Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 26, No. 3, (December 2023) pp. 221–235
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2023.26.3.221

@FMI

On the computation of the quenching time for a nonlinear diffusion equation with singular boundary outfluxes

Ganon Ardjouma, N'Dri Kouakou Cyrille, Edja Kouamé Béranger, Touré Kidjégbo Augustin

Reprinted from the
Annals of Fuzzy Mathematics and Informatics
Vol. 26, No. 3, December 2023

Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 26, No. 3, (December 2023) pp. 221–235
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr
https://doi.org/10.30948/afmi.2023.26.3.221

@FMI

On the computation of the quenching time for a nonlinear diffusion equation with singular boundary outfluxes

Ganon Ardjouma, N'Dri Kouakou Cyrille, Edja Kouamé Béranger, Touré Kidjégbo Augustin

Received 17 August 2023; Revised 22 September 2023; Accepted 22 October 2023

Abstract. This work is concerned with the study of the numerical approximation for the nonlinear diffusion equation $(u^m)_t = u_{xx}$, $0 < x < 1,\ t > 0$, with a singular boundary outfluxes $u_x(0,t) = u^{-p}(0,t)$, $u_x(1,t) = -u^{-q}(1,t),\ t > 0$. We use the finite differences method to obtain a semidiscrete scheme of the above problem. First, we give appropriate conditions under which the semidiscrete solution quenches in a finite time and estimate its semidiscrete quenching time. Then, we establish the convergence of the semidiscrete quenching time. Finally, we illustrate our analysis with some numerical experiments.

2020 AMS Classification: 35K55, 35K20, 65M06

Keywords: Nonlinear diffusion equation, Numerical quenching, Singular boundary outfluxes, Arc length transformation, Aitken Δ^2 method.

Corresponding Author: Ganon Ardjouma(ardjganon@gmail.com)

1. Introduction

Consider the following nonlinear diffusion equation with singular boundary outfluxes :

$$(1.1)\qquad \begin{cases} (u^m)_t = u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = u^{-p}(0,t),\ u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le 1, \end{cases}$$

where m, p and q are positive given constants, and the initial function u_0 is a positive smooth function satisfying the compatibility conditions

$$u_0'(0) = u_0^{-p}(0) \text{ and } u_0'(1) = -u_0^{-q}(1).$$

Since u_0 satisfies these compatibility conditions, there exists $a_0 \in (0,1)$ such that $u_0' > 0$ in $[0, a_0)$, $u_0' < 0$ in $(a_0, 1]$ and $u_0'(a_0) = 0$. The concept of quenching was first introduced by Kawarada [1] in 1975 and has since been extensively investigated by many authors in recent decades (See [2, 3, 4, 5, 6, 7] and the references therein). But in the literature, there are a few studies about quenching problems with a singular boundary outflux (See [2, 6, 8]).

Definition 1.1. We say that the solution u of (1.1) quenches in a finite time if there exists a finite time T_q such that $\min\{u(x,t) : 0 \leq x \leq 1\} > 0$ for $t \in [0, T_q)$, but

$$\lim_{t\to T_q} \min\{u(x,t) : 0 \leq x \leq 1\} = 0.$$

The time T_q is called the quenching time of the solution u.

B. Selcuk and N. Ozalp [6] prove that for $m \geq 2q/(q+1)$, the solution of (1.1) quenches in finite time at the boundary $x = 1$. They also show that the time derivative blows up at the quenching time, which means that there exists sequence $(x_n, t_n) \to (1, T_q)$ such that $u_t(x_n, t_n) \to \infty$ as $n \to \infty$. Finally they establish results on quenching time and rate.
Problem (1.1) can be rewritten in the following form

$$(1.2) \qquad \begin{cases} u_t = \dfrac{1}{m} u^{1-m} u_{xx}, \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = u^{-p}(0,t),\ u_x(1,t) = -u^{-q}(1,t), \quad t > 0, \\ u(x,0) = u_0(x), \quad 0 \leq x \leq 1. \end{cases}$$

The rest of the paper is organised as follows : in the next section, we present a semidiscrete scheme of (1.2). In the section 3, we give some properties of the semidiscrete scheme. In section 4, under appropriate conditions, we prove that the semidiscrete solution quenches in a finite time and that this time converges to the real one. Finally, in the last section, we give some numerical results.

2. Semidiscrete problem

Let $I > 3$ be an integer and define the grid $x_i = ih,\ i = 0, \ldots, I$, where $h = \dfrac{1}{I}$ is the mesh parameter. We approximate the solution $(u(x_0,t), \ldots, u(x_I,t))^T$ of the problem (1.2) by the solution $U_h(t) = (U_0(t), \ldots, U_I(t))^T$ of the following semidiscrete scheme

$$(2.1) \qquad \frac{dU_i(t)}{dt} = \frac{1}{m} U_i^{1-m}(t) \delta^2 U_i(t), \quad i = 1, \ldots, I-1, \quad t \in (0, T_h),$$

$$(2.2) \qquad \frac{dU_0(t)}{dt} = \frac{1}{m} U_0^{1-m}(t) \left(\delta^2 U_0(t) - \frac{2}{h} U_0^{-p}(t) \right), \quad t \in (0, T_h),$$

$$(2.3) \qquad \frac{dU_I(t)}{dt} = \frac{1}{m} U_I^{1-m}(t) \left(\delta^2 U_I(t) - \frac{2}{h} U_I^{-q}(t) \right), \quad t \in (0, T_h),$$

$$(2.4) \qquad U_i(0) = \varphi_i > 0, \quad i = 0, \ldots, I,$$

where for $t \in (0, T_h)$,

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad i = 1, \dots, I-1,$$
$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \qquad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

and $[0, T_h)$, the maximal time interval on which $U_h(t)$ satisfies $\|U_h(t)\|_{inf} > 0$ and $\lim_{t \to T_h} \|U_h(t)\|_{inf} = 0$, with $\|U_h(t)\|_{inf} = \min_{0 \le i \le I} U_i(t)$. We say that $U_h(t)$ quenches in finite time if T_h is finite. In this case, T_h stands for the quenching time of the solution $U_h(t)$.
Denote

$$\delta_*^2 U_i(t) = \begin{cases} \delta^2 U_i(t) & \text{if } \ i = 1, \dots, I-1, \\ \delta^2 U_0(t) - \frac{2}{h} U_0^{-p}(t) & \text{if } \ i = 0, \\ \delta^2 U_I(t) - \frac{2}{h} U_I^{-q}(t) & \text{if } \ i = I. \end{cases}$$

3. Properties of the semidiscrete scheme

We give in this section some important results which will be used later. The following lemma is a semidiscrete form of the maximum principle.

