St_{\langle\Theta,\Psi,\Omega\rangle}}{\rightarrow} \eta$ and $\Theta\left(\Delta t_w - \Delta p_w; r\right) > 1-\sigma$ and $\Psi\left(\Delta t_w - \Delta p_w; r\right) < \sigma$, $\Omega\left(\Delta t_w - \Delta p_w; r\right) < \sigma$ for all $w \in \mathbb{N}$. For given $\sigma \in (0,1)$, establish

$$K = \{w \in \mathbb{N} : \Theta\left(\Delta p_w - \eta; \xi\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta p_w - \eta; \xi\right) \geq \sigma \text{ or } \Omega\left(\Delta p_w - \eta; \xi\right) \geq \sigma\},$$

$$L = \{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_w; r\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta t_w - \Delta p_w; r\right) \geq \sigma \\ \text{ or } \Omega\left(\Delta t_w - \Delta p_w; r\right) \geq \sigma\}.$$

Obviously, $\delta(K) = 0$ and $\delta(L) = 0$. For $w \in K^c \cap L^c$, we get

$$\Theta\left(\Delta t_w - \eta; r+\xi\right) \geq \min\left[\Theta\left(\Delta t_w - \Delta p_w; r\right), \Theta\left(\Delta p_w - \eta; \xi\right)\right] > 1-\sigma,$$

$$\Psi\left(\Delta t_w - \eta; r+\xi\right) \leq \max\left[\Psi\left(\Delta t_w - \Delta p_w; r\right), \Psi\left(\Delta p_w - \eta; \xi\right)\right] < \sigma$$

and

$$\Omega\left(\Delta t_w - \eta; r+\xi\right) \leq \max\left[\Omega\left(\Delta t_w - \Delta p_w; r\right), \Omega\left(\Delta p_w - \eta; \xi\right)\right] < \sigma.$$

Then

$$\Theta\left(\Delta t_w - \eta; r+\xi\right) > 1-\sigma,\ \Psi\left(\Delta t_w - \eta; r+\xi\right) < \sigma,\ \Omega\left(\Delta t_w - \eta; r+\xi\right) < \sigma$$

for all $w \in K^c \cap L^c$. This gives that

$$\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \eta; r+\xi\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma \\ \text{ or } \Omega\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma\} \subseteq K \cup L.$$

Thus

$$\delta\left(\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \eta; r+\xi\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma \\ \text{ or } \Omega\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma\}\right) \leq \delta(K) + \delta(L).$$

So we get

$$\delta\left(\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \eta; r+\xi\right) \leq 1-\sigma \text{ or } \Psi\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma \\ \text{ or } \Omega\left(\Delta t_w - \eta; r+\xi\right) \geq \sigma\}\right) = 0.$$

As a result, we obtain $\Delta t_w \overset{r-St_{\langle\Theta,\Psi,\Omega\rangle}}{\rightarrow} \eta$. □

Theorem 3.11. *Let $\Delta t = (\Delta t_w)$ be a sequence in an NNS. Then there does not exist elements $u,\ v \in St_{\langle\Theta,\Psi,\Omega\rangle}\!-\!LIM^r_{\Delta t_w}$ for some $r \geq 0$ and each $\sigma \in (0,1)$ such that $\Theta(u-v; sr) \leq 1-\sigma$ or $\Psi(u-v; sr) \geq \sigma$ or $\Omega(u-v; sr) \geq \sigma$ for $s > 2$.*

Proof. We obtain this result by contradiction. Assume that there are elements $u,\ v \in St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$ such that

$$\Theta\left(u-v;sr\right)\leq 1-\sigma \text{ or } \Psi\left(u-v;sr\right)\geq\sigma \text{ or } \Omega(u-v;sr)\geq\sigma \text{ for } s>2. \tag{3.2}$$

