Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 26, No. 2, (October 2023) pp. 165–175
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2023.26.2.165

A new form of fuzzy set and its application in BCK-algebras and BCI-algebras

Young Bae Jun

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 26, No. 2, October 2023

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 26, No. 2, (October 2023) pp. 165–175
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2023.26.2.165

@FMI

A new form of fuzzy set and its application in BCK-algebras and BCI-algebras

Young Bae Jun

Received 9 July 2023; Revised 23 July 2023; Accepted 25 August 2023

Abstract. The notion of the J-operator in the closed interval $[0, 1]$ is introduced and several properties are investigated. Using the J-operator, a new fuzzy set called the Y_J^ε-fuzzy set is established and it is applied it to subalgebras in BCK/BCI-algebras. The concept of the Y_J^ε-fuzzy subalgebra is introduced and its properties are discussed. Conditions for a fuzzy set to be a Y_J^ε-fuzzy subalgebra are provided, and the relationship between the fuzzy subalgebra and the Y_J^ε-fuzzy subalgebra is discussed.

2020 AMS Classification: 03G25, 06F35, 08A72

Keywords: Subalgebra, J-operator, Y_J^ε-fuzzy set, nonconstant factor, Y_J^ε-fuzzy subalgebra.

Corresponding Author: Y. B. Jun (skywine@gmail.com)

1. Introduction

The fuzzy set is a mathematical framework that can express and manipulate the uncertainty and ambiguity of data with applications in control systems, pattern recognition, decision making, image processing, expert systems, data mining, natural language processing, risk assessment and decision analysis, etc. The study of fuzzy sets in BCK-algebra began in 1991 and various studies have been conducted since then. Fuzzy set studies of subalgebra of BCK algebra are being conducted in various forms (See [1, 2, 3]).

In this paper, we introduce the notion of the J-operator in the closed interval $[0, 1]$ and investigate several properties. We use the J-operator to create a new fuzzy set called the Y_J^ε-fuzzy set and apply it to subalgebras in BCK/BCI-algebras. We introduce the concept of the Y_J^ε-fuzzy subalgebra and investigate its properties. We provide conditions for a fuzzy set to be a Y_J^ε-fuzzy subalgebra. We discuss the relationship between the fuzzy subalgebra and the Y_J^ε-fuzzy subalgebra.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (See [4] and [5]) and was extensively investigated by several researchers.

We recall the definitions and basic results required in this paper. See the books [6, 7] for further information regarding BCK/BCI-algebras.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,
(II) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,
(III) $(\forall a \in X)\ (a * a = 0)$,
(IV) $(\forall a, b \in X)\ (a * b = 0,\ b * a = 0\ \Rightarrow\ a = b)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(V) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*.

The order relation "$\leq_X$" in a BCK/BCI-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq_X b\ \Leftrightarrow\ a * b = 0). \tag{2.1}$$

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall a \in X)\,(a * 0 = a)\,, \tag{2.2}$$

$$(\forall a, b, c \in X)\,(a \leq_X b \Rightarrow a * c \leq_X b * c,\ c * b \leq_X c * a)\,, \tag{2.3}$$

$$(\forall a, b, c \in X)\,((a * b) * c = (a * c) * b)\,. \tag{2.4}$$

Every BCK-algebra X satisfies:

$$(\forall x, a \in X)(x * a \leq_X x). \tag{2.5}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X (See [7]) if $a * y \in S$ for all $a, y \in S$.

The concept of fuzzy set was introduced by Zadeh [8] in 1965.

A *fuzzy set* in a set X is defined to be a function $f : X \to [0, 1]$. Denote by $FS(X)$ the collection of all fuzzy sets in X. Define a relation " $\subseteq$ " on $FS(X)$ by

$$(\forall f, g \in FS(X))(f \subseteq g\ \Leftrightarrow\ (\forall a \in X)(f(a) \leq g(a))).$$