Lemma 3.1. *Let* $a_h(t),\ b_h(t) \in C^0([0,T], \mathbb{R}^{I+1}),\ a_h(t) \ge 0$ *and* $V_h(t) \in C^1([0,T], \mathbb{R}^{I+1})$ *such that*

$$\frac{dV_i(t)}{dt} - a_i(t)\delta^2 V_i(t) + b_i(t)V_i(t) \ \ge \ 0, \quad 0 \le i \le I,\ t \in (0,T], \tag{3.1}$$
$$V_i(0) \ \ge \ 0, \quad 0 \le i \le I. \tag{3.2}$$

Then we have $V_i(t) \ge 0, \quad 0 \le i \le I,\ t \in [0,T]$.

Proof. Define the vector $Z_h(t) = V_h(t)e^{-\lambda t}$, where λ is a real such that $b_i(t) - \lambda > 0,\ 0 \le i \le I,\ t \in [0,T]$. Let $m = \min_{0 \le i \le I,\ 0 \le t \le T} Z_i(t)$. Since for $i \in \{0, \dots, I\}$, $Z_i(t)$ is a continuous function, there exists $t_0 \in [0,T]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in \{0, \cdots, I\}$. If $t_0 = 0$, $Z_i(t) \ge m = V_{i_0}(0) \ge 0$, then $V_i(t) \ge 0, \quad 0 \le i \le I,\ t \in [0,T]$.
Else, it is easy to see that

$$\frac{dZ_{i_0}(t_0)}{dt} \ = \ \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \tag{3.3}$$
$$\delta^2 Z_{i_0}(t_0) \ \ge \ 0. \tag{3.4}$$

Using (3.1), we compute that

$$\frac{dZ_{i_0}(t_0)}{dt} - a_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + (b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \ge 0. \tag{3.5}$$

From (3.3)–(3.5), we deduce that $(b_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \ge 0$, which implies that $m = Z_{i_0}(t_0) \ge 0$ because $b_{i_0}(t_0) - \lambda > 0$. Hence $V_h(t) \ge 0$ for $t \in [0,T]$, and we obtain the expected result. □

Lemma 3.2. *Let $f \in C^0(\mathbb{R},\mathbb{R})$. If $V_h,\ W_h \in C^1([0,T],\mathbb{R}^{I+1})$ and $a_h \in C^0([0,T],\mathbb{R}_+^{I+1})$ are such that*

$$\frac{dV_i(t)}{dt} - a_i(t)\delta^2 V_i(t) + f(V_i(t)) < \frac{dW_i(t)}{dt} - a_i(t)\delta^2 W_i(t) + f(W_i(t)), \quad 0 \le i \le I,\ t \in (0,T], \tag{3.6}$$

$$V_i(0) < W_i(0), \quad 0 \le i \le I. \tag{3.7}$$

Then we have $V_i(t) < W_i(t), \quad 0 \le i \le I, \quad t \in [0,T]$.

Proof. Let us define the vector $Z_h(t) = W_h(t) - V_h(t)$. Let t_0 be the first $t \in (0,T]$ such that $Z_i(t) > 0$ for $t \in [0,t_0)$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0,\ldots,I\}$. It is not hard to see that

$$\begin{aligned} \frac{d}{dt}Z_{i_0}(t_0) &= \lim_{k\to 0}\frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \le 0,\\ \delta^2 Z_{i_0}(t_0) &\ge 0, \end{aligned}$$

which implies that

$$\frac{d}{dt}Z_{i_0}(t_0) - a_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + f(W_{i_0}(t_0)) - f(V_{i_0}(t_0)) \le 0,$$

but this inequality contradicts (3.6) and the proof is complete. □

The following lemma shows that the solution of the semidiscrete scheme is a non-increasing function of t.

Lemma 3.3. *Let U_h be a solution of (2.1)–(2.4) and the initial data at (2.4) satisfies $\delta_*^2\varphi_i \le 0,\ 0 \le i \le I$. Then*

$$\frac{dU_i(t)}{dt} \le 0 \quad \text{for} \ \ 0 \le i \le I,\ t \in [0,T_h).$$

Proof. Take $T_0 < T_h$ fixed. Let us define the vector $V_h(t)$ such that $V_i(t) = \dfrac{dU_i(t)}{dt}$ for $0 \le i \le I,\ t \in [0,T_0]$. We have

$$\frac{dV_i(t)}{dt} = \left(\frac{1-m}{m}U_i^{-m}(t)\delta^2 U_i(t)\right)V_i(t) + \frac{1}{m}U_i^{1-m}(t)\delta^2 V_i(t),\ 1 \le i \le I-1, \tag{3.8}$$

$$\frac{dV_0(t)}{dt} = \left(\frac{1-m}{m}U_0^{-m}(t)\delta^2 U_0(t) - \frac{2(1-m-p)}{mh}U_0^{-m-p}(t)\right)V_0(t) + \frac{1}{m}U_0^{1-m}(t)\delta^2 V_0(t) \tag{3.9}$$

$$\frac{dV_I(t)}{dt} = \left(\frac{1-m}{m}U_I^{-m}(t)\delta^2 U_I(t) - \frac{2(1-m-q)}{mh}U_I^{-m-q}(t)\right)V_I(t) + \frac{1}{m}U_I^{1-m}(t)\delta^2 V_I(t). \tag{3.10}$$