As $u,v \in St_{(\Theta,\Psi,\Omega)}-\mathrm{LIM}^r_{\Delta t_w}$, for given $\sigma\in(0,1)$ and all $\xi>0$, we get $\delta(K)=\delta(L)=0$, where

$$K=\Big\{w\in\mathbb{N}:\Theta\left(\Delta t_w-u;r+\tfrac{\xi}{2}\right)\leq 1-\sigma \text{ or } \Psi\left(\Delta t_w-u;r+\tfrac{\xi}{2}\right)\geq\sigma \text{ or } \Omega\left(\Delta t_w-u;r+\tfrac{\xi}{2}\right)\geq\sigma\Big\}$$

and

$$L=\Big\{w\in\mathbb{N}:\Theta\left(\Delta t_w-v;r+\tfrac{\xi}{2}\right)\leq 1-\sigma \text{ or } \Psi\left(\Delta t_w-v;r+\tfrac{\xi}{2}\right)\geq\sigma \text{ or } \Omega\left(\Delta t_w-v;r+\tfrac{\xi}{2}\right)\geq\sigma\Big\}.$$

For $w\in K^c\cap L^c$, we obtain

$$\Theta(u-v;2r+\xi)\geq\min\left\{\Theta\left(\Delta t_w-v;r+\frac{\xi}{2}\right),\Theta\left(\Delta t_w-u;r+\frac{\xi}{2}\right)\right\}>1-\sigma,$$

$$\Psi(u-v;2r+\xi)\leq\max\left\{\Psi\left(\Delta t_w-v;r+\frac{\xi}{2}\right),\Psi\left(\Delta t_w-u;r+\frac{\xi}{2}\right)\right\}<\sigma$$

and

$$\Omega(u-v;2r+\xi)\leq\max\left\{\Omega\left(\Delta t_w-v;r+\frac{\xi}{2}\right),\Omega\left(\Delta t_w-u;r+\frac{\xi}{2}\right)\right\}<\sigma.$$

Then we have

$$\Theta\left(u-v;2r+\xi\right)>1-\sigma,\ \Psi\left(u-v;2r+\xi\right)<\sigma,\ \Omega(u-v;2r+\xi)<\sigma. \tag{3.3}$$

Thus from (3.3), we get

$$\Theta(u-v;sr)>1-\sigma,\ \Psi(u-v;sr)<\sigma,\ \Omega(u-v;sr)<\sigma \text{ for } s>2$$

which is a contradiction to (3.2). So there are not elements $u,\ v\in St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$ such that $\Theta(u-v;sr)\leq 1-\sigma$ or $\Psi(u-v;sr)\geq\sigma$ or $\Omega(u-v;sr)\geq\sigma$ for $s>2$. □

Now, we investigate statistical cluster point of a difference sequence in NNS and obtain various results related to it.

Definition 3.12. Let $\mathcal{X}=(F,\mathcal{N},\circ,\bullet)$ be an NNS. Then $\lambda\in\mathcal{X}$ is called a *rough statistical cluster point* of a sequence $\Delta t=(\Delta t_w)$ in $\mathcal{X}$ w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ for some $r\geq 0$, if for each $\xi>0$ and $\sigma\in(0,1)$,

$$\delta\left(\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\lambda;r+\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\lambda;r+\xi\right)<\sigma, \Omega\left(\Delta t_w-\lambda;r+\xi\right)<\sigma\}\right)>0,$$

i.e.,

$$\delta\left(\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\lambda;r+\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\lambda;r+\xi\right)<\sigma, \Omega\left(\Delta t_w-\lambda;r+\xi\right)<\sigma\}\right)\neq 0.$$

In this case, λ is called a $r-St_{\langle\Theta,\Psi,\Omega\rangle}$-*cluster point* of a sequence $\Delta t=(\Delta t_w)$ w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$.

Let $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$ indicates the set of all $r-St_{\langle\Theta,\Psi,\Omega\rangle}$-cluster points w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ of a sequence $\Delta t=(\Delta t_w)$ in an NNS. If $r=0$, then we obtain an ordinary statistical cluster point w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ in an NNS, i.e., $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)=\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$.

Theorem 3.13. *Let $\mathcal{X}=(F,\mathcal{N},\circ,\bullet)$ be an NNS. Then, the set $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$, defined as the collection of all $r-St_{\langle\Theta,\Psi,\Omega\rangle}$-cluster points w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ for any sequence $\Delta t=(\Delta t_w)$, is closed for some $r\geq 0$.*

Proof. When $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)=\varnothing$, then we have to prove nothing.