The *join* ($\vee$) and *meet* ($\wedge$) of f and g are defined by

$$(f \vee g)(a) = \max\{f(a), g(a)\},$$
$$(f \wedge g)(a) = \min\{f(a), g(a)\},$$

respectively for all $a \in X$. The *complement* of f, denoted by f^c, is defined by

$$(\forall a \in X)(f^c(a) = 1 - f(a)).$$

A fuzzy set f in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X (See [9]), if it satisfies:

$$(\forall x, a \in X)(f(x * a) \geq f(x) \wedge f(a)). \tag{2.6}$$

3. The fuzzy set induced by the J-operator in $[0,1]$

We use the notation I instead of $[0,1]$. Let "$\ll$" be the order relation in I^2 defined as follows:

$$(\forall (m,n),(j,i)\in I^2)((m,n)\ll (j,i) \;\Leftrightarrow\; m\leq j,\, n\leq i)$$

For every $m,\varepsilon\in I$, we define $m\wedge\varepsilon := \min\{m,\varepsilon\}$ and $m\vee\varepsilon := \max\{m,\varepsilon\}$.

Consider a binary operation Y_J in I given as follows:

$$Y_J : I^2\to I,\ (m,\varepsilon)\mapsto (1-m)\wedge(1-\varepsilon).$$

We will call this binary operation Y_J as the *J-operator* in I.

Proposition 3.1. *The J-operator Y_J in I satisfies:*

$$(\forall m\in I)(Y_J(1,m)=0), \tag{3.1}$$

$$(\forall (m,\varepsilon)\in I^2)(Y_J(m,\varepsilon)=Y_J(\varepsilon,m)), \tag{3.2}$$

$$(\forall (m_1,\varepsilon_1),(m_2,\varepsilon_2)\in I^2)\left(\begin{array}{c}(m_1,\varepsilon_1)\ll(m_2,\varepsilon_2)\;\Rightarrow\\ Y_J(m_1,m_2)\geq Y_J(\varepsilon_1,\varepsilon_2)\end{array}\right). \tag{3.3}$$

Proof. Straightforward. □

Using the J-operator Y_J in I, we induce a new fuzzy set.

Definition 3.2. Let X be a set. Given a fuzzy set f in X and $\varepsilon\in I$, let $\varepsilon(f)$ be a mapping defined by

$$\varepsilon(f) : X\to I,\ x\mapsto Y_J(\varepsilon, f(x)).$$

It is clear that $\varepsilon(f)$ is a fuzzy set in X determined by the J-operator and ε. So we can say that $\varepsilon(f)$ is a Y_J^{ε}*-fuzzy set* of f in X.

Example 3.3. Let f be a fuzzy set in $\mathbb{R}$ given by

$$f:\mathbb{R}\to[0,1],\ a\mapsto\begin{cases}0.77 & \text{if } a\in\{x\in\mathbb{R}\mid x<0\},\\ 0.42 & \text{if } a=0,\\ 0.59 & \text{if } a\in\{x\in\mathbb{R}\mid x>0\},\end{cases}$$

where $\mathbb{R}$ is the set of all real numbers. Then the Y_J^{ε}-fuzzy set of f with $\varepsilon=0.51$ is described as follows:

$$\varepsilon(f):\mathbb{R}\to[0,1],\ a\mapsto\begin{cases}0.23 & \text{if } a\in\{x\in\mathbb{R}\mid x<0\},\\ 0.49 & \text{if } a=0,\\ 0.41 & \text{if } a\in\{x\in\mathbb{R}\mid x>0\}.\end{cases}$$

Proposition 3.4. *If $\varepsilon=0$, then the Y_J^{ε}-fuzzy set $\varepsilon(f)$ is the same as the complement of f. If $\varepsilon=1$, then the Y_J^{ε}-fuzzy set $\varepsilon(f)$ is the zero fuzzy set, i.e., $1(f)(x)=0$ for all $x\in X$.*

Proof. Let $\varepsilon=0$. Then $\varepsilon(f)(x)=0(f)(x)=Y_J(0,f(x))=(1-0)\wedge(1-f(x))=1-f(x)=f^c(x)$ for all $x\in X$. Thus the Y_J^{ε}-fuzzy set with $\varepsilon=0$ is the complement of f. If $\varepsilon=1$, then $1(f)(x)=Y_J(1,f(x))=(1-1)\wedge(1-f(x))=0$ for all $x\in X$. □

Proposition 3.5. *Given a fuzzy set f in a set X, if $\varepsilon\in I$ satisfies $\varepsilon\geq f(x)$ for all $x\in X$, then the Y_J^{ε}-fuzzy set of f is constant on X.*

Proof. Let f be a fuzzy set in X that satisfies $\varepsilon \geq f(x)$ for all $x \in X$. Then $1-\varepsilon \leq 1-f(x)$ for all $x \in X$, and thus

$$\varepsilon(f)(x) = Y_J(\varepsilon, f(x)) = (1-\varepsilon) \wedge (1-f(x)) = 1-\varepsilon$$

for all $x \in X$. This shows that $\varepsilon(f)$ is constant on X. □

Given a fuzzy set f in X and $\varepsilon \in (0,1)$, if the Y_J^{ε}-fuzzy set $\varepsilon(f)$ of f is not constant on X, then ε is is said to be a *nonconstant factor* in $(0,1)$.