Set

$$K_1 = \max_{1\le i\le I-1,\ 0\le t\le T_0}\left\{\left|\frac{1-m}{m}U_i^{-m}(t)\delta^2 U_i(t)\right|\right\},$$

$$K_2 = \max_{0\le t\le T_0}\left\{\left|\frac{1-m}{m}U_0^{-m}(t)\delta^2 U_0(t)\right| - \frac{2(1-m-p)}{mh}U_0^{-m-p}(t)\right\}$$

and

$$K_3 = \max_{0\le t\le T_0} \left\{ \left| \frac{1-m}{m} U_I^{-m}(t)\delta^2 U_I(t) \right| - \frac{2(1-m-q)}{mh} U_I^{-m-q}(t) \right\}.$$

Let K be a positive constant satisfying

$$K > \max\{K_1, K_2, K_3\}.$$

Denote $V_h(t) = (V_0(t), \dots, V_I(t))^T$ and consider the vector $W_h(t) = V_h(t)e^{-Kt}$. Note that $W_h(0) \le 0$ because $V_h(0) \le 0$.

Let t_0 be the first $t \in (0, T_0]$ such that $W_i(t) \le 0$ for $t \in [0, t_0)$, but $W_{i_0}(t_0) > 0$ for a certain $i_0 \in \{0, \dots, I\}$. Without lost of generality, we suppose that i_0 is such that $W_{i_0}(t_0) = \max_{0\le i\le I}\{W_i(t_0)\}$. Then we have

$$\frac{dW_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{W_{i_0}(t_0) - W_{i_0}(t_0-k)}{k} \ge 0, \tag{3.11}$$

$$\delta^2 W_{i_0}(t_0) \le 0. \tag{3.12}$$

From relations (3.8)–(3.10) and (3.12), we can easily show that

$$\frac{dW_{i_0}(t_0)}{dt} \le \left(\frac{1-m}{m} U_{i_0}^{-m}(t_0)\delta^2 U_{i_0}(t_0) - K \right) W_{i_0}(t_0) < 0, \quad 1 \le i_0 \le I-1,$$

$$\frac{dW_{i_0}(t_0)}{dt} \le \left(\frac{1-m}{m} U_0^{-m}(t_0)\delta^2 U_0(t_0) - \frac{2(1-m-p)}{mh} U_0^{-m-p}(t_0) - K \right) W_0(t_0) < 0, \ i_0 = 0,$$

$$\frac{dW_{i_0}(t_0)}{dt} \le \left(\frac{1-m}{m} U_I^{-m}(t_0)\delta^2 U_I(t_0) - \frac{2(1-m-q)}{mh} U_I^{-m-q}(t_0) - K \right) W_I(t_0) < 0, \ i_0 = I,$$

which is a contradiction with (3.11) and the lemma is completely proved. □

The lemma below reveals that for a positive initial data that satisfies the compatibility conditions, the semidiscrete solution can not be monotone on the space.

Lemma 3.4. *Let U_h be a solution of (2.1)–(2.4) and the initial condition at (2.4) verifies*

$$\varphi_i < \varphi_{i+1}, \ 0 \le i \le k_0 - 1,$$

$$\varphi_i > \varphi_{i+1}, \ k_0 \le i \le I - 1,$$

where $k_0 \in \{2, \dots, I-2\}$. Then

(1) $U_i(t) < U_{i+1}(t)$ *for* $0 \le i \le k_0 - 2, \ t \in [0, T_h)$,

(2) $U_i(t) > U_{i+1}(t)$ *for* $k_0 + 1 \le i \le I - 1, \ t \in [0, T_h)$.

Proof. (1) Define the functions $Z_i(t) = U_i(t) - U_{i+1}(t)$. Because $\varphi_i - \varphi_{i+1} < 0$, $0 \le i \le k_0 - 1$, let t_0 be the first $t \in (0, T_h)$ such that $Z_i(t) < 0$ for $t \in [0, t_0)$, $0 \le i \le k_0 - 1$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, k_0 - 1\}$. Without lost of generality, we suppose that i_0 is the smallest integer which satisfies the above equality. It is not hard to see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k\to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0-k)}{k} \ge 0, \tag{3.13}$$

$$\delta^2 Z_{i_0}(t_0) < 0 \quad \text{if } 1 \le i_0 \le k_0 - 2, \tag{3.14}$$

$$\delta^2 Z_{i_0}(t_0) \le 0 \quad \text{if } i_0 = 0. \tag{3.15}$$

Using relations (2.1), (2.2), (3.14) and (3.15), we have

$$\frac{dZ_{i_0}(t_0)}{dt} = \frac{1}{m}U_{i_0}^{1-m}(t_0)\delta^2 Z_{i_0}(t_0) < 0 \quad \text{if } 1 \le i_0 \le k_0 - 2,$$

$$\frac{dZ_{i_0}(t_0)}{dt} = \frac{1}{m}U_0^{1-m}(t_0)\delta^2 Z_0(t_0) - \frac{2}{mh}U_0^{1-m-p}(t_0) < 0 \quad \text{if } i_0 = 0,$$

which contradict (3.13) and the desired result follows.
By an analogous argument, we prove the latter part of the lemma. □

Lemma 3.5. *Assume $p \le q$. Let U_h be a solution of (2.1)–(2.4) and the initial condition at (2.4) verifies*

$$\varphi_I \le 1 \textit{ and } \varphi_i > \varphi_{I-i},\ 0 \le i \le k_0 + 1,$$

where k_0 is defined in the previous Lemma. Then $U_i(t) > U_{I-i}(t)$ for $0 \le i \le k_0,\ t \in [0, T_h)$.

