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# Maximal and minimal neighborhoods of rough fuzzy sets

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Abstract. It has a comment on two patterns of generalizations of the roughness of some fuzzy set. The authors in 2022, based on the minimal neighborhoods of rough fuzzy sets, introduced a generalization of rough fuzzy sets. This paper will be used maximal neighborhoods in defining a new generalization of rough fuzzy sets in a pattern similar to this generalization introduced by the authors in 2022. Mainly, it is shown that the new boundary region set computed using maximal neighborhoods does not depend on that set computed using minimal neighborhoods as given by the authors in 2022. As an application, it is shown that the connectedness of rough fuzzy topological spaces could be defined using maximal neighborhoods. Still, this connectedness is not related to the connectedness expressed by the authors in 2022 using minimal neighborhoods.

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## 1. Introduction and Preliminaeies

Pawlak basically defined the notion of the rough set in [1] based on an equivalence relation $R$ defined on a universal finite set $X$. Based on the resulting equivalence classes, some objects are found in a vague area called the boundary region that could not be determined by the set or its complement. The pair $(X, R)$ was called an approximation space. Many authors studied the roughness notion but based on a more generalized relation on $X$ cited in [2, 3, 4, 5, 6]. In [7], it was studied the maximal right neighborhoods related to the arbitrary relation defined on $X$. Also, in [8], it was studied the maximal left neighborhoods related to the arbitrary relation defined on $X$, and both could be as a dual work to the other one. Many researchers made a wide research on rough sets and its application as given in [9, 10, 11, 12, 13]. The boundary region area became an essential role in artificial intelligence, granular computing and decision analysis. The topology $\tau_R$ generated in an approximation space $(X, R)$ ensuring some topological properties of rough sets were studied by some authors cited in [14, 15]. In [16], the authors based on the minimal neighborhoods made a generalization of the roughness notion for the fuzzy sets and consequently for the crisp sets. Accompanying a fuzzy ideal $\ell$ ([17]) on a fuzzy approximation space $(X, R)$, is a generalization in defining the rough fuzzy sets as shown in [16]. It was explained in [16] a generalization of the definitions of [1, 2, 4, 5, 6]. A clearer form of the accuracy fuzzy set of roughness was given in [16]. Many applications of rough sets are used in real life problems such as given in [18, 19, 20].

In this paper, we modified our Definitions in [16], to give a similar pattern of definitions but based on the maximal neighborhoods. About the accuracy of roughness, we clarified the computations into a suitable form in the crisp case different from the accuracy of roughness in the fuzzy case. The main result is that: Using the maximal neighborhoods in defining the fuzzy roughness (in a pattern similar to that defined in [16] we produced a new generalized form of fuzzy roughness. But the resulting boundary region fuzzy set is not related to that found in [16]. While in the crisp case, using minimal neighborhoods as defined in [16] produces a boundary region set fewer than that produced if we used maximal neighborhoods as given in this paper. Also, we explained even if constructed a fuzzy topological space and studied some topological notions like fuzzy connectedness based on maximal neighborhoods, we get that this fuzzy connectedness is not related to that fuzzy connectedness defined in [16] by minimal neighborhoods. While in the crisp case, if $(X, \tau_R)$ is a connected space as shown in [16], then it is a connected space which is defined by using maximal neighborhoods but not converse. The core of that difference between crisp case and fuzzy case is coming from $A \cap A^c = \phi$ may not be correct in the fuzzy case. Thus, both $A$ and $A^c$ may belong to the ideal $\ell$ and both may not belong to $\ell$.

Let $X$ be a finite set of objects and $I$ the closed unit interval $[0, 1]$. Then $I^X$ denotes all fuzzy subsets of $X$ and $2^X$ denotes all crisp subsets (ordinary sets) of $X$. For any $\lambda,\ \mu \in I^X$, the *complement* $\lambda^c$ of $\lambda$, the *union* $\lambda \vee \mu$ of $\lambda$ and $\mu$, and the *intersection* $\lambda \wedge \mu$ of $\lambda$ and $\mu$, are respectively defined as follows (See [21]): for each $x \in X$,

$$\lambda^c(x) = 1 - \lambda(x),\ (\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x),\ (\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x).$$

For any $\lambda,\ \mu \in I^X$, the *order* $\lambda \leq \mu$ between $\lambda$ and $\mu$, and for each $t \in I$, a *constant fuzzy set* $\bar{t}$ in $X$ are respectively defined by: for each $x \in X$,

$$\lambda \leq \mu \Leftrightarrow \lambda(x) \leq \mu(x),\ \bar{t}(x) = t.$$

For each $\lambda \in I^X$, the Infimum and the supremum of $\lambda$ are respectively given as:

$$\inf \lambda = \bigwedge_{x \in X} \lambda(x) \text{ and } \sup \lambda = \bigvee_{x \in X} \lambda(x).$$

Recall that the *fuzzy difference* between two fuzzy sets was defined ([22]) as:

$$(\lambda \mathbin{\bar{\wedge}} \mu) = \begin{cases} \bar{0} & \text{if } \lambda \leq \mu, \\ \lambda \wedge \mu^c & \text{otherwise.} \end{cases} \tag{1.1}$$

A subset $\ell \subset I^X$ is called a *fuzzy ideal* ([17]) on $X$, if it satisfies the following conditions: for any $\mu,\ \nu \in I^X$,

(i) $\overline{0} \in \ell$,

(ii) if $\nu \leq \mu$ and $\mu \in \ell$, then $\nu \in \ell$,

(iii) if $\mu \in \ell$ and $\nu \in \ell$, then $(\mu \vee \nu) \in \ell$.

Usually, we consider the proper fuzzy ideal $\ell$ $(\overline{1} \notin \ell)$. Denote the fuzzy ideal $\ell^\circ$ for a fuzzy ideal including only $\overline{0}$.

## 2. Rough fuzzy sets

First of all, we recall the concept of the lower approximations and the upper approximations proposed by some researchers.

Pawlak [1]: Let $R$ be an equivalence relation on $X$ and let $[x]_R$ be the equivalence class containing $x$. Then for any subset $A$ of $X$, the *lower approximation* $\underline{R}(A)$ and the *upper approximation* $\bar{R}(A)$ are defined by:

$$\underline{R}(A) = \{x \in X : [x]_R \subseteq A\}, \quad \bar{R}(A) = \{x \in X : [x]_R \cap A \neq \varnothing\}.$$

Yao [6]: Let $R$ be a binary relation on $X$. Then for any subset $A$ of $X$, the *lower approximation* $\underline{R}(A)$ and the *upper approximation* $\bar{R}(A)$ are defined by:

$$\underline{R}(A) = \{x \in X : xR \subseteq A\}, \quad \bar{R}(A) = \{x \in X : xR \cap A \neq \varnothing\},$$

where $xR$ is called the *after set* of $x$ defined by:

$$xR = \{y \in X : xRy\}.$$

Moreover, $Rx$ is called the *fore set* of $x$ defined by:

$$Rx = \{y \in X : yRx\}.$$

It is clear that Pawlak's Definition is a special case of Yao's Definition.