Proposition 3.6. *Let f be a fuzzy set in a set X and $(\varepsilon_1, \varepsilon_2) \in I^2$. If $\varepsilon_1 \leq \varepsilon_2$, then $\varepsilon_1(f) \supseteq \varepsilon_2(f)$.*

Proof. For every $x \in X$, we have

$$\begin{aligned}\varepsilon_1(f)(x) &= Y_J(\varepsilon_1, f(x)) = (1-\varepsilon_1) \wedge (1-f(x)) \\ &\geq (1-\varepsilon_2) \wedge (1-f(x)) = Y_J(\varepsilon_2, f(x)) \\ &= \varepsilon_2(f)(x).\end{aligned}$$

Then $\varepsilon_1(f) \supseteq \varepsilon_2(f)$. □

Proposition 3.7. *Let f and g be fuzzy sets in a set X and $\varepsilon \in I$. If $f \subseteq g$, then $\varepsilon(f) \supseteq \varepsilon(g)$.*

Proof. Suppose that $f \subseteq g$. Then $f(x) \leq g(x)$ for all $x \in X$, and thus

$$\begin{aligned}\varepsilon(f)(x) &= Y_J(\varepsilon, f(x)) = (1-\varepsilon) \wedge (1-f(x)) \\ &\geq (1-\varepsilon) \wedge (1-g(x)) = Y_J(\varepsilon, g(x)) \\ &= \varepsilon(g)(x)\end{aligned}$$

for all $x \in X$. So $\varepsilon(f) \supseteq \varepsilon(g)$. □

Proposition 3.8. *If f is a fuzzy set in a set X, then $\varepsilon(f^c) \subseteq \varepsilon(f)^c$ for all $\varepsilon \in I$.*

Proof. Let $x \in X$ and $\varepsilon \in I$. For ε and $f(x)$, we have to consider the following cases:

(i) $\varepsilon \leq f(x) \leq 0.5$,	(ii) $f(x) \leq \varepsilon \leq 0.5$,
(iii) $0.5 < \varepsilon \leq f(x)$,	(iv) $0.5 < f(x) \leq \varepsilon$,
(v) $\varepsilon \leq_X 0.5 \leq f(x)$,	(vi) $f(x) \leq 0.5 \leq_X \varepsilon$.

(i) Suppose $\varepsilon \leq f(x) \leq 0.5$ Then we have

$$\begin{aligned}\varepsilon(f)^c(x) &= 1-\varepsilon(f)(x) = 1-Y_J(\varepsilon, f(x)) \\ &= 1-((1-\varepsilon) \wedge (1-f(x))) \\ &= 1-(1-f(x)) = f(x)\end{aligned}$$

and

$$\begin{aligned}\varepsilon(f^c)(x) &= Y_J(\varepsilon, f^c(x)) = (1-\varepsilon) \wedge (1-f^c(x)) \\ &= (1-\varepsilon) \wedge f(x) = f(x).\end{aligned}$$

Thus $\varepsilon(f)^c = \varepsilon(f^c)$.

(ii) Suppose $f(x) \leq \varepsilon \leq 0.5$ Then $1 - f(x) \geq 1 - \varepsilon \geq 0.5$ and $f(x) \leq 1 - \varepsilon$. Thus

$$\begin{aligned}\varepsilon(f)^c(x) &= 1 - \varepsilon(f)(x) = 1 - Y_J(\varepsilon, f(x))\\ &= 1 - ((1-\varepsilon) \wedge (1 - f(x)))\\ &= 1 - (1 - \varepsilon) = \varepsilon.\end{aligned}$$

So

$$\begin{aligned}\varepsilon(f^c)(x) &= Y_J(\varepsilon, f^c(x)) = (1-\varepsilon) \wedge (1 - f^c(x))\\ &= (1-\varepsilon) \wedge f(x) = f(x) \leq \varepsilon = \varepsilon(f)^c(x).\end{aligned}$$

Hence $\varepsilon(f^c) \subseteq \varepsilon(f)^c$.