Suppose $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\neq\varnothing$ and let $\Delta p=(\Delta p_w)$ be a sequence in $\mathcal{X}$ such that

$$\Delta p\subseteq\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\text{ and }(\Delta p_w)\overset{\langle\Theta,\Psi,\Omega\rangle}{\to}p^*.$$

It is adequate to denote that $p^*\in\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. Since $\Delta p_w\overset{\langle\Theta,\Psi,\Omega\rangle}{\to}p^*$, for each $\xi>0$ and $\sigma\in(0,1)$, there is a $w_\xi\in\mathbb{N}$ such that for $w\geq w_\xi$,

$$\Theta\left(\Delta p_w-p^*;\frac{\xi}{2}\right)>1-\sigma,\ \Psi\left(\Delta p_w-p^*;\frac{\xi}{2}\right)<\sigma,\ \Omega\left(\Delta p_w-p^*;\frac{\xi}{2}\right)<\sigma.$$

Now select $w_0\in\mathbb{N}$ such that $w_0\geq w_\xi$. Then we get

$$\Theta\left(\Delta p_{w_0}-p^*;\frac{\xi}{2}\right)>1-\sigma,\ \Psi\left(\Delta p_{w_0}-p^*;\frac{\xi}{2}\right)<\sigma,\ \Omega\left(\Delta p_{w_0}-p^*;\frac{\xi}{2}\right)<\sigma.$$

Again as $\Delta p\subseteq\Gamma^r_{(\Theta,\Psi,\Omega)}(\Delta t)$, $\Delta p_{w_0}\in\Gamma^r_{(\Theta,\Psi,\Omega)}(\Delta t)$. Thus we have

(3.4)
$$\delta\left(\left\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)>1-\sigma,\Psi\left(\Delta t_w-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)<\sigma,\right.\right.$$
$$\left.\left.\Omega\left(\Delta t_w-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)<\sigma\right\}\right)>0.$$

Now let

$$j\in\left\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)>1-\sigma,\Psi\left(\Delta t_w-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)<\sigma,\right.$$
$$\left.\Omega\left(\Delta t_w-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)<\sigma\right\}.$$

Then we get

$$\Theta\left(\Delta t_j-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)>1-\sigma,\Psi\left(\Delta t_j-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)<\sigma,$$
$$\Omega\left(\Delta t_j-\Delta p_{w_0};r+\tfrac{\xi}{2}\right)<\sigma.$$

Thus we have

$$\begin{aligned}\Theta\left(\Delta t_j-p^*;r+\xi\right)&\geq\min\left\{\Theta\left(\Delta t_j-\Delta p_{w_0};r+\tfrac{\xi}{2}\right),\Theta\left(\Delta p_{w_0}-p^*;\tfrac{\xi}{2}\right)\right\}\\&>1-\sigma,\\\Psi\left(\Delta t_j-p^*;r+\xi\right)&\leq\max\left\{\Psi\left(\Delta t_j-\Delta p_{w_0};r+\tfrac{\xi}{2}\right),\Psi\left(\Delta p_{w_0}-p^*;\tfrac{\xi}{2}\right)\right\}\\&<\sigma\end{aligned}$$

and

$$\begin{aligned}\Omega\left(\Delta t_j-p^*;r+\xi\right)&\leq\max\left\{\Omega\left(\Delta t_j-\Delta p_{w_0};r+\tfrac{\xi}{2}\right),\Omega\left(\Delta p_{w_0}-p^*;\tfrac{\xi}{2}\right)\right\}\\&<\sigma.\end{aligned}$$

So we have

$$j \in \{\Theta(\Delta t_w - p^*; r+\xi) > 1-\sigma, \Psi(\Delta t_w - p^*; r+\xi) < \sigma, \\ \Omega(\Delta t_w - p^*; r+\xi) < \sigma\}.$$

