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Ogunfolu Olusola B.

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On the number of subgroups and distinct fuzzy subgroups of a group defined by a presentation

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Abstract. In this paper, we determined the number of subgroups of a group $\rho_2 \in G$ defined by a presentation $\rho_2 = \langle a, b \mid a^{27} = b^{27} = e, [a, b] = a^9 \rangle$, where a, b are generators and G is a class of finite minimal nonabelian p-groups with generators of the same order. The form and the order of elements of ρ_2 were identified. We determined chains of subgroups and the number of distinct fuzzy subgroups by defining a natural equivalence relation on the subgroup lattice $L(\rho_2 \in G)$ associated with counting number of chains of subgroups of ρ_2 that ends in ρ_2.

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Corresponding Author: Olusola B. Ogunfolu(teminiyidele@gmail.com)

1. Introduction

The idea of fuzzy sets was introduced by Zadeh [1] and Rosenfeld [2] brought about the concept of fuzzy subgroups. There are uncountably many distinct fuzzy sets on the same (finite or infinite) domain and uncountable codomain (that is, the real interval $[0, 1]$). Also, algebraic structures like groups have uncountably many fuzzy subgroups. Many authors have treated the particular case of finite cyclic group. Zhang and Zou [3] also determined the number of fuzzy subgroups of cyclic groups of the order p^n, where p is a prime number. In addition, Murali and Makamba [4] found the number of fuzzy subgroups of abelian groups of the order $p^n q^m$ where p and q are different primes (For more references [5, 6, 7]). Calugaceanu [8] used Goursat's lemma for groups and thus, derived an explicit formula to find the number of subgroups of a finite abelian group.

Tarnauceanu and Bentea [9] on their part, established a recurrence relation which was used to count the number of distinct fuzzy subgroups for two classes of abelian groups: finite cyclic groups and finite elementary abelian p-groups. Furthermore, Tarnauceanu [10] established a connection between the fuzzy subgroups of a finite cyclic group with k direct factors and the lattice paths of $\mathbb{Z}^k$ which leads to an explicit formula for the well-known central Delannoy numbers. Tarnauceanu [11] similarly used the arithmetic method to count the number of subgroups of a finite abelan group. More of Tarnauceanu works is in [12]. Petrillo [13] however, used structure theorem by Goursat and also apply the fundamental theorem of finite abelian groups to calculate an explicit formula for finding the number of subgroups in a direct product of finite cyclic groups. Onasanya and Ilori [14] worked on fuzzy subgroups and fuzzy cosets.

Amit and Yogesh [15] worked on the number of subgroups of finite abelian group $Z_m \bigotimes Z_n$ while Gideon [16] classified fuzzy subgroups of finite abelian groups $G = \mathbb{Z}_{p^n + \mathbb{Z}_{q^m}}$, where $\mathbb{Z}$ is an integer, p and q are distinct primes and m, n are natural numbers by using the notion of equivalence classes. Humera and Raza [17] as well as Imanparast and Darabi [18] determined the number of fuzzy subgroups of finite abelian group $G = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_n}$, where $p_1, p_2, \cdots, p_n$ are all distinct primes and $G = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$ whereas Hampejset al. [19] established a formula for counting the number of subgroups of the group $Z_m \times Z_n$ by using simple group-theoretic and number theoretic arguments.

Gautami [20] obtained a general formula for counting the number of subgroups of a p-group of arbitrary rank by using combinatorial arguments. Hampejs and Laszlo [21] on the other hand, studied the number of subgroups of finite abelian groups of rank three by using the simple group-theoretic and number-theoretic arguments, while Laszlo [22] deduced an explicit formula for counting the number of subgroups of finite abelian groups having rank two. Tarnauceanu and Laszlo [23] however, determined the number of subgroups of a given exponent in a finite abelian p-groups of rank two and of rank three. Appiah and Makamba [24] on their part classified fuzzy subgroups of a rank-3 abelian group $G = Z_{p^n} + Z_p + Z_p$ for any fixed prime p and any positive integer n, using a natural equivalence relation.

Sulaiman and Ahmad [25] classified the number of fuzzy subgroups of finite cyclic groups and also counted the number of fuzzy subgroups of group $G(\rho)$ defined by presentation $\rho = \langle a, b \mid a^2, b^q, ab = b^r a\rangle$ with q prime number and $r < q$ and also Humera and Raza [26] determined the number of subgroups of finite nonabelian group defined by a presentation $G = \langle x, y \mid x^2 = y^{16} = 1, yx = xy^7\rangle$. Darabi and Imanparast [27] computed number of fuzzy subgroups of some dihedral groups such as D_{2p^n} where p is a prime number and $D_{2p_1 \times p_2 \times \cdots \times p_n}$ where $p_1, p_2, \cdots, p_n$ are distinct prime numbers. EniOluwafe [28] on his own part gave an explicit formula for counting subgroups of type: $D_{2^{n-1}} \times C_2, n \geqslant 3$. Onasanya and Ilori [29] studied the action of a group on a fuzzy set via fuzzy membership function.