(iii) Suppose $0.5 < \varepsilon \leq f(x)$. Then $1 - f(x) \leq 1 - \varepsilon < 0.5$ and $1 - \varepsilon < f(x)$. Thus

$$\begin{aligned}\varepsilon(f)^c(x) &= 1 - \varepsilon(f)(x) = 1 - Y_J(\varepsilon, f(x))\\ &= 1 - ((1-\varepsilon) \wedge (1 - f(x)))\\ &= 1 - (1 - f(x)) = f(x)\end{aligned}$$

and

$$\begin{aligned}\varepsilon(f^c)(x) &= Y_J(\varepsilon, f^c(x)) = (1-\varepsilon) \wedge (1 - f^c(x))\\ &= (1-\varepsilon) \wedge f(x) = 1 - \varepsilon < 0.5 < f(x)\\ &= \varepsilon(f)^c(x).\end{aligned}$$

So $\varepsilon(f^c) \subseteq \varepsilon(f)^c$.

(iv) Suppose $0.5 < f(x) \leq \varepsilon$. Then we get $1 - \varepsilon \leq 1 - f(x) < 0.5$ and $1 - \varepsilon < f(x)$. Thus

$$\begin{aligned}\varepsilon(f)^c(x) &= 1 - \varepsilon(f)(x) = 1 - Y_J(\varepsilon, f(x))\\ &= 1 - ((1-\varepsilon) \wedge (1 - f(x)))\\ &= 1 - (1 - \varepsilon) = \varepsilon\end{aligned}$$

and

$$\begin{aligned}\varepsilon(f^c)(x) &= Y_J(\varepsilon, f^c(x)) = (1-\varepsilon) \wedge (1 - f^c(x))\\ &= (1-\varepsilon) \wedge f(x) = 1 - \varepsilon < f(x) \leq \varepsilon\\ &= \varepsilon(f)^c(x).\end{aligned}$$

So $\varepsilon(f^c) \subseteq \varepsilon(f)^c$.

(v) Suppose $\varepsilon \leq 0.5 \leq f(x)$. Then $1 - f(x) \leq 0.5 \leq 1 - \varepsilon$ and either $1 - \varepsilon \leq f(x)$ or $1 - \varepsilon > f(x)$. It follows that

$$\begin{aligned}\varepsilon(f)^c(x) &= 1 - \varepsilon(f)(x) = 1 - Y_J(\varepsilon, f(x))\\ &= 1 - ((1-\varepsilon) \wedge (1 - f(x)))\\ &= 1 - (1 - f(x)) = f(x)\end{aligned}$$

and

$$\begin{aligned}\varepsilon(f^c)(x) &= Y_J(\varepsilon, f^c(x)) = (1-\varepsilon) \wedge (1 - f^c(x)) = (1-\varepsilon) \wedge f(x)\\ &= \begin{cases} 1 - \varepsilon & \text{if } 1 - \varepsilon \leq f(x),\\ f(x) & \text{if } 1 - \varepsilon > f(x).\end{cases}\end{aligned}$$

Thus $\varepsilon(f^c) \subseteq \varepsilon(f)^c$.

(vi) Suppose $f(x) \leq 0.5 \leq_X \varepsilon$. Then $1-f(x) \geq 0.5 > 1-\varepsilon$ and either $1-\varepsilon \geq f(x)$ or $1-\varepsilon < f(x)$. It follows that

$$\begin{aligned}\varepsilon(f)^c(x) &= 1-\varepsilon(f)(x) = 1-Y_J(\varepsilon, f(x)) \\ &= 1-((1-\varepsilon)\wedge(1-f(x))) \\ &= 1-(1-\varepsilon) = \varepsilon \geq f(x)\end{aligned}$$

and

$$\begin{aligned}\varepsilon(f^c)(x) &= Y_J(\varepsilon, f^c(x)) = (1-\varepsilon)\wedge(1-f^c(x)) = (1-\varepsilon)\wedge f(x) \\ &= \begin{cases} f(x) & \text{if } 1-\varepsilon \geq f(x), \\ 1-\varepsilon & \text{if } 1-\varepsilon < f(x). \end{cases}\end{aligned}$$

Thus $\varepsilon(f^c) \subseteq \varepsilon(f)^c$. This completes the proof. □

Given two fuzzy sets f and g in a set X and $\varepsilon \in I$, we can naturally anticipate the next operations:

$$\varepsilon(f \wedge g) = \varepsilon(f) \wedge \varepsilon(g) \text{ and } \varepsilon(f \vee g) = \varepsilon(f) \vee \varepsilon(g).$$

However, the following example shows that these things are not generally true.