Hence we get

$$\begin{aligned} &\left\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_{w_0}; r+\tfrac{\xi}{2}\right) > 1-\sigma, \Psi\left(\Delta t_w - \Delta p_{w_0}; r+\tfrac{\xi}{2}\right) < \sigma, \\ &\qquad \Omega\left(\Delta t_w - \Delta p_{w_0}; r+\tfrac{\xi}{2}\right) < \sigma\right\} \\ &\subseteq \{w \in \mathbb{N} : \Theta(\Delta t_w - p^*; r+\xi) > 1-\sigma, \Psi(\Delta t_w - p^*; r+\xi) < \sigma, \\ &\qquad \Omega(\Delta t_w - p^*; r+\xi) < \sigma\}. \end{aligned}$$

On the other hand, we have

(3.5)

$$\begin{aligned} &\delta\left(\left\{w \in \mathbb{N} : \Theta\left(\Delta t_w - \Delta p_{w_0}; r+\tfrac{\xi}{2}\right) > 1+\sigma, \Psi\left(\Delta t_w - \Delta p_{w_0}; r+\tfrac{\xi}{2}\right) < \sigma, \\ &\qquad \Omega\left(\Delta t_w - \Delta p_{w_0}; r+\tfrac{\xi}{2}\right) < \sigma\right\}\right) \\ &\leq \delta(\{w \in \mathbb{N} : \Theta(\Delta t_w - p^*; r+\xi) > 1+\sigma, \Psi(\Delta t_w - p^*; r+\xi) < \sigma, \\ &\qquad \Omega(\Delta t_w - p^*; r+\xi) < \sigma\}). \end{aligned}$$

Using equation (3.4), we have determined that the set on the left-hand side of equation (3.5) possesses a natural density greater than zero.

$$\delta(\{w \in \mathbb{N} : \Theta(\Delta t_w - p^*; r+\xi) > 1+\sigma, \Psi(\Delta t_w - p^*; r+\xi) < \sigma, \\ \Omega(\Delta t_w - p^*; r+\xi) < \sigma\}) > 0.$$

As a result, $p^* \in \Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. □

Theorem 3.14. *Let $\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$ be the set of all statistical cluster points w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ of a sequence $\Delta t = (\Delta t_w)$ in an NNS $\mathcal{X} = (F, \mathcal{N}, \circ, \bullet)$ and $r \geq 0$. Then for an arbitrary $\lambda \in \Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$ and $\sigma \in (0,1)$, we have $\Theta(\gamma-\lambda) > 1+\sigma$, $\Psi(\gamma-\lambda) < \sigma$, $\Omega(\gamma-\lambda) < \sigma$ for all $\gamma \in \Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$.*

Proof. Let $\lambda \in \Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. Then for all $\xi > 0$ and $\sigma \in (0,1)$, we get

$$\delta(\{w \in \mathbb{N} : \Theta(\Delta t_w - \lambda; \xi) > 1-\sigma, \Psi(\Delta t_w - \lambda; \xi) < \sigma, \\ \Omega(\Delta t_w - \lambda; \xi) < \sigma\}) > 0. \tag{3.6}$$

Now, we will demonstrate that if $\gamma \in \mathcal{X}$ has the following conditions:

$$\Theta(\gamma-\lambda; r) > 1-\sigma, \; \Psi(\gamma-\lambda; r) < \sigma, \; \Omega(\gamma-\lambda; r) < \sigma,$$

then $\gamma \in \Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. Let

$$j \in \{w \in \mathbb{N} : \Theta(\Delta t_w - \lambda; \xi) > 1-\sigma, \Psi(\Delta t_w - \lambda; \xi) < \sigma, \\ \Omega(\Delta t_w - \lambda; \xi) < \sigma\}.$$

Then $\Theta(\Delta t_j - \lambda; \xi) > 1-\sigma$, $\Psi(\Delta t_j - \lambda; \xi) < \sigma$, $\Omega(\Delta t_j - \lambda; \xi) < \sigma$. On the other hand, we get

$$\begin{aligned} \Theta(\Delta t_j - \gamma; r+\xi) &\geq \min\{\Theta(\Delta t_j - \lambda; \xi), \Theta(\gamma-\lambda; r)\} \\ &> 1-\sigma, \\ \Psi(\Delta t_j - \gamma; r+\xi) &\leq \max\{\Psi(\Delta t_j - \lambda; \xi), \Psi(\gamma-\lambda; r)\} \\ &< \sigma \end{aligned}$$

and

$$\begin{aligned}\Omega\left(\Delta t_{w_j}-\gamma;r+\xi\right) &\leq \max\left\{\Omega\left(\Delta t_{w_j}-\lambda;\xi\right),\Omega(\gamma-\lambda;r)\right\}\\ &< \sigma.\end{aligned}$$