Proof. We set $Z_i(t) = U_i(t) - U_{I-i}(t),\ 0 \le i \le k_0 + 1$. Let t_0 be the first $t \in (0, T_h)$ such that $Z_i(t) > 0$ for $t \in [0, t_0),\ 0 \le i \le k_0 + 1$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \dots, k_0 + 1\}$. Without lost of generality, we may suppose that i_0 is the greatest integer which satisfies the above equality. One can check that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \tag{3.16}$$

$$\delta^2 Z_{i_0}(t_0) > 0 \quad \text{if } 0 \le i_0 \le k_0. \tag{3.17}$$

From relations (2.1), (2.2) and using (3.17), we obtain by a simple computation

$$\frac{dZ_{i_0}(t_0)}{dt} = \frac{1}{m}U_{i_0}^{1-m}(t_0)\delta^2 Z_{i_0}(t_0) > 0 \quad \text{if } 1 \le i_0 \le k_0,$$

$$\frac{dZ_{i_0}(t_0)}{dt} = \frac{1}{m}U_0^{1-m}(t_0)\delta^2 Z_0(t_0) + \frac{2}{mh}U_I^{1-m-q}(t_0)\Big(1 - U_I^{q-p}(t_0)\Big) > 0 \quad \text{if } i_0 = 0,$$

which contradict (3.16) and we conclude the proof. □

Remark 3.6. Under the assumptions of Lemmas (3.4) and (3.5),

$$\min_{0 \le i \le I} U_i(t) = U_I(t) \ for\ t \in [0, T_h).$$

4. Convergence of the semidiscrete quenching time

In this section, we give suitable assumptions under which, the semidiscrete solution quenches in finite time and its quenching time converges to the theoretical one when the mesh size goes to zero. The next theorem shows that the semidiscrete solution approximates the continuous one under condition (4.1).

Theorem 4.1. *Assume that the problem (1.2) has a solution $u \in C^{4,1}([0,1] \times [0,T])$ and the initial condition at (2.4) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \ \textit{as}\ h \to 0, \tag{4.1}$$

where $u_h(t) = \big(u(x_0, t), \dots, u(x_I, t)\big)^T$. Then for h small enough, the semidiscrete problem (2.1)–(2.4) has a unique solution $U_h \in C^1([0,T], \mathbb{R}^{I+1})$ such that

$$\max_{0 \le t \le T} \|U_h(t) - u_h(t)\|_\infty = O\big(\|\varphi_h - u_h(0)\|_\infty + h^2\big) \ \textit{as}\ h \to 0. \tag{4.2}$$

Proof. Let $\alpha > 0$ be such that

$$\|u(\cdot,t)\|_{\inf} \geq \alpha \quad \text{for} \quad t \in [0,T]. \tag{4.3}$$

Then the problem (2.1)–(2.4) has for each h, a unique solution $U_h \in C^1([0,T_h),\mathbb{R}^{I+1})$. Let $t(h) \leq \min\{T, T_h\}$ be the greatest value of $t > 0$ such that

$$\|U_h(t) - u_h(t)\|_\infty < \alpha/2, \; t \in (0,t(h)). \tag{4.4}$$

Note that, because of (4.1), $t(h) > 0$ for h small enough. Using the fact that $U_i(t) = u(x_i,t) - (-U_i(t) + u(x_i,t))$, we get
$\|U_h(t)\|_{inf} \geq \|u(\cdot,t)\|_{inf} - \|U_h(t) - u_h(t)\|_\infty, \; t \in (0,t(h))$, which implies that

$$\|U_h(t)\|_{\inf} \geq \alpha - \alpha/2 = \alpha/2, \; t \in (0,t(h)). \tag{4.5}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the discretization error. Using the Taylor's expansion, we have for $t \in (0,t(h))$

$$\begin{aligned}
\frac{de_i(t)}{dt} - \frac{1}{m}U_i^{1-m}(t)\delta^2 e_i(t) &= \frac{1-m}{m}\xi_i^{-m}(t)\delta^2 u(x_i,t)e_i(t) + O(h^2), \quad 1 \leq i \leq I-1,\\
\frac{de_0(t)}{dt} - \frac{1}{m}U_0^{1-m}(t)\delta^2 e_0(t) &= \left(\frac{1-m}{m}\xi_0^{-m}(t)\delta^2 u(x_0,t) - \frac{2(1-m-p)}{mh}\theta_0^{-m-p}(t)\right)e_0(t)\\
&\quad + O(h^2),\\
\frac{de_I(t)}{dt} - \frac{1}{m}U_I^{1-m}(t)\delta^2 e_I(t) &= \left(\frac{1-m}{m}\xi_I^{-m}(t)\delta^2 u(x_I,t) - \frac{2(1-m-q)}{mh}\theta_I^{-m-q}(t)\right)e_I(t)\\
&\quad + O(h^2),
\end{aligned}$$

where $\xi_i(t)$ is the intermediate value between $U_i(t)$ and $u(x_i,t)$ for $i \in \{0,\ldots,I\}$ and $\theta_0(t)$ the one between $U_0(t)$ and $u(x_0,t)$. Since $u \in C^{4,1}([0,1]\times[0,t(h)])$ and the fact that relation (4.4) holds, there exist K and L positive constants such that

$$\begin{aligned}
\frac{d}{dt}e_i(t) - \frac{1}{m}U_i^{1-m}(t)\delta^2 e_i(t) &\leq K|e_i(t)| + Lh^2, \quad 1 \leq i \leq I-1,\\
\frac{d}{dt}e_i(t) - \frac{1}{m}U_i^{1-m}(t)\delta^2 e_i(t) &\leq \frac{K}{h}|e_i(t)| + Lh^2, \quad i \in \{0;I\}.
\end{aligned}$$

On the other hand, we consider the function

$$Z(x,t) = \left(\|\varphi_h - u_h(0)\|_\infty + Mh^2\right)e^{(Q+1)t+Rx^2},$$

and we denote by $Z(x_i,t)$ the discretization in space of $Z(x,t)$. Since $\|U_h(t)\|_{inf} > 0, \; t \in (0,t(h))$, we obtain for suitable non-negative constants $M, \; Q, \; R$ that

$$\begin{aligned}
\frac{d}{dt}Z(x_i,t) - \frac{1}{m}U_i^{1-m}(t)\delta^2 Z(x_i,t) &> K|Z(x_i,t)| + Lh^2, \quad 1 \leq i \leq I-1,\\
\frac{d}{dt}Z(x_i,t) - \frac{1}{m}U_i^{1-m}(t)\delta^2 Z(x_i,t) &> \frac{K}{h}|Z(x_i,t)| + Lh^2, \quad i \in \{0;I\},\\
Z(x_i,0) &> |e_i(0)|.
\end{aligned}$$