Example 3.9. Let f and g be constant fuzzy sets in any set X given by$f(x) = 0.3$ and $g(x) = 0.8$ for all $x \in X$. If we take $\varepsilon := 0.4 \in I$, then

$$\begin{aligned}\varepsilon(f\wedge g)(x) &= Y_J(\varepsilon, (f\wedge g)(x)) = Y_J(\varepsilon, f(x)\wedge g(x)) \\ &= (1-\varepsilon)\wedge(1-f(x)\wedge g(x)) \\ &= (1-0.4)\wedge(1-0.3\wedge 0.8) = 0.6\wedge 0.7 = 0.6\end{aligned}$$

and

$$\begin{aligned}(\varepsilon(f)\wedge\varepsilon(g))(x) &= \varepsilon(f)(x)\wedge\varepsilon(g)(x) \\ &= Y_J(\varepsilon, f(x))\wedge Y_J(\varepsilon, g(x)) \\ &= ((1-\varepsilon)\wedge(1-f(x)))\wedge((1-\varepsilon)\wedge(1-g(x))) \\ &= (1-\varepsilon)\wedge(1-f(x))\wedge(1-g(x)) \\ &= (1-0.4)\wedge(1-0.3)\wedge(1-0.8) = 0.2.\end{aligned}$$

Thus $\varepsilon(f\wedge g) \neq \varepsilon(f)\wedge\varepsilon(g)$. Also,

$$\begin{aligned}\varepsilon(f\vee g)(x) &= Y_J(\varepsilon, (f\vee g)(x)) = Y_J(\varepsilon, f(x)\vee g(x)) \\ &= (1-\varepsilon)\wedge((1-f(x)\vee g(x))) \\ &= (1-0.4)\wedge(1-0.3\vee 0.8) = 0.6\wedge 0.2 = 0.2\end{aligned}$$

and

$$\begin{aligned}(\varepsilon(f)\vee\varepsilon(g))(x) &= \varepsilon(f)(x)\vee\varepsilon(g)(x) \\ &= Y_J(\varepsilon, f(x))\vee Y_J(\varepsilon, g(x)) \\ &= ((1-\varepsilon)\wedge(1-f(x)))\vee((1-\varepsilon)\wedge(1-g(x))) \\ &= (1-0.4)\wedge(1-0.3))\vee((1-0.4)\wedge(1-0.8)) \\ &= 0.6\vee 0.2 = 0.6.\end{aligned}$$

So $\varepsilon(f \vee g) \neq \varepsilon(f) \vee \varepsilon(g)$.

Proposition 3.10. *If f and g are fuzzy sets in a set X and $\varepsilon \in I$, then $\varepsilon(f) \wedge \varepsilon(g) = \varepsilon(f \vee g)$ and $\varepsilon(f) \vee \varepsilon(g) = \varepsilon(f \wedge g)$.*

Proof. For every $x \in X$, we obtain

$$\begin{aligned}(\varepsilon(f) \wedge \varepsilon(g))(x) &= \varepsilon(f)(x) \wedge \varepsilon(g)(x) = Y_J(\varepsilon, f(x)) \wedge Y_J(\varepsilon, g(x)) \\ &= ((1-\varepsilon) \wedge (1-f(x))) \wedge ((1-\varepsilon) \wedge (1-g(x))) \\ &= (1-\varepsilon) \wedge ((1-f(x)) \wedge (1-g(x))) \\ &= (1-\varepsilon) \wedge (1-f(x) \vee g(x)) \\ &= (1-\varepsilon) \wedge (1-(f \vee g)(x)) \\ &= Y_J(\varepsilon, (f \vee g)(x)) = \varepsilon(f \vee g)(x)\end{aligned}$$