Thus we have

$$\Theta\left(\Delta t_j-\gamma;r+\xi\right)>1-\sigma,\ \Psi\left(\Delta t_j-\gamma;r+\xi\right)<\sigma,\ \Omega\left(\Delta t_j-\gamma;r+\xi\right)<\sigma.$$

So we obtain

$$j\in\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\gamma;r+\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\gamma;r+\xi\right)<\sigma,\\ \Omega\left(\Delta t_w-\gamma;r+\xi\right)<\sigma\}.$$

Furthermore, the following inclusion holds:

$$\begin{aligned}&\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\lambda;\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\lambda;\xi\right)<\sigma,\,\Omega\left(\Delta t_w-\lambda;\xi\right)<\sigma\}\\ &\subseteq\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\gamma;r+\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\gamma;r+\xi\right)<\sigma,\\ &\qquad\Omega\left(\Delta t_w-\gamma;r+\xi\right)<\sigma\}.\end{aligned}$$

Hence we get

$$\begin{aligned}&\delta\left(\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\lambda;\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\lambda;\xi\right)<\sigma,\,\Omega\left(\Delta t_w-\lambda;\xi\right)<\sigma\}\right)\\ &\leq\delta\left(\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\gamma;r+\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\gamma;r+\xi\right)<\sigma,\right.\\ &\qquad\left.\Omega\left(\Delta t_w-\gamma;r+\xi\right)<\sigma\}\right).\end{aligned}$$

By employing equation (3.6), we obtain

$$\delta\left(\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\gamma;r+\xi\right)>1-\sigma,\Psi\left(\Delta t_w-\gamma;r+\xi\right)<\sigma,\right.\\ \left.\Omega\left(\Delta t_w-\gamma;r+\xi\right)<\sigma\}\right)>0.$$

As a result, $\gamma\in\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. □

Theorem 3.15. *If*

$$\overline{B(\kappa,\sigma,r)}=\{\Delta t\in\mathcal{X}:\Theta(\Delta t-\kappa;r)\geq 1-\sigma,\Psi(\Delta t-\kappa;r)\leq\sigma,\Omega(\Delta t-\kappa;r)\leq\sigma\}$$

represents the closure of open ball

$$B(\kappa,\sigma,r)=\{\Delta t\in\mathcal{X}:\Theta(\Delta t-\kappa;r)>1-\sigma,\Psi(\Delta t-\kappa;r)<\sigma,\Omega(\Delta t-\kappa;r)<\sigma\}$$

for some $r>0$, $\sigma\in(0,1)$ *and a fixed* $\kappa\in\mathcal{X}$*, then*

$$\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)=\bigcup_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}.$$

Proof. Let $\lambda\in\bigcup_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}$. Then there is a $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$ for some $r\geq 0$ and given $\sigma\in(0,1)$ such that $\Theta(\kappa-\lambda;r)>1-\sigma$, $\Psi(\kappa-\lambda;r)<\sigma$, $\Omega(\kappa-\lambda;r)<\sigma$. Fix $\xi>0$. As $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$, there is a set

$$K=\{w\in\mathcal{X}:\Theta\left(\Delta t_w-\kappa;\xi\right)>1-\sigma\text{ and }\Psi\left(\Delta t_w-\kappa;\xi\right)<\sigma,\,\Omega\left(\Delta t_w-\kappa;\xi\right)<\sigma\},$$

with $\delta(K)>0$. Now, for $w\in K$,

$$\begin{aligned}\Theta\left(\Delta t_w-\lambda;r+\xi\right) &\geq\min\left\{\Theta\left(\Delta t_w-\kappa;\xi\right),\Theta(\kappa-\lambda;r)\right\}\\ &>1-\sigma,\end{aligned}$$

$$\begin{aligned}\Psi\left(\Delta t_w-\lambda;r+\xi\right) &\leq\max\left\{\Psi\left(\Delta t_w-\kappa;\xi\right),\Psi(\kappa-\lambda;r)\right\}\\ &<\sigma\end{aligned}$$

and

$$\begin{aligned}\Omega\left(\Delta t_w-\lambda;r+\xi\right) &\leq \max\left\{\Omega\left(\Delta t_w-\kappa;\xi\right),\Omega(\kappa-\lambda;r)\right\}\\ &< \sigma.\end{aligned}$$