Allam [2]: Let $R$ be a reflexive binary relation on $X$ and let $[p]R$ b the intersection of all the after sets $xR$ containing $p \in X$. Then for any subset $A$ of $X$, the *lower approximation* $\underline{R}(A)$ and the *upper approximation* $\bar{R}(A)$ are defined by:

$$\underline{R}(A) = \{x \in X : [x]R \subseteq A\}, \quad \bar{R}(A) = \{x \in X : [x]R \cap A \neq \varnothing\},$$

where $[p]R$ is defined by:

$$[p]R \ = \begin{cases} \bigcap\limits_{p \in xR} xR & \text{if } \exists x : p \in xR, \\ \varnothing & \text{otherwise} \end{cases} \tag{2.1}$$

and $R[p]$ is defined by:

$$R[p] \ = \begin{cases} \bigcap\limits_{p \in Rx} Rx & \text{if } \exists x : p \in Rx, \\ \varnothing & \text{otherwise.} \end{cases} \tag{2.2}$$

In this case, $[p]R$, $R[p]$ are called *minimal right* and *minimal left neighborhoods* of $p \in X$.

It is obvious that Allam's Definition is a refinement of both of Pawlak's Definition and Yao's Definition.

Kandil [4]: Let $R$ be a reflexive relation on $X$ and let $\ell$ be an ideal on $X$. Then for any subset $A$ of $X$, the *lower approximation* $\underline{R}(A)$ and the *upper approximation* $\bar{R}(A)$ are defined by:

$$\underline{R}(A) = \{x \in A : [x]R \cap A^c \in \ell\},\ \bar{R}(A) = A \cup \{x \in X : [x]R \cap A \notin \ell\}.$$

It is clear that the definition of Kandil is a generalization of Allam's Definition (if $\ell = \ell^\circ$, then both are equivalent).

Kozae [5]: Let $R$ be a binary relation on $X$. Then for any subset $A$ of $X$, the *lower approximation* $\underline{R}(A)$ and the *upper approximation* $\bar{R}(A)$ are defined by:

$$\underline{R}(A) = \{x \in X :\ R[x]R \subseteq A\},\ \bar{R}(A) = \{x \in X :\ R[x]R \cap A \neq \varnothing\},$$

where $R[x]R$ is defined by $R[x]R\ =\ [x]R\ \cap\ R[x]$.
It is a refinement of Allam's Definition, and if $R$ was taken reflexive and symmetric, it will be a special case of Kandil's Definition.

The previous definitions are given for the roughness of an approximation space $(X, R)$ in the ordinary case. Ibedou and Abbas [16] introduced a generalization of all the definitions given in [1, 2, 4, 5, 6] in the ordinary case (crisp case) and in the fuzzy case as given down.

**Definition 2.1** (See [16])**.** Let $X$ be a finite set, $R$ a fuzzy relation on $X$ that has at least one value $R(x, y) > 0$ for some $x,\ y \in X$ and $\ell$ a fuzzy ideal on $X$. Then for any $x \in X$, defined the fuzzy sets $xR,\ Rx \in I^X$ as follow: for each $y \in X$,

$$xR(y) = R(x, y)\ \text{ and }\ Rx(y) = R(y, x).$$

For any $a \in X$, the fuzzy sets $[a]R,\ \ R[a] \in I^X$ are defined by: for each $x \in X$,

$$[a]R(x) = \bigwedge_{R(x,a)>0} xR(a) \quad\text{and}\quad R[a](x) = \bigwedge_{R(a,x)>0} Rx(a). \tag{2.3}$$

Equation 2.3 defines the *minimal right* and the *minimal left fuzzy neighborhoods* of $a \in X$.

For any $a \in X$, defined $R[a]R : X \to I$ as follows: for each $x \in X$,

$$R[a]R(x) = [a]R(x) \wedge R[a](x). \tag{2.4}$$

For every $x \in X$, defined $\lambda_*,\ \lambda^* \in I^X$ of a fuzzy set $\lambda \in I^X$ by: for each $x \in X$,

$$\lambda_*(x) = \begin{cases} (\bigvee\limits_{z\in X} R[z]R\,(x))^c & \text{if } R[x]R \wedge \lambda^c \notin \ell \text{ and } R[x]R \wedge \lambda \notin \ell \\ 1 & \text{if } R[x]R \wedge \lambda^c \in \ell \\ 0 & \text{if } R[x]R \wedge \lambda^c \notin \ell \text{ and } R[x]R \wedge \lambda \in \ell \end{cases} \tag{2.5}$$

$$\lambda^*(x) = \begin{cases} \bigvee\limits_{z\in X} R[z]R\,(x) & \text{if } R[x]R \wedge \lambda \notin \ell \text{ and } R[x]R \wedge \lambda^c \notin \ell \\ 0 & \text{if } R[x]R \wedge \lambda \in \ell \\ 1 & \text{if } R[x]R \wedge \lambda \notin \ell \text{ and } R[x]R \wedge \lambda^c \in \ell. \end{cases} \tag{2.6}$$

Let $R$ be a fuzzy equivalence relation on a set $X$ and let $\lambda \in I^X$. Then the *lower fuzzy set*, the *upper fuzzy set* and the *boundary fuzzy region* of $\lambda$, denoted by $\lambda_R$, $\lambda^R$ and $\lambda^B$ are respectively defined by:

$$\lambda_R = \lambda \wedge \lambda_* \ \text{ and }\ \lambda^R = \lambda \vee \lambda^*, \tag{2.7}$$

$$\lambda^B = \lambda^R \,\bar{\wedge}\, \lambda_R. \tag{2.8}$$

The pair $(X, R)$ is called a rough fuzzy approximation space, and $\lambda_R$ and $\lambda^R$ are called *rough fuzzy sets* of $\lambda$.

For every rough fuzzy set $\lambda \in I^X$, the *accuracy fuzzy set* $\alpha(\lambda) \in I^X$ of $\lambda$ is defined as follows: for all $x \in X$,

$$\alpha(\lambda)(x) = \begin{cases} 0 & \text{if } \lambda^R = \overline{1} \text{ and } \lambda_R = \overline{0} \\ ((\lambda^R(x) - \lambda(x)) \vee (\lambda(x) - \lambda_R(x)))^c & \text{if } \lambda^R \not\leq \lambda_R \\ 1 & \text{otherwise,} \end{cases} \tag{2.9}$$

The *accuracy value* of the rough fuzzy set $\lambda$ is denoted by $\mathrm{Inf}(\alpha(\lambda))$.