It follows from Lemma 3.2 that

$$e_i(t) < Z_i(t), \quad 0 \leq i \leq I, \; t \in (0,t(h)).$$

By the same argument, we also prove that

$$-e_i(t) < Z(x_i, t), \quad 0 \le i \le I,\ t \in (0, t(h)).$$

Which lead to

$$\|U_h(t) - u_h(t)\|_\infty \le \big(\|\varphi_h - u_h(0)\|_\infty + Mh^2 \big) e^{(Q+1)t+R}, \quad t \in (0, t(h)). \tag{4.6}$$

Let us show now that $t(h) = T$. Suppose $t(h) < T$. From (4.4) and (4.6), we have

$$\frac{\alpha}{2} = \|U_h(t(h)) - u_h(t(h))\|_\infty \le \Big(\|U_h(0) - u_h(0)\|_\infty + Mh^2 \Big) e^{(Q+1)T+R}.$$

Since $\Big(\|U_h(0) - u_h(0)\|_\infty + Mh^2 \Big) e^{(Q+1)T+R}$ goes to zero as h tends to zero, we deduce that $\alpha/2 \le 0$, which is impossible. □

Lemma 4.2. *Let $U_h \in \mathbb{R}^{I+1}$ such that $U_h > 0$. Then*

$$\delta^2 U_i^{-\beta} \ge -\beta U_i^{-\beta-1} \delta^2 U_i, \quad 0 \le i \le I,$$

where $\beta > 0$.

Proof. We refer to [5]. □

Theorem 4.3. *Let $m \ge 1$ and the initial data φ_h at (2.4) satisfies $\delta_*^2 \varphi_I \ne 0$. Under the assumptions of Lemmas (3.3), (3.4) and (3.5), the solution U_h of (2.1)–(2.4) quenches in a finite time T_h with the following estimation*

$$T_h \le \frac{1}{A} \frac{\|\varphi_h\|_{inf}^{q+1}}{q+1} \quad \textit{where} \quad A \in \Big(0; \frac{\delta_*^2 \varphi_I}{-m\varphi_I^{m-q-1}}\Big]. \tag{4.7}$$

Proof. Since $[0, T_h)$ is the maximal time interval on which $\|U_h(t)\|_{\inf} > 0$, our goal is to show that T_h is finite and satisfies the inequality (4.7). For $t \in [0, T_h)$, let us Introduce the vector $J_h(t)$ defined as follows

$$J_I(t) = \frac{dU_I(t)}{dt} + AU_I^{-q}, \quad J_i(t) = \frac{dU_i(t)}{dt}, \quad 0 \le i \le I-1. \tag{4.8}$$

Notice that

$$\begin{cases} \delta_*^2 \varphi_I \le -mA\varphi_I^{m-q-1} \quad \text{since} \quad A \in \Big(0; \frac{\delta_*^2 \varphi_I}{-m\varphi_I^{m-q-1}}\Big] \\ \text{and} \quad \delta_*^2 \varphi_i \le 0, \ 0 \le i \le I-1 \quad \text{(assumption of Lemma (3.3))}. \end{cases} \tag{4.9}$$

A straightforward calculation using (2.1)-(2.3) yields for $t \in (0, T_h)$

$$\frac{dJ_i(t)}{dt} - \frac{1}{m} U_i^{1-m}(t) \delta^2 J_i(t) = \frac{1-m}{m} U_i^{-m}(t) \delta^2 U_i(t) \frac{dU_i(t)}{dt}, \quad 1 \le i \le I-1, \tag{4.10}$$

$$\begin{aligned} \frac{dJ_0(t)}{dt} - \frac{1}{m} U_0^{1-m}(t) \delta^2 J_0(t) = {} & \frac{2p}{mh} U_0^{-m-p}(t) \frac{dU_0(t)}{dt} \\ & + \frac{1-m}{m} U_0^{-m}(t) \left(\delta^2 U_0(t) - \frac{2}{h} U_0^{-p}(t) \right) \frac{dU_0(t)}{dt}, \end{aligned} \tag{4.11}$$

$$\begin{aligned}(4.12)\quad \frac{dJ_I(t)}{dt} - \frac{1}{m}U_I^{1-m}(t)\delta^2 J_I(t) &= \frac{2q}{mh}U_I^{-m-q}(t)\frac{dU_I(t)}{dt}\\ &+ \frac{1-m}{m}U_I^{-m}(t)\left(\delta^2 U_I(t) - \frac{2}{h}U_I^{-q}(t)\right)\frac{dU_I(t)}{dt}\\ &- \frac{A}{m}U_I^{1-m}(t)\delta^2 U_I^{-q}(t) - qAU_I^{-q-1}(t)\frac{dU_I(t)}{dt}\end{aligned}$$

$$(4.13)\quad \le \frac{2q}{mh}U_I^{-m-q}(t)\frac{dU_I(t)}{dt} + \frac{1-m}{m}U_I^{-m}(t)\left(\delta^2 U_I(t) - \frac{2}{h}U_I^{-q}(t)\right)\frac{dU_I(t)}{dt} + \frac{2Aq}{mh}U_I^{-m-2q}(t).$$

We obtain inequality (4.13) by applying Lemma 4.2 to equality (4.12).

Now, using Lemma 3.3 and the fact that $1-m \le 0$, we deduce from relations (4.10), (4.11) and (4.13) that

$$\begin{aligned}\frac{dJ_i(t)}{dt} - \frac{1}{m}U_i^{1-m}(t)\delta^2 J_i(t) &\le 0, \quad 0 \le i \le I-1,\\ \frac{dJ_I(t)}{dt} - \frac{1}{m}U_I^{1-m}(t)\delta^2 J_I(t) &\le \frac{2q}{mh}U_I^{-m-q}(t)J_I(t).\end{aligned}$$

We observe from (4.9) that $J_i(0) \le 0,\ 0 \le i \le I$.
Applying Lemma 3.1, we obtain $J_h(t) \le 0$ for $t \in [0, T_h)$, which implies that

$$\frac{dU_I(t)}{dt} + AU_I^{-q}(t) \le 0 \quad \text{for} \quad t \in [0, T_h).$$

This estimate may be rewritten in the following manner

$$U_I^q(t)dU_I(t) \le -Adt \quad \text{for} \quad t \in [0, T_h).$$

Integrating the above inequality over (t, T_h) to get

$$(4.14)\qquad T_h - t \le \frac{1}{A}\frac{U_I^{q+1}(t)}{q+1}.$$

From Remark 3.6 and taking $t = 0$ in (4.14), we get the desired result. □

Remark 4.4. Using (4.14) and taking account Remark 3.6, we have

$$T_h - t \le \frac{1}{A}\frac{U_i^{q+1}(t)}{q+1} \quad \text{for} \ \ 0 \le i \le I,\ t \in [0, T_h),$$

and there exists a constant $C > 0$ such that

$$U_i(t) \ge C\Big(T_h - t\Big)^{1/(q+1)} \quad \text{for} \ \ 0 \le i \le I,\ t \in [0, T_h).$$

Theorem 4.5. *Suppose that the solution u of (1.2) quenches in a finite time T_q such that $u \in C^{4,1}([0,1] \times [0, T_q))$ and the initial condition at (2.4) satisfies $\|\varphi_h - u_h(0)\|_\infty = o(1)$ as $h \to 0$. Under the assumptions of Theorem 4.3, the solution U_h of (2.1)–(2.4) quenches in a finite time T_h and we have*

$$\lim_{h\to 0} T_h = T_q.$$

Proof. Set $\epsilon > 0$. There exists ρ such that

$$\frac{1}{A}\frac{y^{q+1}}{(q+1)} \leq \frac{\epsilon}{2}, \quad 0 \leq y \leq \rho. \tag{4.15}$$

Since $u(x,t)$ quenches in a finite time T_q, there exists a time $T_0 < T_q$ such that $|T_0 - T_q| < \epsilon/2$ and $0 < \|u(x,t)\|_{\inf} \leq \rho/2$ for $t \in [T_0, T_q)$. Setting $T_1 = (T_0 + T_q)/2$, it is not hard to see that $\|u(x,t)\|_{\inf} > 0$ for $t \in [0, T_1]$. From Theorem 4.1, we have $\|U_h(t) - u_h(t)\|_\infty \leq \rho/2$ for $t \in [0, T_1]$, which implies that $\|U_h(T_1) - u_h(T_1)\|_\infty \leq \rho/2$. Applying the triangle inequality, we get

$$\|U_h(T_1)\|_{\inf} \leq \|U_h(T_1) - u_h(T_1)\|_\infty + \|u_h(T_1)\|_{\inf} \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho.$$

From Theorem 4.3, U_h quenches in a finite time T_h. We deduce from Remark 4.4 and relation (4.15) that

$$|T_h - T_q| \leq |T_h - T_1| + |T_1 - T_q| \leq \frac{1}{A}\frac{\|U_h(T_1)\|_{\inf}^{q+1}}{(q+1)} + \frac{\epsilon}{2} \leq \epsilon,$$

and the proof is complete. □

5. Numerical experiments

Before doing simulation, we transform the semidiscrete problem (2.1)–(2.4) into the following one by setting $V_h = \dfrac{1}{U_h}$:

$$\frac{dV_i(t)}{dt} = g(V_i(t)), \quad i = 0, \ldots, I, \quad t \in (0, T_h), \tag{5.1}$$

$$V_i(0) = (\varphi_i)^{-1}, \quad i = 0, \ldots, I, \tag{5.2}$$

where

$$\begin{aligned} g(V_i) &= \frac{V_i^{m+1}}{mh^2}\left(\frac{2}{V_i} - \frac{1}{V_{i+1}} - \frac{1}{V_{i-1}}\right), \quad i = 1, \ldots, I-1, \quad t \in (0, T_h), \\ g(V_0) &= \frac{2V_0^{m+1}}{mh^2}\left(hV_0^p + \frac{1}{V_0} - \frac{1}{V_1}\right), \quad t \in (0, T_h), \\ g(V_I) &= \frac{2V_I^{m+1}}{mh^2}\left(hV_I^p + \frac{1}{V_I} - \frac{1}{V_{I-1}}\right), \quad t \in (0, T_h). \end{aligned}$$

We know from ([9, 10]) that the solution V_h of (5.1)–(5.2) blows up at the quenching time T_h of U_h.

Hence, we estimate the numerical blow-up time of (2.1)–(2.4) by using the algorithm proposed by C. Hirota and K. Ozawa [11]. Firstly, we transform the semidiscrete scheme (5.1)-(5.2) by the arc length transformation technique into the following form :

$$\begin{cases} \dfrac{d}{d\ell}\begin{pmatrix} t \\ V_0 \\ \vdots \\ V_I \end{pmatrix} = \dfrac{1}{\sqrt{1+\sum_{i=0}^{I} g_i^2}}\begin{pmatrix} 1 \\ g_0 \\ \vdots \\ g_I \end{pmatrix}, \quad 0 < \ell < \infty, \\ t(0) = 0, \quad V_i(0) = (\varphi_i)^{-1} > 0, \quad 0 \le i \le I, \end{cases} \tag{5.3}$$

where
"ℓ" is such that $d\ell^2 = dt^2 + \sum_{i=0}^{I} dV_i^2$ and is called the arc length.
The variables t and V_i are fonctions of ℓ, and C. Hirota and K. Ozawa [11] proved that

$$\lim_{\ell\to\infty} t(\ell) = T_h \quad \text{and} \quad \lim_{\ell\to\infty} \|V_h(\ell)\|_\infty = \infty.$$