Olapade and EniOluwafe [30] counted the number of subgroups for a class of finite nonabelian p-groups. Olayiwola and EniOluwafe [31] counted the number of distinct fuzzy subgroups of certain dihedral group. Olayiwola [32] classified fuzzy subgroups of some finite dihedral groups. Sulaiman and Ahmad [33] computed number of fuzzy subgroups of symmetric groups S_2, S_3 and alternating group A_4. Sulaiman [34] defined an equivalence relation on the set of all fuzzy subgroups of a symmetric group S_4. He used the diagram of subgroups lattice of S_4 to determine the number of fuzzy subgroups while Ogiugo and EniOluwafe [35] classified the fuzzy subgroups of certain finite nonabelian symmetric groups. Adebisi [36] computed the number of distinct fuzzy subgroups of the modular nilpotent group formed by taking the cartesian product of the modular p-group M_{p^n} of order p^n and a cyclic group C_p of order p, where p is a prime, Tarnauceanu [37] introduced a new equivalence relation. He classified fuzzy subgroups with respect to this new equivalence relation defined on the lattice of fuzzy subgroups of a finite group. [38] and [39] discussed extensively on subgroups and p-groups.

In this paper, we determined the order of elements of a presentation group $\rho_2 \in G$, all subgroups, chains of subgroups and number of distinct fuzzy subgroups of the presentation group by defining a natural equivalence relation on the subgroup lattice $L(\rho_2 \in G)$ associated with counting number of chains of subgroups of ρ_2 that ends in ρ_2 [12]. The package GAP - Groups, Algorithms, and Programming [40] was also used.

2. Preliminaries

In this section, we shall discuss fuzzy algebraic structure via elements.

Definition 2.1 ([38])**.** A *p-group* is a group in which every element has order equal to a power of p, where p is a prime number. Then a finite group is a p-group, if its order is a power of p. Let the order of a nontrivial finite group G be $n = \prod_{i=1}^{t} p_i^{n_i}$. A subgroup $H \subseteq G$ is a *p_i-group*, if its order is some power of p_i.

Remark 2.2 ([39])**.** Let the order of a nontrivial finite group G be $n = \prod_{i=1}^{t} p_i^{n_i}$. A sylow p_i-subgroup of a group G is a maximal p_i-subgroup of G that is not a proper subgroup of any other p_i-subgroup of G. The set of all sylow p-subgroups is denoted by $Syl_p(G)$.

Definition 2.3 ([38])**.** Let G be a finite group and let p^β be the maximal power of prime p dividing $\mid G \mid$. Then

- (i) every p-subgroup of G belongs to some subgroup of order p^β, i.e., there exists a sylow p-subgroup of G of order p^β,
- (ii) the number of sylow p-subgroups $n_p \cong 1(mod p)$,
- (iii) any two sylow p-subgroups are conjugate in G.

Definition 2.4 ([39])**.** A *subgroup series* is a chain of subgroups

$$1 = H_0 \leq H_1 \leq \cdots \leq H_r = G.$$

A series with the additional property that $H_j \neq H_{j+1}$ for all j is called a *series without repetition*, equivalently, each H_j is a proper subgroup of H_{j+1}.

Remark 2.5 ([39])**.** The length of a series is the number of strict inclusion $H_j < H_{j+1}$. If the series has no repetition the length is r. Series can be noted in either ascending order:

$$1 = H_0 \leq H_1 \leq \cdots \leq H_r = G$$

or descending order:

$$G = H'_0 \geq H'_1 \geq \cdots \geq H'_r = 1.$$

Definition 2.6 ([7])**.** A *finite r-chain* is a collection of numbers on closed interval $[0,1]$ of the form $1 > \alpha_1 > \alpha_2 > \alpha_3 > \cdots > \alpha_{r-1} > \alpha_r$. We simply write $1_{\alpha_1\alpha_2\alpha_3\cdots\alpha_{r-1}\alpha_r}$ in the descending order for the above r-chain. The *length* of an r-chain is $(r+1)$. The numbers $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{r-1}, \alpha_r$ are called *points*. An r-chain is called a *key chain*, if $1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_{r-1} \geq \alpha_r \geq 0$.

Definition 2.7 ([7])**.** A chain $\{e\} \subseteq G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots \subseteq G_n = G$ of subgroups of G is a *maximal chain*, if no new subgroups can be inserted in the chain. Such a chain is also called a *flag* of G.