This gives that

$$\delta\left(\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\lambda;r+\xi\right)>1-\sigma \text{ and } \Psi\left(\Delta t_w-\lambda;r+\xi\right)<\sigma \\ \Omega\left(\Delta t_w-\lambda;r+\xi\right)<\sigma\}\right)>0.$$

Thus $\lambda\in\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. As a result, $\bigcup\limits_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}\subseteq\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)(\Delta t)$.

Conversely, let $\lambda\in\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. Then clearly, $\lambda\in\bigcup\limits_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}$. Assume that $\lambda\notin\bigcup\limits_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}$, i.e., $\lambda\notin\overline{B(\kappa,\sigma,r)}$ for all $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. Then we have: for all $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$,

$$\Theta(\lambda-\kappa;r)\leq 1-\sigma \text{ or } \Psi(\lambda-\kappa;r)\geq\sigma \text{ or } \Omega(\lambda-\kappa;r)\geq\sigma.$$

Let $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}\left(\Delta t\right)$. Then clearly, $\kappa\in\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}\left(\Delta t\right)$. Thus by Theorem 3.14, we get

$$\Theta\left(\lambda-\kappa;r\right)>1-\sigma,\ \Psi\left(\lambda-\kappa;r\right)<\sigma,\ \Omega\left(\lambda-\kappa;r\right)<\sigma.$$

This is a contradiction to the supposition. So $\lambda\in\bigcup\limits_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B\left(\kappa,\sigma,r\right)}$. Hence $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}\left(\Delta t\right)\subseteq\bigcup\limits_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B\left(\kappa,\sigma,r\right)}$. □

Theorem 3.16. *Let $\Delta t=(\Delta t_w)$ be a sequence in an NNS $\mathcal{X}=(F,\mathcal{N},\circ,\bullet)$. Then for any $\sigma\in(0,1)$,*

(1) *if $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}\left(\Delta t\right)$, then $St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}\subseteq\overline{B\left(\kappa,\sigma,r\right)}$,*

(2) $St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}=\bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}$
$=\left\{\gamma\in\mathcal{X}:\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subseteq\overline{B(\gamma,\sigma,r)}\right\}.$

Proof. (1) Suppose $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}\left(\Delta t\right)$ and let $\gamma\in St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}$. Then for all $\xi>0$ and $\sigma\in(0,1)$, we establish the sets

$$K=\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\gamma;r+\xi\right)>1-\sigma,\ \Psi\left(\Delta t_w-\gamma;r+\xi\right)<\sigma,\\ \Omega\left(\Delta t_w-\gamma;r+\xi\right)<\sigma\}$$

with $\delta\left(K^c\right)=0$
and

$$L=\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\kappa;\xi\right)>1-\sigma \text{ and } \Psi\left(\Delta t_w-\kappa;\xi\right)<\sigma,\ \Omega\left(\Delta t_w-\kappa;\xi\right)<\sigma\}$$

with $\delta(L)\neq 0$.
Then for each $w\in K\cap L$, we have

$$\begin{aligned}\Theta(\gamma-\kappa;r) &\geq \min\left\{\Theta\left(\Delta t_w-\kappa;\xi\right),\Theta\left(\Delta t_w-\gamma;r+\xi\right)\right\}\\ &>1-\sigma,\end{aligned}$$

$$\begin{aligned}\Psi(\gamma-\kappa;r) &\leq \max\left\{\Psi\left(\Delta t_w-\kappa;\xi\right),\Psi\left(\Delta t_w-\gamma;r+\xi\right)\right\}\\ &<\sigma\end{aligned}$$

and

$$\begin{aligned}\Omega(\gamma-\kappa;r) &\leq \max\left\{\Omega\left(\Delta t_w-\kappa;\xi\right),\Omega\left(\Delta t_w-\gamma;r+\xi\right)\right\}\\ &<\sigma.\end{aligned}$$

Thus $\gamma\in\overline{B(\kappa,\sigma,r)}$. So $St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}\subseteq\overline{B(\kappa,\sigma,r)}$.