Secondly, we introduce $\{v_j\}$ which is a sequence of the arc length and we apply an ODE solver (DOP54) to (5.3) for each value of $\{v_j\}$. In this way, we generate a linearly convergent sequence to the blow-up time, which sequence is finally accelerated by the Aitken Δ^2 method. The three tolerances parameters, AbsTol, RelTol and InitialStep of the DOP54 (See [11, 12] for more details) are set as follows AbsTol = RelTol = 1.d–15, InitialStep = 0, and the sequence of the arc length is define by $v_j = 2^4 \cdot 2^j$ $(j = 0, \dots, 10)$. In the following Tables, T_h is the approximate quenching time corresponding to meshes of $I = 16, 32, 64, 128, 256, 512$; and n, the numbers of iterations required to obtain T_h.

In accordance with the quenching condition of the continuous solution u, we take in our simulations $m \ge \frac{2q}{q+1}$, see [6].

Case 1 : $\varphi_i = \cos(\frac{\pi}{2} * i * h) + (\frac{\pi}{2} - 1) * i * h + (2 - \frac{\pi}{2})$, $0 \le i \le I$ and $p = -\ln(\frac{\pi}{2} - 1)/\ln(3 - \frac{\pi}{2})$.
Tables 1-3 are obtained for various values of parameters m and q in the case 1.

Table 1. For $m = 2.5,\ q = 3$

I	T_h	n
16	0.271 265 072 978	1897
32	0.268 864 903 153	3676
64	0.268 158 507 779	7136
128	0.267 961 338 713	13900
256	0.267 908 075 669	27536
512	0.267 893 994 001	60141

Table 2. For $m = 4,\ q = 3$

I	T_h	n
16	0.483 811 738 426	2018
32	0.483 510 685 640	4081
64	0.483 434 668 354	8121
128	0.483 415 565 675	16052
256	0.483 410 776 155	32164
512	0.483 409 576 764	71987

Table 3. For $m = 4,\ q = 4$

I	T_h	n
16	0.349 720 316 923	1527
32	0.347 514 613 795	3035
64	0.346 894 845 834	5988
128	0.346 728 719 634	11778
256	0.346 685 351 831	23461
512	0.346 674 207 714	51313

Remark 5.1. We observe from the above tables that there is a relationship between T_h, m and q. In fact, when the parameter of the flux on the boundary $x_I = 1$ is a constant ($q = 3$) and that m increases from $m = 2.5$ to 4, the quenching time also increases (from $T_h = 0.267$ to 0.483) see Tables 1, 2. Whereas when m remains constant ($m = 4$) and q increases (by $q = 3$ to 4), the quenching time diminishes (from $T_h = 0.483$ to 0.346) see Tables 2, 3.

Below, we give some plots in figures 1-3 to illustrate the evolution of U_h in the case 1 for $I = 256,\ m = 2.5,\ q = 3$.

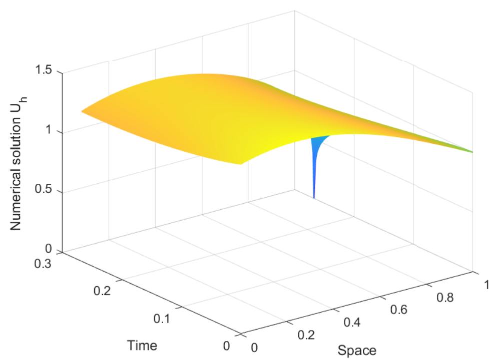

Figure 1. Evolution of the numerical solution U_h

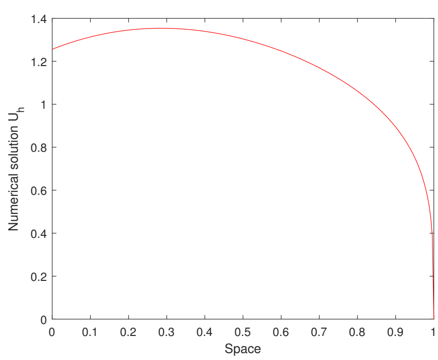

Figure 2. Evolution of U_h according to the space at quenching time.

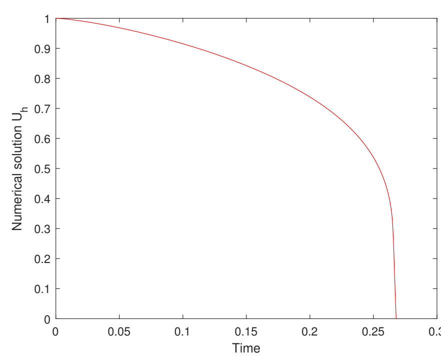

Figure 3. Evolution of U_h according to the time.

Case 2 : $\varphi_i = -1.2 * (i * h)^2 + i * h + 1,\ 0 \le i \le I$ and $q = \ln(1.4)/\ln(1.25)$.
We obtained tables 4-6 for various values of parameter p and $m = 2$ in the case 2.

Table 4. For $m = 2,\ p = 0.5$

I	T_h	n
16	0.096 371 886 086	1776
32	0.095 364 023 857	3519
64	0.095 072 620 314	6924
128	0.094 992 159 745	13517
256	0.094 970 588 501	26456
512	0.094 964 918 356	52896

Table 5. For $m = 2,\ p = 1$

I	T_h	n
16	0.096 371 525 733	1776
32	0.095 363 755 042	3519
64	0.095 072 372 870	6924
128	0.094 991 917 552	13518
256	0.094 970 347 616	26456
512	0.094 964 677 797	52896

Table 6. For $m = 2,\ p = 1.4$

I	T_h	n
16	0.096 371 212 474	1777
32	0.095 363 521 385	3520
64	0.095 072 157 799	6925
128	0.094 991 707 047	13518
256	0.094 970 138 248	26457
512	0.094 964 468 713	52898

Remark 5.2. Tables 4-6 reveal that the flux on the boundary $x_0 = 0$ does not have a significant effect on the quenching time.

Others illustrations are given in the below figures to show the evolution of the numerical solution U_h for $I = 256,\ m = 2,\ p = 0.5$ according to the case 2.

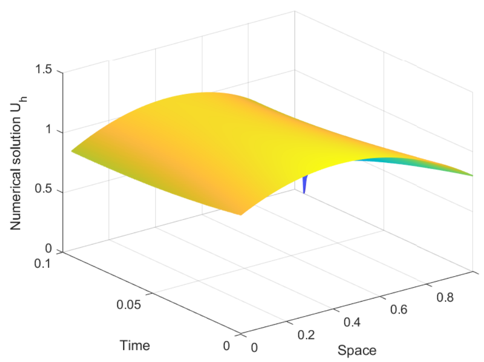

Figure 4. Evolution of the numerical solution U_h.

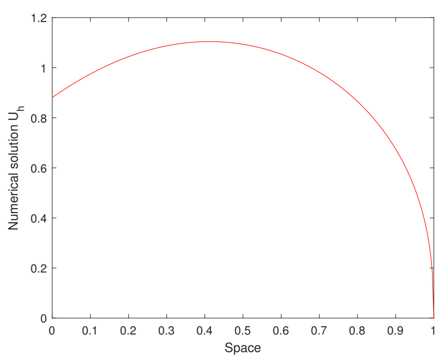

Figure 5. Evolution of U_h according to the space at quenching time.

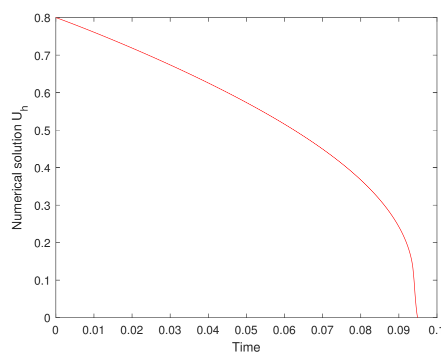

Figure 6. Evolution of U_h according to the time.

Remark 5.3. From figures 1-6, we observe that the evolution of the numerical solution is in agreement with the theoretical results obtained.

6. Conclusion

In this work, we have studied the numerical quenching of the solution of the nonlinear diffusion equation with singular boundary outfluxes (1.1). We have used the finite difference method to construct the semidiscrete problem (2.1)-(2.4) related to the continuous problem. We have also proved that the semidiscrete solution reproduces the qualitative and quenching properties of the continuous one. Better, we have shown that, under some assumptions, the semidiscrete solution and its quenching time converge respectively to the continuous solution and the theoretical quenching time, when the mesh parameter goes to zero. Finally, some numerical experiments have been presented to illustrate our analysis. we can extend this work to a higher dimensional space.

References

[1] H. Kawarada, On solutions of initial boundary problem for $u_t = u_{xx} + 1/(1-u)$, Publ. RIMS, Kyoto Univ. 10 (1975) 729–736.

[2] K. Deng and M. Xu, Quenching for a nonlinear diffusion equation with a singular boundary condition, Z. Angew. Math. Phys. 50 (1999) 574–584.

[3] M. Fila and B. Kawohl, Asymptotic analysis of quenching problems, Rocky Mt. J. Math. 22 (2) (1992) 563–577.

[4] K. C. N'Dri, A. K. Touré and G. Yoro, Numerical quenching versus blow-up for a nonlinear parabolic equation with nonlinear boundary outflux, Adv. Math., Sci. J. 9 (1) (2020) 151–171.

[5] D. Nabongo and T. K. Boni, Numerical quenching for a nonlinear diffusion equation with a singular boundary condition, Bull. Belg. Math. Soc. Simon Stevin. 16 (2009) 289–303.

[6] B. Selcuk and N. Ozalp, Quenching for A Nonlinear Diffusion Equation with Singular Boundary Outfluxes, Turk. J. Math. Comput. Sci. 8 (2018) 65–69.

[7] M. Winkler, Quenching phenomena in strongly degenerate diffusion equations with strong absorption, J. Math. Anal. Appl. 288 (2003) 481–504.

[8] Y. Yang, Finite time quenching for a nonlinear diffusion equation with singular boundary flux, Journal of Physics: Conference Series 814 (1) (2017) 1–6.

[9] N. Ozalp and B. Selcuk, Blow up and quenching for a problem with nonlinear boundary conditions, Electron. J. Differ. Equ. 2015 (192) (2015) 1–11.

[10] T. Salin, On quenching with logarithmic singularity, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods. 52 (2023) 261–289.

[11] C. Hirota and K. Ozawa, Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations–An application to the blow-up problems of partial differential equations,J. Comput. Appl. Math. 193 (2006) 614–637.

[12] A. Ganon, M. M. Taha and A. K. Touré: Blow-up for Semidiscretization of Semilinear Parabolic Equation With Nonlinear Boundary Condition, J. Math. Res. 11 (2019) 1–10.

Ganon Ardjouma (ardjganon@gmail.com)
UMRI Mathématiques et Nouvelles Technologies de l'Information, Institut National Polytechnique Félix HOUPHOUËT-BOIGNY, Yamoussoukro, Côte d'Ivoire

N'Dri Kouakou Cyrille (ndri.pack@gmail.com)
UMRI Mathématiques et Nouvelles Technologies de l'Information, Institut National Polytechnique Félix HOUPHOUËT-BOIGNY, Yamoussoukro, Côte d'Ivoire

Edja Kouamé Béranger (kouame.edja@uvci.edu.ci)
Université Virtuelle de Côte d'Ivoire, 28 BP 536 Abidjan 28, Côte d'Ivoire

Touré Kidjégbo Augustin (latoureci@gmail.com)
UMRI Mathématiques et Nouvelles Technologies de l'Information, Institut National Polytechnique Félix HOUPHOUËT-BOIGNY, Yamoussoukro, Côte d'Ivoire