Definition 2.8 ([15])**.** Let $(G, \cdot, e)$ be a group with a binary operation $\cdot$ and identity element $e \in G$ and let the collection of all fuzzy subsets of G be denoted by $F(G)$. Then $\mu \in F(G)$ is a *fuzzy subgroup* of G, if for any $x,\ y \in G$,

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,
(ii) $\mu(x^{-1}) \geq \mu(x)$.

Proposition 2.9 (See Propositions 5.6 and 5.4, [2])**.** *Let e denote the identity element of G. If μ is a fuzzy subgroup of G, then*

(i) $\mu(x) = \mu(x^{-1})$ *for all* $x \in G$
(ii) $\mu(e) \geq \mu(x)$ *for all* $x \in G$ *and* $e \in G$.

Definition 2.10 ([12])**.** Let μ be a fuzzy subgroup of a group G and for each $\alpha \in \mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ and $\alpha_1, \alpha_2, \cdots, \alpha_r \in [0,1]$. Then the subset of G

$$\mu_\alpha = \{x \in G \mid \mu(x) \geq \alpha\}$$

is a classical subgroup of G and is called an *α-level subset* of G or simply level subset of G under μ_α or ${}_\mu G_\alpha$.

If μ is a fuzzy subgroup of a group G and $\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ then we assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_r$

Theorem 2.11. *See Tarnauceanu [12]] Let $M_1, M_2, \cdots, M_k$ be the maximal subgroups of G. Let $g(G)$ be the number of maximal chains of subgroups in G and $h(G)$ be the number of chains of subgroups of G ended in G. Let C be the set of chains in G of type*

$$H_1 \subset H_2 \subset \cdots \subset H_r = G,$$

C' be the set of chains in G of type

$$H_1 \subset H_2 \subset \cdots \subset H_r \neq G.$$

Let C_i be the set of chains of C' which are contained in $M_i, i = 1, 2, \cdots, k$. Then

$$h(G) = 2\left(\sum_{i=1}^{k} h(M_i) - \sum_{1\leq i_1<i_2\leq k} h(Mi_1 \cap Mi_2) + \ldots + (-1)^{k-1} h(\bigcap_{i=1}^{k} M_i)\right).$$

Example 2.12. Let M_1, M_2, M_3 and M_4 denote the maximal subgroups in the presentation group ρ_2. Then by using the Exclusion-Inclusion Principle in Theorem 2.11:

$$h(G) = 2\left(\sum_{i=1}^{k} h(M_i) - \sum_{1\leq i_1<i_2\leq k} h(Mi_1 \cap Mi_2) + \ldots + (-1)^{k-1} h(\bigcap_{i=1}^{k} M_i)\right),$$

we obtain,

(a) $h(C_3 \times C_3) = h(Z_3 \times Z_3)$
$$= 2(h(C_3) + h(C_3) + h(C_3) + h(C_3) - h(1) - h(1) - h(1)$$
$$-h(1) - h(1) - h(1) + h(1) + h(1) + h(1) + h(1) - h(1))$$
$$= 2(4h(C_3) + (-6 + 4 - 1)h(1)) = 2(4h(C_3) - 3h(1))$$
$$= 2(4(2) - 3(1)) = 2(5) = 10,$$

(b) $h(C_9) = 2(h(C_3)) = 2(2) = 4$,

(c) $h(C_9 \times C_3) = 2(h(C_9) + h(C_3 \times C_3) + h(C_9) + h(C_9) - h(C_3) - h(C_3)$
$$-h(C_3) - h(C_3) - h(C_3) - h(C_3) + h(C_3) + h(C_3)$$
$$+h(C_3)h(C_3) - h(C_9)) = 32,$$

(d) $h(\rho_1) = h(C_{3^2} : C_{3^2})$
$$= 2(h(C_9 \times C_3) + h(C_9 \times C_3) + h(C_9 \times C_3) + h(C_9 \times C_3)$$
$$-h(C_3 \times C_3) - h(C_3 \times C_3) - h(C_3 \times C_3) - h(C_3 \times C_3)$$
$$-h(C_3 \times C_3) - h(C_3 \times C_3) + h(C_3 \times C_3) + h(C_3 \times C_3)$$
$$+h(C_3 \times C_3) + h(C_3 \times C_3) - h(C_3 \times C_3)$$
$$= 2(4h(C_9 \times C_3) + (-6 + 4 - 1)h(C_3 \times C_3))$$
$$= 2(4(32) - 3(10)) = 2(98) = 196,$$