(2) From (1), we get

$$St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}\subseteq\bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}.$$

Let $s\in\bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}$. Then we have: for all $\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$,

$$\Theta(s-\kappa;r)\geq 1-\sigma,\ \ \Psi(s-\kappa;r)\leq\sigma,\ \ \Omega(s-\kappa;r)\leq\sigma.$$

Thus $\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subseteq\overline{B(s,\sigma,r)}$, i.e.,

$$\bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}\subseteq\left\{\gamma\in\mathcal{X}:\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subseteq\overline{B(\gamma,\sigma,r)}.\right\}$$

In addition, let $s\notin St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}$. Then for $\xi>0$, we obtain

$$\begin{aligned}\delta\left(\left\{w\in\mathbb{N}:\Theta\left(\Delta t_w-s;r+\xi\right)\leq 1-\sigma \text{ or } \Psi\left(\Delta t_w-s;r+\xi\right)\geq\sigma\right.\right.\\ \left.\left.\text{or } \Omega\left(\Delta t_w-s;r+\xi\right)\geq\sigma\right\}\right)\neq 0.\end{aligned}$$

This gives that a statistical cluster point κ exists for the sequence $\Delta t=(\Delta t_w)$ such that

$$\Theta(s-\kappa;r+\xi)\leq 1-\sigma\ \text{ or }\ \Psi(s-\kappa;r+\xi)\geq\sigma\ \text{ or }\ \Omega(s-\kappa;r+\xi)\geq\sigma.$$

Thus $\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subsetneq\overline{B(s,\sigma,r)}$ and $s\notin\left\{\gamma\in\mathcal{X}:\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subseteq\overline{B(\gamma,\sigma,r)}\right\}$. So we have

$$\left\{\gamma\in\mathcal{X}:\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subseteq\overline{B(\gamma,\sigma,r)}\right\}\subseteq St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}.$$

Hence we get

$$\bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}\subseteq St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$$

Therefore we have

$$St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}=\bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)}\overline{B(\kappa,\sigma,r)}=\left\{\gamma\in\mathcal{X}:\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)\subseteq\overline{B(\gamma,\sigma,r)}\right\}.$$

□

Theorem 3.17. *Let $\Delta t=(\Delta t_w)$ be a sequence in an NNS which is statistically convergent to $\eta\in\mathcal{X}$ w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$. Then there exists $\sigma\in(0,1)$ such that*

$$St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}=\overline{B(\eta,\sigma,r)}\ \textit{for some } r\geq 0.$$

Proof. Take $\xi>0$. Since $\Delta t_w\overset{St_{\langle\Theta,\Psi,\Omega\rangle}}{\to}\eta$, there is a set

$$\begin{aligned}K=\left\{w\in\mathbb{N}:\Theta\left(\Delta t_w-\eta;\xi\right)\leq 1-\sigma \text{ or } \Psi\left(\Delta t_w-\eta;\xi\right)\geq\sigma\right.\\ \left.\text{or } \Omega\left(\Delta t_w-\eta;\xi\right)\geq\sigma\right\}\ \text{with } \delta(K)=0.\end{aligned}$$

Let

$$s\in\overline{B(\eta,\sigma,r)}=\{s\in\mathcal{X}:\Theta(s-\eta;\xi)\geq 1-\sigma \text{ or } \Psi(s-\eta;\xi)\leq\sigma \text{ or } \Omega(s-\eta;\xi)\leq\sigma\}.$$