Now, we modify Equation 2.3 to use in defining a new type of roughness of fuzzy sets and roughness of ordinary (crisp) sets. By the previous definitions of $xR,\ Rx \in I^X$, we can define the maximal fuzzy neighborhoods of any $a \in X$.

For any $a \in X$, the fuzzy sets $[a]\check{R},\ \check{R}[a] \in I^X$ are defined as follows: for each $x \in X$,

$$[a]\check{R}(x) = \bigvee_{R(x,a)>0} xR(a) \quad \text{and} \quad \check{R}[a](x) = \bigvee_{R(a,x)>0} Rx(a). \tag{2.10}$$

Equation 2.10 defines the *maximal right* and the *maximal left fuzzy neighborhoods* of $a \in X$.

In the crisp case, Equation 2.10 defines the maximal right and the maximal left neighborhoods as used in [7, 8] of some element $a \in X$, and moreover, Hosney and Al-shami [3] introduced a generalization using the intersection of maximal right and maximal left neighborhoods accompanied with a usual ideal on $X$.

Using Equation 2.10, we can define a new type of rough sets of an approximation space $(X, R)$ in the fuzzy case and the crisp case as follow.

**Definition 2.2.** Let $\ell$ be a fuzzy ideal defined on a fuzzy approximation space $(X, R)$. Then $\lambda_{**},\ \lambda^{**} \in I^X$ of a fuzzy set $\lambda \in I^X$ are defined as follow: for every $x \in X$,

$$\lambda_{**}(x) = \begin{cases} (\bigvee_{z\in X} \check{R}[z]\check{R}(x))^c & \text{if } \check{R}[x]\check{R} \wedge \lambda^c \notin \ell \text{ and } \check{R}[x]\check{R} \wedge \lambda \notin \ell \\ 1 & \text{if } \check{R}[x]\check{R} \wedge \lambda^c \in \ell \\ 0 & \text{if } \check{R}[x]\check{R} \wedge \lambda^c \notin \ell \text{ and } \check{R}[x]\check{R} \wedge \lambda \in \ell, \end{cases} \tag{2.11}$$

$$\lambda^{**}(x) = \begin{cases} \bigvee_{z\in X} \check{R}[z]\check{R}(x) & \text{if } \check{R}[x]\check{R} \wedge \lambda \notin \ell \text{ and } \check{R}[x]\check{R} \wedge \lambda^c \notin \ell \\ 0 & \text{if } \check{R}[x]\check{R} \wedge \lambda \in \ell \\ 1 & \text{if } \check{R}[x]\check{R} \wedge \lambda \notin \ell \text{ and } \check{R}[x]\check{R} \wedge \lambda^c \in \ell, \end{cases} \tag{2.12}$$

where

$$\check{R}[x]\check{R} \ = \ [x]\check{R} \ \wedge \ \check{R}[x]. \tag{2.13}$$

The roughness of the fuzzy set $\lambda \in I^X$ is defined by:

$$\underline{\lambda} = \lambda \wedge \lambda_{**} \quad \text{and} \quad \overline{\lambda} = \lambda \vee \lambda^{**}.$$

Then $\underline{\lambda}$ is called the *lower fuzzy set* of $\lambda$ and $\overline{\lambda}$ is called the *upper fuzzy set* of $\lambda$. The *boundary region fuzzy set* $B(\lambda)$ of $\lambda$ is defined by $B(\lambda) = \overline{\lambda} \,\bar{\wedge}\, \underline{\lambda}$. The pair $(X, R)$

is called a *rough fuzzy approximation space*. For every rough fuzzy set $\lambda \in I^X$, the *accuracy fuzzy set* $acc(\lambda)$ of $\lambda \in I^X$ is defined by: for all $x \in X$,

$$acc(\lambda)(x) = \begin{cases} 0 & \text{if } \overline{\lambda} = \overline{1} \text{ and } \underline{\lambda} = \overline{0} \\ ((\overline{\lambda}(x) - \lambda(x)) \vee (\lambda(x) - \underline{\lambda}(x)))^c & \text{if } \overline{\lambda} \not\leq \underline{\lambda} \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, only in the fuzzy case, the accuracy fuzzy set $acc(\lambda)$ is not equivalent to the complement of the boundary region fuzzy set $B(\lambda)$, and moreover the accuracy value of the rough fuzzy set $\lambda$ is given by $\text{Inf}(acc(\lambda))$. But, in the crisp (ordinary) case, the accuracy crisp set is equivalent to the complement of the boundary region crisp set $(B(\lambda))^c$, and moreover the accuracy value of a rough set $\lambda$ in the crisp case is given by the fraction:

$$\frac{\text{number of nonzero membership values of } \underline{\lambda}}{\text{number of nonzero membership values of } \overline{\lambda}} \tag{2.14}$$

This is the computation of the accuracy value of some rough set in the crisp case according to Definition 2.2 and according to Definition 2.1 as well.

Definition 2.2 is a generalization of rough fuzzy sets in a similar way to that one defined by the authors in [16]. In fact, this new definition in both of the fuzzy case and the crisp case imply a new boundary region fuzzy set. In the fuzzy case, that new boundary region fuzzy set not dependent with the boundary region set given in [16]. To explain this main result about the difference between using the maximal fuzzy neighborhoods as given here and using the minimal fuzzy neighborhoods as given in [16], we completely analyze all the branches in both of Definition 2.1 and Definition 2.2.

**Remark 2.3.** (1) $\check{R}[x]\check{R} \wedge \lambda \;\geq\; R[x]R \wedge \lambda \notin \ell \;\Rightarrow\; \check{R}[x]\check{R} \wedge \lambda \notin \ell$
and $\check{R}[x]\check{R} \wedge \lambda^c \;\geq\; R[x]R \wedge \lambda^c \notin \ell \;\Rightarrow\; \check{R}[x]\check{R} \wedge \lambda^c \notin \ell$.
Then we have

$$\lambda_{**}(x) \leq \lambda_*(x),\; \lambda^{**}(x) \geq \lambda^*(x) \text{ for each } x \in X.$$

Thus we get

$$\underline{\lambda}(x) \leq \lambda_R(x),\; \overline{\lambda}(x) \geq \lambda^R(x) \text{ for each } x \in X.$$

So $\lambda^B(x) \leq B(\lambda)(x)$ for each $x \in X$.
(2) $R[x]R \wedge \lambda^c \;\leq\; \check{R}[x]\check{R} \wedge \lambda^c \in \ell \;\Rightarrow\; R[x]R \wedge \lambda^c \in \ell$
and $R[x]R \wedge \lambda \;\leq\; \check{R}[x]\check{R} \wedge \lambda \in \ell \;\Rightarrow\; R[x]R \wedge \lambda \in \ell$. The we get

$$\lambda_{**}(x) = \lambda_*(x) = 1,\; \lambda^{**}(x) = \lambda^*(x) = 0 \text{ for each } x \in X.$$