(e) $h(C_{27}) = 2(h(C_9)) = 2(4) = 8$,

(f) $h(C_{27} \times C_3) = 2(h(C_{27}) + h(C_9 \times C_3) + h(C_{27}) + h(C_{27})$
$$+h(C_9) + h(C_9) + h(C_9) + h(C_9) + h(C_9) + h(C_9)$$
$$-h(C_9) - h(C_9) - h(C_9) - h(C_9) + h(C_9))$$
$$= 2(3h(C_{27}) + h(C_9 \times C_3) + (-6 + 4 - 1)h(C_9))$$
$$= 2(3(8) + 32 - 3(4)) = 2(44) = 88,$$

(g) $h(C_{27} \times C_9) = 2(3h(C_{27} \times C_3) + h(C_9 \times C_9) - 3h(C_9 \times C_3))$
$$= 2(3(88) + 196 - 3(32)) = 2(364) = 728,$$

(h) $h(\rho_2) = h(C_{3^3} : C_{3^3}) = 2(4h(C_{27} \times C_9) - 3h(C_9 \times C_9))$
$$= 2(4(728) - 3(196)) = 2(2324) = 4648.$$

Thus the number of distinct fuzzy subgroups of the presentation group ρ_2 is 4648. The chain method and the Exclusion-Inclusion Principle therefore gives the same number of fuzzy subgroups.

Lemma 2.13 (See Lemma 3, [27])**.** *The number of fuzzy subgroups of G is equal to the number of chains on the lattice of subgroups of G.*

3. The main results

3.1. **Elements of ρ_2.**

Considering the presentation group

$$\rho_2 \in G = \langle a, b \mid a^{27} = b^{27} = e, [a, b] = a^9 \rangle$$

The elements of $\rho_2 \in G$ with generators a and b are written as $e, a, a^2, \cdots, a^{26}, b, b^2, \cdots, b^{26}, ab, \cdots, ab^{26}, a^2b, \cdots, a^2b^{26}, a^3b, \cdots, a^3b^{26}, a^4b, \cdots, a^4b^{26}, a^5b, \cdots, a^5b^{26}, a^6b, \cdots, a^6b^{26}, a^7b, \cdots, a^7b^{26}, \cdots, a^{25}b, \cdots, a^{25}b^{26}, a^{26}b, \cdots, a^{26}b^{26}$.

3.2. **Subgroups of ρ_2.**

3.2.1. *Subgroups of order 3.* $\langle a^9 \rangle$, $\langle b^9 \rangle$, $\langle a^9b^9 \rangle$, $\langle a^{18}b^9 \rangle$

3.2.2. *Subgroups of order 9.* $\langle a^9, b^9 \rangle$, $\langle a^{24}b^3 \rangle$, $\langle b^3 \rangle$, $\langle a^{18}b^3 \rangle$, $\cdots$, $\langle a^6b^3 \rangle$, $\langle a^{15}b^3 \rangle$

3.2.3. *Subgroups of order 27.* $\langle a^3, b^9\rangle$, $\langle a^3b^3, a^9\rangle$, $\langle a^{24}b^3, a^9\rangle$, $\cdots$, $\langle a^{19}b\rangle$, $\langle ab\rangle$, $\cdots$, $\langle a^{17}b\rangle$, $\langle a^{24}b\rangle$, $\cdots$, $\langle a^{25}b\rangle$, $\langle a^7b\rangle$, $\langle a^{16}b\rangle$

3.2.4. *Subgroups of order 81.* $\langle a, b^9\rangle$, $\langle b, a^9\rangle$,$\cdots$, $\langle a^{23}b, a^9\rangle$, $\langle a^8b, a^9\rangle$,

3.2.5. *Subgroups of Order 243.* $\langle a, b^3\rangle$, $\langle b, a^3\rangle$, $\langle a^{19}b, a^3\rangle$.

3.3. **Chains of subgroups of ρ_2.**
The subgroups of ρ_2 were numbered starting from e (identity) as Number (1) and ρ_2 as Number (76). The number of chains of subgroups of length 1 to chains of subgroups of length 5 were counted in other to determine the number of distinct fuzzy subgroups of ρ_2. The chains of subgroups of length 1 and of length 2 were shown below:

Chains of subgroups of length 1

Order $3, \rho_2$

$2 \subset 76$, $3 \subset 76$, $4 \subset 76$, $5 \subset 76$,

Order $9, \rho_2$

$6 \subset 76$, $7 \subset 76$, $8 \subset 76$, $9 \subset 76$, $10 \subset 76$, $11 \subset 76$, $12 \subset 76$, $13 \subset 76$, $14 \subset 76$, $15 \subset 76$, $16 \subset 76$, $17 \subset 76$, $18 \subset 76$,