Then for each $w \in K^c$, we get

$$\begin{aligned}\Theta\left(\Delta t_w - s; r+\xi\right) &\geq \min\left\{\Theta\left(\Delta t_w - \eta; \xi\right), \Theta(s-\eta; r)\right\}\\ &> 1-\sigma,\\ \Psi\left(\Delta t_w - s; r+\xi\right) &\leq \max\left\{\Psi\left(\Delta t_w - \eta; \xi\right), \Psi(s-\eta; r)\right\}\\ &< \sigma\end{aligned}$$

and

$$\begin{aligned}\Omega\left(\Delta t_w - s; r+\xi\right) &\leq \max\left\{\Omega\left(\Delta t_w - \eta; \xi\right), \Omega(s-\eta; r)\right\}\\ &< \sigma.\end{aligned}$$

Thus $s \in St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$, i.e., $\overline{B(\eta,\sigma,r)} \subseteq St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$. Note that $St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w} \subseteq \overline{B(\eta,\sigma,r)}$. As a result, we obtain $St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w} = \overline{B(\eta,\sigma,r)}$ for some $r \geq 0$. □

Theorem 3.18. *Let $\Delta t = (\Delta t_w)$ be a sequence in an NNS. Then Δt is statistically convergent w.r.t the norm $\langle\Theta,\Psi,\Omega\rangle$ if and only if $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t) = St_{\langle\Theta,\Psi,\Omega\rangle}-LIM^r_{\Delta t_w}$ for some $r \geq 0$.*

Proof. Necessary part: Suppose $\Delta t_w \overset{St_{\langle\Theta,\Psi,\Omega\rangle}}{\rightarrow} \eta$. Then $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t) = \{\eta\}$. Then by Theorem 3.15, for some $r \geq 0$ and $\sigma \in (0,1)$, $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t) = \overline{B(\eta,\sigma,r)}$. Thus by Theorem 3.17, we obtain $\overline{B(\eta,\sigma,r)} = St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$. So $\Gamma^r_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t) = St_{\langle\Theta,\Psi,\Omega\rangle}-\mathrm{LIM}^r_{\Delta t_w}$.

Sufficient part: Suppose the necessary condition holds. Then by Theorem 3.15 and Theorem 3.16 (2), we get

$$\bigcup_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)} \overline{B(\kappa,\sigma,r)} = \bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)} \overline{B(\kappa,\sigma,r)}.$$

This gives that either $\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t) = \varnothing$ for $\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$ is a singleton set. Thus

$$St_{\langle\Theta,\Psi,\Omega\rangle} - \mathrm{LIM}^r_{\Delta t_w} = \bigcap_{\kappa\in\Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)} \overline{B(\kappa,\sigma,r)} = \overline{B(\eta,\sigma,r)}$$

for some $\eta \in \Gamma_{\langle\Theta,\Psi,\Omega\rangle}(\Delta t)$. So by Theorem 3.17, $St_{\langle\Theta,\Psi,\Omega\rangle} - LIM^r_{\Delta t_w} = \{\eta\}$. □

4. Conclusion

In this paper, we have extended the existing theories on sequence convergence in NNSs by introducing the concept of rough statistical convergence for difference sequences in NNSs. We have investigated novel notions of rough convergence and rough statistical convergence, specifically focusing on the behavior of difference sequences in NNSs.

Moreover, we have thoroughly analyzed the properties and characteristics of a mathematical construct denoted as $St_{(\Theta,\Psi,\Omega)} - \mathrm{LIM}^r_{\Delta t_w}$, which represents the r-statistical limit set of the difference sequence (Δt_w). Our examination of these features has provided valuable insights into the behavior and attributes associated with the r-statistical limit set in the context of rough statistical convergence in NNSs.

Two potential future applications stemming from the understanding of convergence behavior in NNSs and their difference sequences are optimized decision-making systems and analysis of data with inherent uncertainty.

The insights gained from this research can be leveraged to develop more efficient decision-making systems that can handle complex and uncertain data in various domains such as finance, healthcare and engineering.

By expanding the existing theories and introducing new concepts, our study contributes to the deeper understanding of sequence convergence in NNSs. The findings presented in this paper pave the way for further research and applications in the field of NNSs, offering potential avenues for exploring the convergence behavior of difference sequences and advancing the theoretical framework in this area.

Additionally, the understanding of rough statistical convergence in NNSs can contribute to the analysis and interpretation of data sets characterized by inherent uncertainties, enabling more accurate predictions and informed decision-making processes in fields such as data science, machine learning and artificial intelligence.

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