Thus we have

$$\underline{\lambda}(x) = \lambda_R(x) = \lambda = \lambda^R(x) = \overline{\lambda}(x) \text{ for each } x \in X.$$

So $\lambda^B(x) = B(\lambda)(x)$ for each $x \in X$.
(3) $\check{R}[x]\check{R} \wedge \lambda^c \;\geq\; R[x]R \wedge \lambda^c \notin \ell \;\Rightarrow\; \check{R}[x]\check{R} \wedge \lambda^c \notin \ell$
but $\check{R}[x]\check{R} \wedge \lambda \;\geq\; R[x]R \wedge \lambda \in \ell \;\not\Rightarrow\; \check{R}[x]\check{R} \wedge \lambda \in \ell$.
Also, $R[x]R \wedge \lambda \;\leq\; \check{R}[x]\check{R} \wedge \lambda \in \ell \;\Rightarrow\; R[x]R \wedge \lambda \in \ell$

but $R[x]R \wedge \lambda^c \ \leq \ \check{R}[x]\check{R} \wedge \lambda^c \notin \ell \ \not\Rightarrow \ R[x]R \wedge \lambda^c \notin \ell$.
Then we have

$$\lambda_{**}(x) \not\leq \lambda_*(x),\ \lambda_{**}(x) \not\geq \lambda_*(x) \text{ and } \lambda^{**}(x) \not\leq \lambda^*(x),\ \lambda^{**}(x) \not\geq \lambda^*(x) \text{ for each } x \in X.$$

Thus we get

$$\underline{\lambda}(x) \not\geq \lambda_R(x),\ \underline{\lambda}(x) \not\leq \lambda_R(x) \text{ and } \overline{\lambda}(x) \not\geq \lambda^R(x),\ \overline{\lambda}(x) \not\leq \lambda^R(x) \text{ for each } x \in X.$$

So $\lambda^B(x) \not\leq B(\lambda)(x)$ and $\lambda^B(x) \not\geq B(\lambda)(x)$ for each $x \in X$.

(4) $\check{R}[x]\check{R} \wedge \lambda \ \geq \ R[x]R \wedge \lambda \notin \ell \ \Rightarrow \ \check{R}[x]\check{R} \wedge \lambda \notin \ell$
but $\check{R}[x]\check{R} \wedge \lambda^c \ \geq \ R[x]R \wedge \lambda^c \in \ell \ \not\Rightarrow \ \check{R}[x]\check{R} \wedge \lambda^c \in \ell$.
Also, $R[x]R \wedge \lambda^c \ \leq \ \check{R}[x]\check{R} \wedge \lambda^c \in \ell \ \Rightarrow \ R[x]R \wedge \lambda^c \in \ell$
but $R[x]R \wedge \lambda \ \leq \ \check{R}[x]\check{R} \wedge \lambda \notin \ell \ \not\Rightarrow \ R[x]R \wedge \lambda \notin \ell$.
Then we get

$$\lambda^{**}(x) \not\leq \lambda^*(x),\ \lambda^{**}(x) \not\geq \lambda^*(x) \text{ and } \lambda_{**}(x) \not\leq \lambda_*(x),\ \lambda_{**}(x) \not\geq \lambda_*(x) \text{ for each } x \in X.$$

Thus we have

$$\overline{\lambda}(x) \not\geq \lambda^R(x),\ \overline{\lambda}(x) \not\leq \lambda^R(x) \text{ and } \underline{\lambda}(x) \not\geq \lambda_R(x),\ \underline{\lambda}(x) \not\leq \lambda_R(x) \text{ for each } x \in X.$$

So $\lambda^B(x) \not\leq B(\lambda)(x)$ and $\lambda^B(x) \not\geq B(\lambda)(x)$ for each $x \in X$.

(5) If $\check{R}[x]\check{R} \in \ell$, then $R[x]R \in \ell$. Thus we have

$$\check{R}[x]\check{R} \wedge \lambda \in \ell,\ \check{R}[x]\check{R} \wedge \lambda^c \in \ell,\ R[x]R \wedge \lambda \in \ell,\ R[x]R \wedge \lambda^c \in \ell.$$

So $\underline{\lambda}(x) = \lambda_R(x) = \lambda(x) = \lambda^R(x) = \overline{\lambda}(x)$ for all $x \in X$.

(6) If $R[x]R \in \ell$, then $R[x]R \wedge \lambda \in \ell$, $R[x]R \wedge \lambda^c \in \ell$. Thus $\lambda_R(x) = \lambda(x) = \lambda^R(x)$ for all $x \in X$.

(7) In the crisp case, we usually consider a proper crisp ideal, i.e., $\overline{1} = \lambda \vee \lambda^c \notin \ell$, and then it could not be both of $\lambda \in \ell$ and $\lambda^c \in \ell$ but in the fuzzy case, we can find both of $\lambda \in \ell$ and $\lambda^c \in \ell$ while $\ell$ is a proper fuzzy ideal. Then in the crisp case, may be either $\lambda \in \ell$ or $\lambda^c \in \ell$, while in the fuzzy case, it could be satisfied both of $\lambda \in \ell$ and $\lambda^c \in \ell$. Only in the fuzzy case, it could be found both of $\lambda \notin \ell$ and $\lambda^c \notin \ell$ (not satisfied in the crisp case from being $X \notin \ell$).

(8) If $\lambda \in \ell$, then $\lambda^{**} = \overline{0}$. Thus $\overline{\lambda} = \lambda$.

(9) If $\lambda^c \in \ell$, then $\lambda_{**} = \overline{1}$. Thus $\underline{\lambda} = \lambda$.

(10) In the crisp case, we see that the first branch in Equations 2.5, 2.11 goes to zero and the first branch in Equations 2.6, 2.12 goes to one all time. Hence, Definition 2.1 using the minimal neighborhoods as used in [16] produces a boundary region set fewer than that boundary region set using the maximal neighborhoods as produced in Definition 2.2. Finally, in the crisp case, using the minimal neighborhoods produces a fewer boundary region area than that produced using the maximal neighborhoods. In the fuzzy case, there is no relation between the two generations because of the first branch in Equations 2.5, 2.6, 2.11, 2.12.

**Remark 2.4.** If $\ell = \ell^\circ$, then (2) and (5) in Remark 2.3 are equivalent and could not be found except $\check{R}[x]\check{R} = \overline{0}$ for $x \in X$. Also, (6) in Remark 2.3 is satisfied only if $R[x]R = \overline{0}$ for $x \in X$. Moreover, (8), (9) in Remark 2.3 will be trivial.

Moreover, while it was proved that the definitions in [1, 2, 4, 5, 6] were special cases of the definitions in [16], we notice that Definition 2.2, even under the suitable restrictions, produces a boundary region not related to any of those boundary regions given by [2, 4, 5]. This is because these definitions in [2, 4, 5] were based on minimal neighborhoods not maximal neighborhoods.

The following example produces different computations of roughness upon Definition 2.2 and Definition 2.1 of a fuzzy set $\lambda$ in an approximation space $(X, R)$.

**Example 2.5** (See Example 2.6, [16]). Let $X = \{a, b, c, d\}$, let $\lambda = \{0.1, 0.8, 0.4, 0.6\}$ be a fuzzy set in $I^X$ and let $R$ be a fuzzy relation on $X$ as shown down.

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 0 | 0.2 | 1 | 0.5 |
| $b$ | 0.6 | 0 | 0.8 | 0.5 |
| $c$ | 1 | 0.5 | 0.6 | 0.6 |
| $d$ | 0.9 | 0.6 | 1 | 1 |

Then we get $[a]\check{R} = \{1, 0.6, 1, 1\}$, $[b]\check{R} = \{1, 0.6, 1, 1\}$, $[c]\check{R} = \{1, 0.6, 1, 1\}$, $[d]\check{R} = \{1, 0.6, 1, 1\}$ and

$\check{R}[a] = \{1, 0.8, 0.6, 1\}$, $\check{R}[b] = \{1, 0.8, 1, 1\}$, $\check{R}[c] = \{1, 0.8, 1, 1\}$, $\check{R}[d] = \{1, 0.8, 1, 1\}$.

Thus we have

$\check{R}[a]\check{R} = \{1, 0.6, 0.6, 1\}$, $\check{R}[b]\check{R} = \{1, 0.6, 1, 1\}$,

$\check{R}[c]\check{R} = \{1, 0.6, 1, 1\}$, $\check{R}[d]\check{R} = \{1, 0.6, 1, 1\}$.

Consider a fuzzy ideal $\ell$ on $X$ such that $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.4}$. Then $\lambda_{**} = \{0, 0.4, 0, 0\}$, $\lambda^{**} = \{1, 0.6, 1, 1\}$. Thus $\underline{\lambda} = \{0, 0.4, 0, 0\}$ and $\overline{\lambda} = \{1, 0.8, 1, 1\}$. So $B(\lambda) = \{1, 0.6, 1, 1\}$, $acc(\lambda) = \{0.1, 0.6, 0.4, 0.4\}$, $\text{Inf}(acc(\lambda)) = 0.1$.

While the computations in [16] give us $\lambda_* = \{0.8, 0.8, 0.4, 0.5\}$, $\lambda^* = \{0.2, 0.2, 0.6, 0.5\}$. Then $\lambda_R = \{0.1, 0.8, 0.4, 0.5\}$, $\lambda^R = \{0.2, 0.8, 0.6, 0.6\}$. Thus $\lambda^B = \{0.2, 0.2, 0.6, 0.5\}$, $\alpha(\lambda) = \{0.9, 1, 0.8, 0.9\}$, $\text{Inf}\alpha(\lambda) = 0.8$. Here, the boundary region fuzzy set $\lambda^B$ is fewer than the boundary region fuzzy set $B(\lambda)$.

In case of $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.5}$, we get $\lambda_{**} = \{0, 0.4, 0, 0\}$, $\lambda^{**} = \{1, 0.6, 1, 1\}$. Then $\underline{\lambda} = \{0, 0.4, 0, 0\}$ and $\overline{\lambda} = \{1, 0.8, 1, 1\}$. Thus $B(\lambda) = \{1, 0.6, 1, 1\}$, $acc(\lambda) = \{0.1, 0.6, 0.4, 0.4\}$, $\text{Inf}(acc(\lambda)) = 0.1$.

While the computations in [16] give us $\lambda_* = \{1, 0, 1, 1\}$, $\lambda^* = \overline{0}$. Then $\lambda^R = \{0.1, 0.8, 0.4, 0.6\}$, $\lambda_R = \{0.1, 0, 0.4, 0.6\}$. Thus $\lambda^B = \{0.1, 0.8, 0.4, 0.4\}$. So $\alpha(\lambda) = \{1, 0.2, 1, 1\}$ and $\text{Inf}(\alpha(\lambda)) = 0.2$. Here, the boundary region fuzzy set $\lambda^B$ is not fewer than the boundary region fuzzy set $B(\lambda)$. This is because of:
$\check{R}[b]\check{R} \wedge \lambda \notin \ell$ and $\check{R}[b]\check{R} \wedge \lambda^c \notin \ell$. Then $\lambda_{**}(b) = 0.4$. Thus $\underline{\lambda}(b) = 0.4$. But $R[b]R \wedge \lambda \in \ell$ and $R[b]R \wedge \lambda^c \notin \ell$. So $\lambda_*(b) = 0$. Hence $\lambda_R(b) = 0$. This is a case similar to case (3) in Remark 2.3.

In case of $\nu \in \ell \Leftrightarrow \nu \leq \overline{0.6}$, we get that $\lambda_{**} = \overline{0} = \lambda^{**}$. Then $\underline{\lambda} = \overline{0}$, $\overline{\lambda} = \lambda = \{0.1, 0.8, 0.4, 0.6\}$. Thus $B(\lambda) = \lambda$, $acc(\lambda) = \{0.9, 0.2, 0.6, 0.4\}$, $\text{Inf}(acc(\lambda)) = 0.2$.

While the computations in [16] give us $\lambda_* = \overline{1}$ and $\lambda^* = \overline{0}$. Then $\lambda_R = \lambda = \lambda^R$. Thus $\lambda^B = \overline{0}$, $\alpha(\lambda) = \overline{1}$. So $\text{Inf}(\alpha(\lambda)) = 1$.

In the crisp case, we give the following examples.