Order $27, \rho_2$

$19 \subset 76$, $20 \subset 76$, $21 \subset 76$, $22 \subset 76$, $23 \subset 76$, $24 \subset 76$, $25 \subset 76$, $26 \subset 76$, $27 \subset 76$, $28 \subset 76$,, $29 \subset 76$, $30 \subset 76$, $31 \subset 76$, $32 \subset 76$, $33 \subset 76$, $34 \subset 76$, $35 \subset 76$, $36 \subset 76$, $37 \subset 76$, $38 \subset 76$, $39 \subset 76$, $40 \subset 76$, $41 \subset 76$, $42 \subset 76$, $43 \subset 76$, $44 \subset 76$, $45 \subset 76$, $46 \subset 76$, $47 \subset 76$, $48 \subset 76$, $49 \subset 76$, $50 \subset 76$, $51 \subset 76$, $52 \subset 76$, $53 \subset 76$, $54 \subset 76$, $55 \subset 76$, $56 \subset 76$, $57 \subset 76$, $58 \subset 76$,

Order $81, \rho_2$

$59 \subset 76$, $60 \subset 76$, $61 \subset 76$, $61 \subset 76$, $62 \subset 76$, $63 \subset 76$, $64 \subset 76$, $65 \subset 76$, $66 \subset 76$, $67 \subset 76$, $68 \subset 76$, $69 \subset 76$, $70 \subset 76$, $71 \subset 76$,

Order $243, \rho_2$

$72 \subset 76$, $73 \subset 76$, $74 \subset 76$, $75 \subset 76$,

Chains of subgroups of length 2

Order $3, 9, \rho_2$

$2 \subset 6 \subset 76$,. $2 \subset 11 \subset 76$, $2 \subset 12 \subset 76$, $2 \subset 13 \subset 76$, $3 \subset 6 \subset 76$, $3 \subset 8 \subset 76$, $3 \subset 9 \subset 76$, $3 \subset 10 \subset 76$, $4 \subset 6 \subset 76$, $4 \subset 14 \subset 76$, $4 \subset 15 \subset 76$, $4 \subset 16 \subset 76$, $5 \subset 6 \subset 76$,. $5 \subset 7 \subset 76$, $5 \subset 17 \subset 76$, $5 \subset 18 \subset 76$,

Order $3, 27, \rho_2$

$2 \subset 19 \subset 76$, $2 \subset 20 \subset 76$, $2 \subset 21 \subset 76$, $2 \subset 22 \subset 76$, $2 \subset 23 \subset 76$, $2 \subset 30 \subset 76$, $2 \subset 31 \subset 76$, $2 \subset 32 \subset 76$, $2 \subset 36 \subset 76$, $2 \subset 37 \subset 76$, $2 \subset 38 \subset 76$, $2 \subset 42 \subset 76$, $2 \subset 43 \subset 76$, $3 \subset 19 \subset 76$, $3 \subset 20 \subset 76$, $3 \subset 21 \subset 76$, $3 \subset 22 \subset 76$, $3 \subset 39 \subset 76$, $3 \subset 40 \subset 76$, $3 \subset 41 \subset 76$, $3 \subset 47 \subset 76$, $3 \subset 48 \subset 76$, $3 \subset 49 \subset 76$, $3 \subset 53 \subset 76$, $3 \subset 54 \subset 76$, $3 \subset 55 \subset 76$, $4 \subset 19 \subset 76$, $4 \subset 20 \subset 76$, $4 \subset 21 \subset 76$, $4 \subset 22 \subset 76$, $4 \subset 24 \subset 76$, $4 \subset 25 \subset 76$, $4 \subset 26 \subset 76$, $4 \subset 33 \subset 76$, $4 \subset 34 \subset 76$, $4 \subset 35 \subset 76$, $4 \subset 56 \subset 76$, $4 \subset 57 \subset 76$, $4 \subset 58 \subset 76$, $5 \subset 19 \subset 76$, $5 \subset 20 \subset 76$, $5 \subset 21 \subset 76$, $5 \subset 22 \subset 76$, $5 \subset 27 \subset 76$, $5 \subset 28 \subset 76$, $5 \subset 29 \subset 76$, $5 \subset 44 \subset 76$, $5 \subset 45 \subset 76$, $5 \subset 46 \subset 76$, $5 \subset 50 \subset 76$, $5 \subset 51 \subset 76$, $5 \subset 52 \subset 76$,

Order $3, 81, \rho_2$

$2 \subset 59 \subset 76$, $2 \subset 60 \subset 76$, $2 \subset 61 \subset 76$, $2 \subset 62 \subset 76$, $2 \subset 63 \subset 76$, $2 \subset 64 \subset 76$,