and

$$\begin{aligned}(\varepsilon(f) \vee \varepsilon(g))(x) &= \varepsilon(f)(x) \vee \varepsilon(g)(x) = Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, g(x)) \\ &= ((1-\varepsilon) \wedge (1-f(x))) \vee ((1-\varepsilon) \wedge (1-g(x))) \\ &= (1-\varepsilon \vee f(x)) \vee (1-\varepsilon \vee g(x)) \\ &= 1-\varepsilon \vee (f(x) \wedge g(x)) \\ &= (1-\varepsilon) \wedge (1-f(x) \wedge g(x)) \\ &= Y_J(\varepsilon, f(x) \wedge g(x)) \\ &= Y_J(\varepsilon, (f \wedge g)(x)) = \varepsilon(f \wedge g)(x).\end{aligned}$$

Then $\varepsilon(f) \wedge \varepsilon(g) = \varepsilon(f \vee g)$ and $\varepsilon(f) \vee \varepsilon(g) = \varepsilon(f \wedge g)$. □

4. Y_J^{ε}-fuzzy subalgebras

In what follows, let $(X, *, 0)$ be a BCK-algebra or a BCI-algebra, and $\varepsilon \in (0, 1)$ unless otherwise specified.

Definition 4.1. A fuzzy set f in X is called a *Y_J^{ε}-fuzzy subalgebra* of $(X, *, 0)$, if the next inequality is valid.

$$(\forall x, a \in X)(Y_J(\varepsilon, f(x * a)) \leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(a))). \tag{4.1}$$

Example 4.2. (1) Consider a $BCK-$algebra $X = \{0, 1, 2, 3, 4\}$ (See [7]) with the binary operation "$*$" given by Table 1:

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x = 0, \\ 0.63 & \text{if } x = 1, \\ 0.69 & \text{if } x = 2, \\ 0.47 & \text{if } x = 3, \\ 0.47 & \text{if } x = 4. \end{cases}$$

Then f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for all ε.

(2) Let $X = \{0, 1, 2, a, b\}$ be a set with the binary operation "$*$" given by Table 2:

Table 2. Cayley table for the binary operation "$*$"

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0

Then $(X, *, 0)$ is a BCI-algebra (See [6]). Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ y \mapsto \begin{cases} 0.46 & \text{if } y = 0, \\ 0.61 & \text{if } y = 1, \\ 0.53 & \text{if } y = 2, \\ 0.54 & \text{if } y = a, \\ 0.38 & \text{if } y = b. \end{cases}$$

Then f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for $\varepsilon = 0.61$. But it is not a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for $\varepsilon = 0.43$ since

$$\begin{aligned} Y_J(\varepsilon, f(2 * a)) &= Y_J(\varepsilon, f(b)) = Y_J(0.43, 0.38) = 0.57 \\ &\nleq 0.47 = Y_J(0.43, 0.53) \vee Y_J(0.43, 0.54) \\ &= Y_J(\varepsilon, f(2)) \vee Y_J(\varepsilon, f(a)). \end{aligned}$$

Proposition 4.3. *Every Y_J^ε-fuzzy subalgebra f of $(X, *, 0)$ satisfies:*

$$(\forall x \in X)(Y_J(\varepsilon, f(0)) \leq Y_J(\varepsilon, f(x))), \tag{4.2}$$

$$(\forall x, a \in X)\left(\begin{array}{c} Y_J(\varepsilon, f(0)) = Y_J(\varepsilon, f(x)) \ \Leftrightarrow \\ Y_J(\varepsilon, f(x * a)) \leq Y_J(\varepsilon, f(a)). \end{array} \right). \tag{4.3}$$

Proof. Using (III) and (4.1), we have

$$Y_J(\varepsilon, f(0)) = Y_J(\varepsilon, f(x * x)) \leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(x)) = Y_J(\varepsilon, f(x))$$

for all $x \in X$. Thus (4.2) is valid. Assume that $Y_J(\varepsilon, f(0)) = Y_J(\varepsilon, f(x))$ for all $x \in X$. Then

$$\begin{aligned} Y_J(\varepsilon, f(x * a)) &\leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(a)) \\ &= Y_J(\varepsilon, f(0)) \vee Y_J(\varepsilon, f(a)) \\ &= Y_J(\varepsilon, f(a)) \end{aligned}$$

for all $x, a \in X$. Conversely, if $Y_J(\varepsilon, f(x * a)) \leq Y_J(\varepsilon, f(a))$ for all $x, a \in X$, then $Y_J(\varepsilon, f(x)) = Y_J(\varepsilon, f(x * 0)) \leq Y_J(\varepsilon, f(0))$. Thus $Y_J(\varepsilon, f(0)) = Y_J(\varepsilon, f(x))$ for all $x \in X$. □