**Example 2.6.** Let $R$ be a crisp relation on a set $X = \{a, b, c, d\}$ as shown down.

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 1 | 0 | 0 |
| $b$ | 0 | 1 | 1 | 0 |
| $c$ | 1 | 0 | 0 | 1 |
| $d$ | 0 | 1 | 1 | 0 |

Then we have

$$aR = \{1, 1, 0, 0\},\ bR = \{0, 1, 1, 0\},\ cR = \{1, 0, 0, 1\},\ dR = \{0, 1, 1, 0\}$$

and

$$Ra = \{1, 0, 1, 0\},\ Rb = \{1, 1, 0, 1\},\ Rc = \{0, 1, 0, 1\},\ Rd = \{0, 0, 1, 0\}.$$

Thus we get

$$[a]\check{R} = \{1, 1, 0, 1\},\ [b]\check{R} = \{1, 1, 1, 0\},\ [c]\check{R} = \{0, 1, 1, 0\},\ [d]\check{R} = \{1, 0, 0, 1\}$$

and

$$\check{R}[a] = \{1, 1, 1, 1\},\ \check{R}[b] = \{1, 1, 0, 1\},\ \check{R}[c] = \{1, 0, 1, 0\},\ \check{R}[d] = \{1, 1, 0, 1\}.$$

So we have

$$\check{R}[a]\check{R} = \{1, 1, 0, 1\},\ \check{R}[b]\check{R} = \{1, 1, 0, 0\},\ \check{R}[c]\check{R} = \{0, 0, 1, 0\},\ \check{R}[d]\check{R} = \{1, 0, 0, 1\}.$$

(1) Consider a crisp ideal $\ell$ on $X$ such that $\ell = \{\overline{0}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 1, 1, 0\}\}$. Then for a crisp set $\lambda = \{1, 1, 0, 0\}$ (Note that $\lambda \notin \ell$ and $\lambda^c \notin \ell$), we compute $\lambda_{**},\ \lambda^{**}$ as follow:

$$\lambda_{**} = \{0, 1, 1, 0\},\ \lambda^{**} = \{1, 1, 0, 1\}.$$

Thus $\underline{\lambda} = \{0, 1, 0, 0\}$, $\overline{\lambda} = \{1, 1, 0, 1\}$. So $B(\lambda) = \{1, 0, 0, 1\}$, $acc(\lambda) = \{0, 1, 1, 0\}$. Hence $\text{Inf}(acc(\lambda)) = \frac{1}{3}$.

While according to the computations in [16], we get

$$[a]R = \{1, 0, 0, 0\},\ [b]R = \{0, 1, 0, 0\},\ [c]R = \{0, 1, 1, 0\},\ [d]R = \{1, 0, 0, 1\}$$

and

$$R[a] = \{1, 0, 0, 0\},\ R[b] = \{0, 1, 0, 1\},\ R[c] = \{0, 0, 1, 0\},\ R[d] = \{0, 1, 0, 1\}.$$

Then we have

$$R[a]R = \{1, 0, 0, 0\},\ R[b]R = \{0, 1, 0, 0\},\ R[c]R = \{0, 0, 1, 0\},\ R[d]R = \{0, 0, 0, 1\}.$$

Thus $\lambda_* = \{1, 1, 1, 0\},\ \lambda^* = \{1, 0, 0, 0\}$. So $\lambda_R = \lambda^R = \lambda = \{1, 1, 0, 0\}$. Hence $\lambda^B = \overline{0}$, $\alpha(\lambda) = \overline{1}$. Therefore $\text{Inf}(acc(\lambda)) = \frac{2}{2} = 1$.

(2) Consider the same ideal and a crisp set $\mu = \{0, 0, 1, 0\} \in \ell$. Then we compute $\mu_{**}, \mu^{**}$ as follow:

$$\mu_{**} = \{0, 0, 1, 0\} = \mu,\ \mu^{**} = \overline{0}.$$

Thus $\underline{\mu} = \overline{\mu} = \mu$. So $B(\mu) = \overline{0}$, $acc(\mu) = \overline{1}$. Hence $\text{Inf}(acc(\mu)) = 1$.

While according to the computations in [16], we have $\mu_* = \{0, 1, 1, 0\},\ \mu^* = \overline{0}$. Then $\mu_R = \mu^R = \mu = \{0, 0, 1, 0\}$. Thus $\mu^B = \overline{0}$, $\alpha(\mu) = \overline{1}$. So $\text{Inf}(acc(\mu)) = 1$.

(3) For a fuzzy ideal $\ell$ on $X$ such that $\ell = \{\overline{0}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}, \{0, 0, 1, 1\}\}$ and the crisp set $\lambda = \{1, 1, 0, 0\}$, we have $\lambda^c \in \ell$. Then $\lambda_{**} = \overline{1},\ \lambda^{**} = \{1, 1, 0, 1\}$.

Thus $\underline{\lambda} = \{1,1,0,0\} = \lambda$, $\overline{\lambda} = \{1,1,0,1\}$. So $B(\lambda) = \{0,0,0,1\}$, $acc(\lambda) = \{1,1,1,0\}$. Hence $\text{Inf}(acc(\lambda)) = \frac{2}{3}$.

While according to the computations in [16], we have $\lambda_* = \overline{1}$, $\lambda^* = \{1,1,0,0\} = \lambda$. Then $\lambda_R = \lambda^R = \lambda = \{1,1,0,0\}$. Thus $\lambda^B = \overline{0}$, $\alpha(\lambda) = \overline{1}$. So $\text{Inf}(acc(\lambda)) = \frac{2}{2} = 1$.

**Example 2.7.** Let $R$ be a crisp reflexive relation on a set $X = \{a,b,c,d\}$ as shown down.

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 1 | 1 | 1 | 0 |
| $c$ | 0 | 0 | 1 | 0 |
| $d$ | 1 | 1 | 0 | 1 |

Then we have

$$[a]\check{R} = \overline{1},\ [b]\check{R} = \overline{1},\ [c]\check{R} = \{1,1,1,0\},\ [d]\check{R} = \{1,1,0,1\}$$

and

$$\check{R}[a] = \{1,1,0,1\},\ \check{R}[b] = \overline{1},\ \check{R}[c] = \{0,1,1,0\},\ \check{R}[d] = \{1,1,0,1\}.$$

Thus we get

$$\check{R}[a]\check{R} = \{1,1,0,1\},\ \check{R}[b]\check{R} = \overline{1},\ \check{R}[c]\check{R} = \{0,1,1,0\},\ \check{R}[d]\check{R} = \{1,1,0,1\}.$$

(1) Consider a crisp ideal $\ell$ on $X$ such that $\ell = \{\overline{0}, \{0,0,1,0\}, \{0,0,0,1\}, \{0,0,1,1\}\}$. Then for a crisp set $\lambda = \{1,0,1,0\}$ (Note that $\lambda \notin \ell$ and $\lambda^c \notin \ell$), we compute $\lambda_{**}, \lambda^{**}$ as follow:

$$\lambda_{**} = \overline{0}.\ \lambda^{**} = \{1,1,0,1\}.$$

Then $\underline{\lambda} = \overline{0}$, $\overline{\lambda} = \overline{1}$. So $B(\lambda) = \overline{1}$, $acc(\lambda) = \overline{0}$. Hence $\text{Inf}(acc(\lambda)) = 0$.

While according to the computations in [16], we get that

$$[a]R = \{1,0,0,0\},\ [b]R = \{1,1,0,0\},\ [c]R = \{0,0,1,0\},\ [d]R = \{1,1,0,1\}$$

and

$$R[a] = \{1,1,0,1\},\ R[b] = \{0,1,0,0\},\ R[c] = \{0,1,1,0\},\ R[d] = \{0,0,0,1\}.$$

Then we have

$$R[a]R = \{1,0,0,0\},\ R[b]R = \{0,1,0,0\},\ R[c]R = \{0,0,1,0\},\ R[d]R = \{0,0,0,1\}.$$

Thus $\lambda_* = \{1,0,1,1\}$, $\lambda^* = \{1,0,0,0\}$. So $\lambda_R = \lambda^R = \lambda = \{1,0,1,0\}$. Hence $\lambda^B = \overline{0}$, $\alpha(\lambda) = \overline{1}$. Therefore $\text{Inf}(acc(\lambda)) = \frac{2}{2} = 1$.

(2) Consider the same ideal and a crisp set $\mu = \{0,1,1,1\}$, we compute $\mu_{**}, \mu^{**}$ as follow:

$$\mu_{**} = \{0,0,1,0\},\ \mu^{**} = \overline{1}.$$

Then $\overline{\mu} = \overline{1}$, $\underline{\mu} = \{0,0,1,0\}$. Thus $B(\mu) = \{1,1,0,1\}$, $acc(\mu) = \{0,0,1,0\}$. So $\text{Inf}(acc(\lambda)) = \frac{1}{4}$.

While according to the computations in [16], we have $\mu_* = \{0,1,1,1\}$, $\mu^* = \{0,1,0,0\}$. Then $\mu_R = \mu^R = \mu = \{0,1,1,1\}$. Thus $\mu^B = \overline{0}$, $\alpha(\mu) = \overline{1}$. So $\text{Inf}(acc(\lambda)) = \frac{3}{3} = 1$.

Now, we consider the crisp relation in which every element is only related to itself.

**Example 2.8.** Let $R$ be a crisp reflexive relation on a set $X = \{a, b, c, d\}$ as shown down.

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0 | 0 |
| $b$ | 0 | 1 | 0 | 0 |
| $c$ | 0 | 0 | 1 | 0 |
| $d$ | 0 | 0 | 0 | 1 |

Then clearly, we get $\check{R}[a]\check{R} = R[a]R = \{1, 0, 0, 0\}$, $\check{R}[b]\check{R} = R[b]R = \{0, 1, 0, 0\}$, $\check{R}[c]\check{R} = R[c]R = \{0, 0, 1, 0\}$, $\check{R}[d]\check{R} = R[d]R = \{0, 0, 0, 1\}$.

Considering any crisp ideal $\ell$ on $X$. Then we get $\underline{\lambda} = \lambda_R$ and $\overline{\lambda} = \lambda^R$ for any $\lambda \in 2^X$. Suppose $\ell$ is defined by:

$$\ell = \{\overline{0}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}, \{0, 1, 1, 0\}, \{0, 1, 0, 1\}, \{0, 0, 1, 1\}, \{0, 1, 1, 1\}\}.$$

Then for a crisp set $\lambda = \{1, 1, 0, 0\}$, we compute $\lambda_{**}, \lambda^{**}$ and $\lambda_*, \lambda^*$ as follow:

$$\lambda_{**} = \lambda_* = \overline{1},\ \lambda^{**} = \lambda^* = \{1, 0, 0, 0\}.$$

Then $\underline{\lambda} = \lambda_R = \lambda = \lambda^R = \overline{\lambda}$.

For a crisp set $\mu = \{0, 0, 1, 1\}$, we compute $\mu_{**}, \mu^{**}$ and $\mu_*, \mu^*$ as follow:

$$\mu_{**} = \mu_* = \{0, 1, 1, 1\},\ \mu^{**} = \mu^* = \overline{0}.$$

Then $\underline{\mu} = \mu_R = \mu = \mu^R = \overline{\mu}$. Thus there is no difference between the two definitions Definition 2.1 and Definition 2.2.

**Remark 2.9.** Based on Definition 2.2 and Definition 2.1 respectively, either in the fuzzy case or in the crisp case, whenever $R$ is a reflexive and transitive fuzzy relation on $X$, we have

$$\underline{(\underline{\lambda})} = \underline{\lambda} \text{ and } \overline{(\overline{\lambda})} = \overline{\lambda}\ ((\lambda_R)_R = \lambda_R \text{ and } (\lambda^R)^R = \lambda^R).$$

If $R$ is considered to be a reflexive and transitive fuzzy relation on $X$, then a fuzzy topology $\tau_R$

$$\tau_R = \{\mu \in I^X : \mu = \underline{\mu}\} \ \equiv\ \{\mu \in I^X : \mu^c = \overline{\mu^c}\} \tag{2.15}$$

$$(\tau_R = \{\nu \in I^X : \nu = \nu_R\} \ \equiv\ \{\nu \in I^X : \nu^c = (\nu^c)^R\}) \tag{2.16}$$

is generated on the rough fuzzy approximation space $(X, R)$.

For a fuzzy set $\lambda$ in a rough fuzzy approximation space $(X, R)$ where $R$ is a reflexive and transitive fuzzy relation, we can define, upon Definition 2.2 and Definition 2.1, respectively an interior fuzzy operator $\lambda^{int}$ ($\text{int}_R\lambda$) and a closure fuzzy operator $\lambda^{cl}$ ($\text{cl}_R\lambda$) on $I^X$ as follow:

$$\lambda^{int} = \underline{\lambda},\ \lambda^{cl} = \overline{\lambda}\ (\text{int}_R\lambda = \lambda_R,\ \text{cl}_R\lambda = \lambda^R). \tag{2.17}$$

Then $(X, \tau_R)$ is called a *rough fuzzy topological space*. All topological properties could be studied as in the usual fuzzy topological spaces.

As an application of this type of rough fuzzy topological spaces, we discuss rough fuzzy connected spaces using both of the closure fuzzy operators defined in Equation 2.17 where $R$ is reflexive and transitive fuzzy relation.

## 3. Connectedness in rough (fuzzy) topological spaces

Here, we recall the definitions given in [16] for the connectedness in rough fuzzy topological spaces, and the crisp case will be as restricted definitions for crisp subsets $A \in 2^X$.

**Definition 3.1** (See [16])**.** Let $(X, R)$ be a rough fuzzy approximation space and let $\ell$ be a fuzzy ideal on $X$.

(i) $\mu,\ \nu \in I^X$ are said to be *rough fuzzy approximation separated*, if

$$\mathrm{cl}_R(\mu) \wedge \nu = \mu \wedge \mathrm{cl}_R(\nu) = \overline{0}.$$

(ii) $\eta \in I^X$ is called a *rough fuzzy approximation disconnected set*, if there exist rough fuzzy approximation separated sets $\mu,\ \nu \in I^X$ such that $\mu \vee \nu = \ \eta$. $\eta$ is said to be *rough fuzzy approximation connected*, if it is not rough fuzzy approximation disconnected. In other words, if there are no rough fuzzy approximation separated sets $\mu,\ \nu$ except $\mu = \overline{0}$ or $\nu = \overline{0}$.

(iii) $(X, R)$ is called a *rough fuzzy approximation disconnected space*, if there exist rough fuzzy approximation separated sets $\mu,\ \nu \in I^X$ such that $\mu \vee \nu = \ \overline{1}$. A fuzzy approximation space $(X, R)$ is said to be rough fuzzy approximation connected, if it is not rough fuzzy approximation disconnected.

The above definitions give us the connectedness of rough topological spaces also if we used the crisp case.

**Remark 3.2.** Using the same definitions in [16] but paying attention for the closure fuzzy operator $\lambda^{cl} = \overline{\lambda} = \lambda \vee \lambda^{**}$ and $\lambda^{**}$ is computed with Equation 2.10 not Equation 2.3, we get another type of connectedness of rough fuzzy topological spaces based on the maximal neighborhoods. Since we noticed that $\overline{\lambda}$ is not dependent on $\lambda^R$ defined in [16], then any two rough fuzzy separated sets are not necessary rough fuzzy separated in sense of [16], and also the converse is not true in general. That is, a fuzzy set is disconnected in the new style do not imply that it is disconnected in sense of [16], and the converse is not true in general. Moreover, $(X, R)$ is rough fuzzy connected in sense of [16] do not imply it is rough fuzzy connected in this type, and the converse is not true in general.

In the following, we show the difference between the connectedness as given in [16] and this connectedness explained in Remark 3.2.

**Example 3.3** (See Example 3.3, [16])**.** Let $X = \{a, b, c, d\}$, let $R$ be the reflexive and transitive fuzzy relation defined by

| $R$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | 1 | 0 | 0.1 | 0 |
| $b$ | 0 | 1 | 0.1 | 0 |
| $c$ | 0 | 0 | 1 | 0 |
| $d$ | 0 | 0 | 0.1 | 1 |

Then we have

$$[a]\check{R} = \{1, 0, 0.1, 0\}\ [b]\check{R} = \{0, 1, 0.1, 0\},\ [c]\check{R} = \{1, 1, 1, 1\},\ [d]\check{R} = \{0, 0, 0.1, 1\}$$

and

$\check{R}[a] = \{1, 0, 0, 0\},\ \check{R}[b] = \{0.1, 1, 1, 0.1\},\ \check{R}[c] = \{0.1, 0.1, 1, 0.1\},\ \check{R}[d] = \{0.1, 0.1, 1, 1\}$.

Then we get

$$\check{R}[a]\check{R} = \{1, 0, 0.1, 0\},\ \check{R}[b]\check{R} = \{0, 1, 0.1, 0\},$$
$$\check{R}[c]\check{R} = \{0.1, 0.1, 1, 0.1\},\ \check{R}[d]\check{R} = \{0, 0, 0.1, 1\}.$$

Define a fuzzy ideal $\ell$ on $X$ such that $\nu \in \ell \ \Leftrightarrow\ \nu \leq \overline{0.3}$. Then for all fuzzy sets $\zeta = \{a, b, c, d\} \in \ell$, we can choose $\lambda = \{a, b, 0, 0\}$, $\mu = \{0, 0, c, d\}$, where $a,\ b,\ c,\ d \leq 0.3$ such that $\lambda^{**} = \overline{0},\ \mu^{**} = \overline{0}$. Thus we have

$$\lambda^{cl} = \overline{\lambda} = \lambda = \{a, b, 0, 0\},\ \mu^{cl} = \overline{\mu} = \mu = \{0, 0, c, d\}.$$

So $\lambda^{cl} \wedge \mu = \lambda \wedge \mu^{cl} = \lambda \wedge \mu = \overline{0}$. Hence $\zeta = \lambda \vee \mu = \{a, b, c, d\}$ is a rough fuzzy disconnected set.

From the definitions of $\lambda^{**},\ \mu^{**}$ as given in Equation 2.12, we already have $\lambda^{**} = \overline{0}$, $\mu^{**} = \overline{0}$ for any $\lambda,\ \mu \in \ell$. But, for example, for a fuzzy set $\nu = \{0.5, 0.5, 0.6, 0.6\}$ not included in the fuzzy ideal $\ell$, we can not find $\xi,\ \eta$ with $(\xi \vee \eta) = \nu$ such that $\xi^{cl} \wedge \eta = \xi \wedge \eta^{cl} = \overline{0}$, because of:

$\check{R}[x]\check{R} \wedge \xi \notin \ell$ and $\check{R}[x]\check{R} \wedge \xi^c \notin \ell$, $\check{R}[x]\check{R} \wedge \eta \notin \ell$ and $\check{R}[x]\check{R} \wedge \eta^c \notin \ell\ \forall x \in X$. Then $\xi^{**} = \eta^{**} = \overline{1}$. Thus $\xi^{cl} = \eta^{cl} = \overline{1}$. So $\xi,\ \eta$ could not be as rough separated fuzzy sets to make the set $\nu$ a rough fuzzy disconnected set.

While as shown in [16], for any $\lambda \in I^X$, we have $\lambda^* = \overline{0}$. Then $\lambda^R = \mathrm{cl}_R\lambda = \lambda$ for any $\lambda \in I^X$. Thus we can find $\xi = \{0.5, 0.5, 0, 0\}$, $\eta = \{0, 0, .6, 0.6\}$ so that $\xi \vee \eta = \{0.5, 0.5, 0.6, 0.6\} = \nu$ is a rough fuzzy set for which $\mathrm{cl}_R\xi \wedge \eta = \xi \wedge \mathrm{cl}_R\eta = \xi \wedge \eta = \overline{0}$. So $\nu$ is rough fuzzy disconnected set in sense of [16].

## 4. Conclusion

In this paper, we modified our Definition in [16] that was based on the minimal neighborhoods, to give a new pattern of fuzzy roughness based on the maximal neighborhoods. This new generalization of roughness is not depending on that generalization given in [16] , but at least the definitions in [16] are generalizations of previous definitions as those given in [1, 2, 4, 5, 6]. The crisp boundary region of this new roughness is larger than the crisp boundary region of that introduced in [16]. While in the fuzzy cases, it is not related to that boundary region of roughness as given in [16]. Also, we explained that if constructed a rough fuzzy topological space and studied some topological notion like connectedness, we get that the new type of connectedness is not related to that connectedness defined in [16], but in the crisp case any connected space in sense of [16] is a connected space defined by maximal neighborhoods as well. In future work, we will try to give a generalization of rough fuzzy soft sets.

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