$2\subset 65\subset 76$, $2\subset 66\subset 76$, $2\subset 67\subset 76$, $2\subset 68\subset 76$, $2\subset 69\subset 76$, $2\subset 70\subset 76$, $2\subset 71\subset 76$, $3\subset 59\subset 76$, $3\subset 60\subset 76$, $3\subset 61\subset 76$, $3\subset 62\subset 76$, $3\subset 63\subset 76$, $3\subset 64\subset 76$, $3\subset 65\subset 76$, $3\subset 66\subset 76$, $3\subset 67\subset 76$, $3\subset 68\subset 76$, $3\subset 69\subset 76$, $3\subset 70\subset 76$, $3\subset 71\subset 76$, $4\subset 59\subset 76$, $4\subset 60\subset 76$, $4\subset 61\subset 76$, $4\subset 62\subset 76$, $4\subset 63\subset 76$, $4\subset 64\subset 76$, $4\subset 65\subset 76'$ $4\subset 66\subset 76$, $4\subset 67\subset 76$, $4\subset 68\subset 76$, $4\subset 69\subset 76$, $4\subset 70\subset 76$, $4\subset 71\subset 76$, $5\subset 59\subset 76$, $5\subset 60\subset 76$, $5\subset 61\subset 76$, $5\subset 62\subset 76$, $5\subset 63\subset 76$, $5\subset 64\subset 76$, $5\subset 65\subset 76$, $5\subset 66\subset 76$, $5\subset 67\subset 76$, $5\subset 68\subset 76$, $5\subset 69\subset 76$, $5\subset 70\subset 76$, $5\subset 71\subset 76$,

Order $3, 243, \rho_2$

$2\subset 72\subset 76$, $2\subset 73\subset 76$, $2\subset 74\subset 76$, $2\subset 75\subset 76$, $3\subset 72\subset 76$, $3\subset 73\subset 76$, $3\subset 74\subset 76$, $3\subset 75\subset 76$, $4\subset 72\subset 76$, $4\subset 73\subset 76$, $4\subset 74\subset 76$, $4\subset 75\subset 76$, $5\subset 72\subset 76$, $5\subset 73\subset 76$, $5\subset 74\subset 76$, $5\subset 75\subset 76$,

Order $9, 27, \rho_2$

$6\subset 19\subset 76$, $6\subset 20\subset 76$, $6\subset 21\subset 76$, $6\subset 22\subset 76$, $7\subset 21\subset 76$, $7\subset 44\subset 76$, $7\subset 45\subset 76$, $7\subset 46\subset 76$, $8\subset 22\subset 76$, $8\subset 47\subset 76$, $8\subset 48\subset 76$, $8\subset 49\subset 76$, $9\subset 22\subset 76$, $9\subset 39\subset 76$, $9\subset 40\subset 76$, $9\subset 41\subset 76$, $10\subset 22\subset 76$, $10\subset 53\subset 76$, $10\subset 54\subset 76$, $10\subset 55\subset 76$, $11\subset 19\subset 76$, $11\subset 30\subset 76$, $11\subset 31\subset 76$, $11\subset 32\subset 76$, $12\subset 19\subset 76$, $12\subset 36\subset 76$, $12\subset 37\subset 76$, $12\subset 38\subset 76$, $13\subset 19\subset 76$, $13\subset 23\subset 76$, $13\subset 42\subset 76$, $13\subset 43\subset 76$, $14\subset 20\subset 76$, $14\subset 33\subset 76$, $14\subset 34\subset 76$, $14\subset 35\subset 76$, $15\subset 20\subset 76$, $15\subset 24\subset 76$, $15\subset 25\subset 76$, $15\subset 26\subset 76$, $16\subset 20\subset 76$, $16\subset 56\subset 76$, $16\subset 57\subset 76$, $16\subset 58\subset 76$, $17\subset 21\subset 76$, $17\subset 27\subset 76$, $17\subset 28\subset 76$, $17\subset 29\subset 76$, $18\subset 21\subset 76$, $18\subset 50\subset 76$, $18\subset 51\subset 76$, $18\subset 52\subset 76$,

Order $9, 81, \rho_2$

$6\subset 59\subset 76$, $6\subset 60\subset 76$, $6\subset 61\subset 76$, $6\subset 62\subset 76$, $6\subset 64\subset 76$, $6\subset 65\subset 76$, $6\subset 66\subset 76$, $6\subset 67\subset 76$, $6\subset 68\subset 76$, $6\subset 69\subset 76$, $6\subset 70\subset 76$, $6\subset 71\subset 76$, $7\subset 66\subset 76$, $7\subset 68\subset 76$, $7\subset 70\subset 76$, $7\subset 71\subset 76$, $8\subset 66\subset 76$, $8\subset 68\subset 76$, $8\subset 70\subset 76$, $8\subset 71\subset 76$, $9\subset 60\subset 76$, $9\subset 66\subset 76$, $9\subset 67\subset 76$, $9\subset 69\subset 76$, $10\subset 6o\subset 76$, $10\subset 66\subset 76$, $10\subset 67\subset 76$, $10\subset 69\subset 76$, $11\subset 59\subset 76$, $11\subset 62\subset 76$, $11\subset 64\subset 76$, $11\subset 66\subset 76$, $12\subset 59\subset 76$, $12\subset 62\subset 76$, $12\subset 64\subset 76$, $12\subset 66\subset 76$, $13\subset 59\subset 76$, $13\subset 62\subset 76$, $13\subset 64\subset 76$, $13\subset 66\subset 76$, $14\subset 61\subset 76$, $14\subset 63\subset 76$, $14\subset 65\subset 76$, $14\subset 66\subset 76$, $15\subset 61\subset 76$, $15\subset 63\subset 76$, $15\subset 65\subset 76$, $15\subset 66\subset 76$, $16\subset 61\subset 76$, $16\subset 63\subset 76$, $16\subset 65\subset 76$, $16\subset 66\subset 76$, $17\subset 66\subset 76$, $17\subset 68\subset 76$, $17\subset 70\subset 76$, $17\subset 71\subset 76$, $18\subset 66\subset 76$, $18\subset 68\subset 76$, $18\subset 70\subset 76$, $18\subset 71\subset 76$,

Order $9, 243, \rho_2$

$6\subset 72\subset 76$, $6\subset 73\subset 76$, $6\subset 74\subset 76$, $6\subset 75\subset 76$, $7\subset 72\subset 76$, $7\subset 73\subset 76$, $7\subset 74\subset 76$, $7\subset 75\subset 76$, $8\subset 72\subset 76$, $8\subset 73\subset 76$, $8\subset 74\subset 76$, $8\subset 75\subset 76$, $9\subset 72\subset 76$, $9\subset 73\subset 76$, $9\subset 74\subset 76$, $9\subset 75\subset 76$, $10\subset 72\subset 76$, $10\subset 73\subset 76$, $10\subset 74\subset 76$, $10\subset 75\subset 76$, $11\subset 72\subset 76$, $11\subset 73\subset 76$, $11\subset 74\subset 76$, $11\subset 75\subset 76$, $12\subset 72\subset 76$, $12\subset 73\subset 76$, $12\subset 74\subset 76$, $12\subset 75\subset 76$, $13\subset 72\subset 76$, $13\subset 73\subset 76$, $13\subset 74\subset 76$, $13\subset 75\subset 76$, $14\subset 72\subset 76$, $14\subset 73\subset 76$, $14\subset 74\subset 76$, $14\subset 75\subset 76$, $15\subset 72\subset 76$, $15\subset 73\subset 76$, $15\subset 74\subset 76$, $15\subset 75\subset 76$, $16\subset 72\subset 76$, $16\subset 73\subset 76$, $16\subset 74\subset 76$, $16\subset 75\subset 76$, $17\subset 72\subset 76$, $17\subset 73\subset 76$, $17\subset 74\subset 76$, $17\subset 75\subset 76$,

$18 \subset 72 \subset 76$, $18 \subset 73 \subset 76$, $18 \subset 74 \subset 76$, $18 \subset 75 \subset 76$,
Order $27, 81, \rho_2$
$19 \subset 59 \subset 76$, $19 \subset 62 \subset 76$, $19 \subset 64 \subset 76$, $19 \subset 66 \subset 76$, $20 \subset 61 \subset 76$, $20 \subset 63 \subset 76$, $20 \subset 65 \subset 76$, $20 \subset 66 \subset 76$, $21 \subset 66 \subset 76$, $21 \subset 68 \subset 76$, $21 \subset 70 \subset 76$, $21 \subset 71 \subset 76$, $22 \subset 60 \subset 76$, $22 \subset 66 \subset 76$, $22 \subset 67 \subset 76$, $22 \subset 669 \subset 76$, $23 \subset 64 \subset 76$, $24 \subset 63 \subset 76$, $25 \subset 63 \subset 76$, $26 \subset 63 \subset 76$, $27 \subset 68 \subset 76$, $28 \subset 68 \subset 76$, $29 \subset 68 \subset 76$, $30 \subset 59 \subset 76$, $31 \subset 59 \subset 76$, $32 \subset 59 \subset 76$, $33 \subset 61 \subset 76$, $34 \subset 61 \subset 76$, $35 \subset 61 \subset 76$, $36 \subset 62 \subset 76$, $37 \subset 62 \subset 76$, $38 \subset 62 \subset 76$, $39 \subset 67 \subset 76$, $40 \subset 67 \subset 76$, $41 \subset 67 \subset 76$, $42 \subset 64 \subset 76$, $43 \subset 64 \subset 76$, $44 \subset 70 \subset 76$, $45 \subset 70 \subset 76$, $46 \subset 70 \subset 76$, $47 \subset 60 \subset 76$, $48 \subset 60 \subset 76$, $49 \subset 60 \subset 76$, $50 \subset 71 \subset 76$, $51 \subset 71 \subset 76$, $52 \subset 71 \subset 76$, $53 \subset 69 \subset 76$, $54 \subset 69 \subset 76$, $55 \subset 69 \subset 76$, $56 \subset 65 \subset 76$, $57 \subset 65 \subset 76$, $58 \subset 65 \subset 76$,
Order $27, 243, \rho_2$
$19 \subset 72 \subset 76$, $19 \subset 73 \subset 76$, $19 \subset 74 \subset 76$, $19 \subset 75 \subset 76$, $20 \subset 72 \subset 76$, $20 \subset 73 \subset 76$, $20 \subset 74 \subset 76$, $20 \subset 75 \subset 76$, $21 \subset 72 \subset 76$, $21 \subset 73 \subset 76$, $21 \subset 74 \subset 76$, $21 \subset 75 \subset 76$, $22 \subset 72 \subset 76$, $22 \subset 73 \subset 76$, $22 \subset 74 \subset 76$, $22 \subset 75 \subset 76$, $23 \subset 72 \subset 76$, $24 \subset 75 \subset 76$, $25 \subset 75 \subset 76$, $26 \subset 75 \subset 76$, $27 \subset 74 \subset 76$, $28 \subset 74 \subset 76$, $29 \subset 74 \subset 76$, $30 \subset 72 \subset 76$, $31 \subset 72 \subset 76$, $32 \subset 72 \subset 76$, $33 \subset 75 \subset 76$, $34 \subset 75 \subset 76$, $35 \subset 75 \subset 76$, $36 \subset 72 \subset 76$, $37 \subset 72 \subset 76$, $38 \subset 72 \subset 76$, $39 \subset 73 \subset 76$, $40 \subset 73 \subset 76$, $41 \subset 73 \subset 76$, $42 \subset 72 \subset 76$, $43 \subset 72 \subset 76$, $44 \subset 74 \subset 76$, $45 \subset 74 \subset 76$, $46 \subset 74 \subset 76$, $47 \subset 73 \subset 76$, $48 \subset 73 \subset 76$, $49 \subset 73 \subset 76$, $50 \subset 74 \subset 76$, $51 \subset 74 \subset 76$, $52 \subset 74 \subset 76$, $53 \subset 73 \subset 76$, $54 \subset 73 \subset 76$, $55 \subset 73 \subset 76$, $56 \subset 75 \subset 76$, $57 \subset 75 \subset 76$, $58 \subset 75 \subset 76$
Order $81, 243, \rho_2$
$59 \subset 72 \subset 76$, $60 \subset 73 \subset 76$, $61 \subset 75 \subset 76$, $62 \subset 72 \subset 76$, $63 \subset 75 \subset 76$, $64 \subset 72 \subset 76$, $65 \subset 75 \subset 76$, $66 \subset 72 \subset 76$, $66 \subset 73 \subset 76$, $66 \subset 74 \subset 76$, $66 \subset 75 \subset 76$, $67 \subset 73 \subset 76$, $68 \subset 74 \subset 76$, $69 \subset 73 \subset 76$, $70 \subset 74 \subset 76$,

Lemma 3.1. *The number of distinct fuzzy subgroups of the presentation group*

$$\rho_2 = \langle a, b | a^{27} = b^{27} = e, [a, b] = a^9 \rangle$$

is equal to 4648.

Remark 3.2. The number of chains of lattice subgroups of ρ_2 was counted to be 2324 and now starting each chain of lattice subgroup of ρ_2 that ends in ρ_2 with the identity element e, we got another 2324 chains of lattice subgroups that ends in ρ_2. Thus the number of fuzzy subgroups $h(\rho_2)$ of order 729 is $2 \times 2324 = 4648$.

4. Conclusion

In this paper, we considered the class of finite minimal nonabelian p-groups all of whose generating elements have the same order. We were able to determine the order of elements of a presentation group $\rho_2 \in G$, all subgroups, chains of subgroups and number of distinct fuzzy subgroups of the presentation group by defining a natural equivalence relation on the subgroup lattice $L(\rho_2 \in G)$ associated with counting number of chains of subgroups of ρ_2 that ends in ρ_2.

Recommendations for further research.

(1) Define new equivalence relation induced by the action of automorphism group to classify fuzzy subgroups of some presentation groups.
(2) Developing an algorithm to compute the number of distinct subgroups and fuzzy subgroups of some presentation groups.
(3) Finite abelian group based cryptosystems have been widely used for most public key cryptography. More of nonabelian p-groups will be studied to enhance the safety of a finite nonabelian based public key.

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Ogunfolu Olusola B. (teminiyidele@gmail.com)

*D*epartment of Mathematics, Faculty of Science, University of Ibadan,
Ibadan, Nigeria