Corollary 4.4. *If f is a fuzzy subalgebra of $(X, *, 0)$, then its Y_J^ε-fuzzy set $\varepsilon(f)$ satisfies:*

$$(\forall x \in X)(\varepsilon(f)(0) \leq \varepsilon(f)(x)).$$
$$(\forall x, a \in X)(\varepsilon(f)(0) = \varepsilon(f)(x) \;\Leftrightarrow\; \varepsilon(f)(x * a) \leq \varepsilon(f)(a)).$$

Lemma 4.5. *Every Y_J^ε-fuzzy subalgebra f of a BCI-algebra $(X, *, 0)$ satisfies:*

$$(\forall x \in X)(Y_J(\varepsilon, f(0 * x)) \leq Y_J(\varepsilon, f(x))). \tag{4.4}$$

Proof. It is induced by the combination of (4.1) and (4.2). □

Corollary 4.6. *If f is a fuzzy subalgebra of a BCI-algebra $(X, *, 0)$, then its Y_J^ε-fuzzy set $\varepsilon(f)$ satisfies:*

$$(\forall x \in X)(\varepsilon(f)(0 * x) \leq \varepsilon(f)(x)).$$

Proposition 4.7. *In a BCI-algebra $(X, *, 0)$, every Y_J^ε-fuzzy subalgebra f satisfies:*

$$(\forall x, a \in X)(Y_J(\varepsilon, f(x * (0 * a))) \leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(a))). \tag{4.5}$$

Proof. Using (4.1) and Lemma 4.5, we have

$$\begin{aligned} Y_J(\varepsilon, f(x * (0 * a))) &\leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(0 * a)) \\ &\leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(a)) \end{aligned}$$

for all $x,\ a \in X$. □

Corollary 4.8. *If f is a fuzzy subalgebra of a BCI-algebra $(X, *, 0)$, then its Y_J^ε-fuzzy set $\varepsilon(f)$ satisfies:*

$$(\forall x, a \in X)(\varepsilon(f)(x * (0 * a)) \leq \varepsilon(f)(x) \vee \varepsilon(f)(a)).$$

We provide conditions for a fuzzy set to be a Y_J^ε-fuzzy subalgebra.

Theorem 4.9. *If a fuzzy set f in X satisfies:*

$$(\forall x, y, z \in X)(z \leq_X x \;\Rightarrow\; Y_J(\varepsilon, f(x * y)) \leq Y_J(\varepsilon, f(y)) \vee Y_J(\varepsilon, f(z))), \tag{4.6}$$

then f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$.

Proof. Assume that a fuzzy set f in X satisfies (4.6). Since $x \leq_X x$ for all $x \in X$, we get

$$Y_J(\varepsilon, f(x * y)) \leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(y))$$

for all $x,\ y \in X$. Then f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$. □

We discuss the relationship between the fuzzy subalgebra and the Y_J^ε-fuzzy subalgebra.

Theorem 4.10. *Every fuzzy subalgebra of* $(X, *, 0)$ *is a* Y_J^ε*-fuzzy subalgebra of* $(X, *, 0)$ *for all* $\varepsilon \in (0,1)$.

Proof. Let f be a fuzzy subalgebra of $(X, *, 0)$ and let $\varepsilon \in (0,1)$. Then $f^c(x*a) \leq f^c(x) \vee f^c(a)$ for all $x,\ a \in X$. Thus

$$\begin{aligned} Y_J(\varepsilon, f(x*a)) &= (1-\varepsilon) \wedge f^c(x*a) \\ &\leq (1-\varepsilon) \wedge (f^c(x) \vee f^c(a)) \\ &= ((1-\varepsilon) \wedge f^c(x)) \vee ((1-\varepsilon) \wedge f^c(a)) \\ &= Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(a)) \end{aligned}$$

for all $x,\ a \in X$. So f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for all $\varepsilon \in (0,1)$. □

Question 4.11. Let f be a fuzzy set in X. If f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for some $\varepsilon \in (0,1)$, then is f a fuzzy subalgebra of $(X, *, 0)$?

The answer to Question 4.11 is negative as seen in the following example.

Example 4.12. Let $(X, *, 0)$ be the BCK-algebra in Example 4.2(i) and let f be a fuzzy set in X given as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.69 & \text{if } x = 0, \\ 0.61 & \text{if } x = 1, \\ 0.57 & \text{if } x = 2, \\ 0.48 & \text{if } x = 3, \\ 0.53 & \text{if } x = 4. \end{cases}$$

Then f is a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ for $\varepsilon = 0.69$. But f is not a fuzzy subalgebra of $(X, *, 0)$ because of

$$f(4*1) = f(3) = 0.48 \not\geq 0.53 = f(4) \wedge f(1).$$

Theorem 4.13. *Let f be a fuzzy set in X and let ε be a nonconstant factor in* $(0,1)$*. If f is a* Y_J^ε*-fuzzy subalgebra of* $(X, *, 0)$*, then it is a fuzzy subalgebra of* $(X, *, 0)$.

Proof. Let f be a Y_J^ε-fuzzy subalgebra of $(X, *, 0)$ where ε is a nonconstant factor in $(0,1)$. Then

$$\begin{aligned} &(1-\varepsilon) \wedge (1 - f(x*a)) = Y_J(\varepsilon, f(x*a)) \\ &\leq Y_J(\varepsilon, f(x)) \vee Y_J(\varepsilon, f(a)) \\ &= ((1-\varepsilon) \wedge (1-f(x))) \vee ((1-\varepsilon) \wedge (1-f(a))) \\ &= (1-\varepsilon) \wedge ((1-f(x)) \vee (1-f(a))). \end{aligned}$$

It follows that $1 - f(x*a) \leq (1-f(x)) \vee (1-f(a)) = 1 - f(x) \wedge f(a)$, and thus $f(x*a) \geq f(x) \wedge f(a)$ for all $x,\ a \in X$. So f is a fuzzy subalgebra of $(X, *, 0)$. □

5. Conclusions

In order to develop a new type of fuzzy set, the concept of the J-operator in the closed interval $[0, 1]$ was first introduced and the necessary properties were checked. By using the J-operator, a new fuzzy set called Y_J^ε-fuzzy set was formed, and it was applied to the subalgebra of BCK/BCI-algebras. We introduced the concept of Y_J^ε-fuzzy subalgebras and investigated its properties. We explored conditions for the fuzzy set to be Y_J^ε-fuzzy subalgebra, and further discussed the relationship between fuzzy subalgebra and Y_J^ε-fuzzy subalgebra. Future research will first apply the ideas and results presented in this paper to different types of ideals or filters in BCK/BCI-algebras, and then to other logical algebras. We also want to provide a basis for this Y_J^ε-fuzzy set to be applied to decision making, medical diagnostic systems, state machines, pattern registration, and so on, as the fuzzy set did.

Acknowledgment

The author appreciates the valuable suggestions of anonymous reviewers.

References

[1] C. Jana, T. Senapati and M. Pal, $(\in, \in \vee q)$-intuitionistic fuzzy BCI-subalgebras of a BCI-algebra, J. Intell. Fuzzy Systems, 31 (2016) 613–621. DOI:10.3233/IFS-162175
[2] Y. B. Jun, Fuzzy subalgebras of type (α, β) in BCK/BCI-algebras, Kyungpook Math. J. 47 (2007) 403–410.
[3] Y. B. Jun, Łukasiewicz fuzzy subalgebras in BCK-algebras and BCI-algebras, Ann. Fuzzy Math. Inform. 23 (2) (2022) 213–223. https://doi.org/10.30948/afmi.2022.23.2.213
[4] K. Iséki, On BCI-algebras, Math. Seminar Notes 8 (1980) 125–130.
[5] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 23 (1978) 1–26.
[6] Y. Huang, BCI-algebra, Science Press: Beijing, China 2006.
[7] J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co. Seoul, Korea 1994.
[8] L. A. Zadeh, Fuzzy sets, Information and Control 8(3) (1965) 338–353.
[9] O. G. Xi, Fuzzy BCK-algebras, Math. Japon. 36 (1991) 935–942.

Young Bae Jun (skywine@gmail.com)
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea