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Abstract. Rough group theory is an extension of group theory. It is suitable for the processing of inaccurate data and uncertain informations. This article is an attempt to bring together group cohomology theory and rough group theory. More precisely, we construct the first and the second rough cohomology groups of a rough group.

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Corresponding Author: Yaogan Mensah(mensahyaogan2@gmail.com)

1. Introduction

In 1982, Pawlak [1] introduced the concept of rough set to analyze and to model vague and uncertain data. The notion of rough set appears to be powerful with significant applications in quantum mechanics, software engineering, computer systems, bioinformatics, decision analysis, electrical engineering, finance, chemistry, computer engineering, economics, neurology, medicine, statistics, etc ([2, 3, 4]).

The notion of rough set has been later extended to the group theory setting ([5, 6, 7, 8]). The study of algebraic structures of objets related to rough groups seems to be useful and may conduct to a better understanding of rough groups ([6, 9, 10, 11]). Cohomology theory plays an important rôle in group theory ([12, 13, 14]). This suggests to conduct similar investigation for rough groups.

The main purpose of this work is to construct the first and the second cohomology groups of a rough group.

The rest of the paper is organized as follows. Section 2 collects some definitions and results that we may need. Section 3 states the main results.

2. Preliminaries

In this section, we give some well-known definitions.

Definition 2.1 ([1])**.** Let U be a non-empty set (called the *universe*). Let R be an equivalence relation on U. The pair (U, R) is called an *approximation space*.

Let (U, R) be an approximation space. For $x \in U$, the equivalence class of x is denoted by $[x]$. For $X \subset U$, set

$$\overline{X} = \{x \in U : [x] \cap X \neq \varnothing\} \text{ and } \underline{X} = \{x \in U : [x] \subset X\}.$$

The sets $\overline{X}$ and $\underline{X}$ are called the *upper approximation* and *lower approximation* of X in (U, R) respectively. Clearly, we have $\underline{X} \subset X \subset \overline{X}$.

Assume that U is endowed with a binary operation $U \times U \longrightarrow U$. The product of two elements x and y is denoted by xy. The inverse of x (if it exists) is denoted by x^{-1}.

Definition 2.2 ([7])**.** Let (U, R) be an approximation space. Assume that there is a binary operation on U. A subset G of U is called a *rough group* if all the following properties hold:

(i) $\forall\ x,\ y \in G,\ xy \in \overline{G}$,

(ii) $\forall\ x,\ y,\ z \in \overline{G},\ (xy)z = x(yz)$,

(iii) $\exists\ e \in \overline{G},\ \forall x \in G,\ ex = xe = x$ (e is called a *rough identity element of* G),

(iv) $\forall\ x \in G, \exists\ y \in G, xy = yx = e$ (y is called the *rough inverse of* x and it is denoted by x^{-1}).

Definition 2.3 ([6])**.** A non-empty subset H of a rough group G is called a *rough subgroup* of G, if the following two conditions are satisfied:

(i) $\forall x,\ y \in H,\ xy \in \overline{H}$,

(ii) $\forall x \in H,\ x^{-1} \in H$.

There is only one guaranteed trivial rough subgroup of a rough group G which is G itself. A necessary and sufficient condition for the set $\{e\}$ to be a trivial rough subgroup of the rough group G is that $e \in G$.

A rough normal subgroup can be defined as follows.

Definition 2.4 ([15])**.** Let N be a rough subgroup of a rough group G. Then N is called a *rough normal subgroup* of G, if

$$\forall x \in G, \forall n \in N, xnx^{-1} \in N.$$

Definition 2.5 ([8])**.** Let (U_1, R_1) and (U_2, R_2) be two approximation spaces with binary operations on U_1 and U_2. Suppose that $G_1 \subset U_1, G_2 \subset U_2$ are rough groups. If a mapping $\varphi : \overline{G_1} \to \overline{G_2}$ is such that $\forall x, y \in G_1, \varphi(xy) = \varphi(x)\varphi(y)$, then φ is called a *rough homomorphism*.

3. Rough cohomology groups of a rough group

3.1. **The first rough cohomology group.** Let A be an abelian multiplicative group (with identity element e_A) such that there exists an action of the rough group G on A. That is, there exists a map $\overline{G} \times A \longrightarrow A, (x, a) \longmapsto {}^{x}a$ such that

$$\forall x, y \in \overline{G}, \forall a,\ b \in A,$$

$$ {}^{e}a = a,\ {}^{xy}a = {}^{x}({}^{y}a),\ {}^{x}(ab) = {}^{x}a{}^{x}b, \text{ and } ({}^{x}a)^{-1} = {}^{x}(a^{-1}). \tag{3.1}$$

Definition 3.1. A map $f : \overline{G} \to A$ is called a *rough crossed homomorphism*, if

$$\forall x,\ y \in \overline{G},\ f(xy) = {}^{x}f(y)f(x).$$

Let us denote by $Z^1_r(G, A)$ the set of rough crossed homomorphisms of G over A. For $f,\ g \in Z^1_r(G, A)$, set

$$(f \diamond g)(x) = f(x)g(x), x \in \overline{G}.$$

Theorem 3.2. $(Z^1_r(G, A), \diamond)$ *is an abelian group.*

Proof. (i) Let $f,\ g$ be in $Z^1_r(G, A)$ and let $x,\ y$ be in $\overline{G}$. Then we have

$$\begin{aligned}(f \diamond g)(xy) &= f(xy)g(xy)\\ &= ({}^{x}f(y)f(x))({}^{x}g(y)g(x))\\ &= ({}^{x}f(y)g(y))(f(x)g(x))\\ &= {}^{x}(fg)(y)(fg)(x).\end{aligned}$$

Thus $f \diamond g \in Z^1_r(G, A)$.

(ii) The law $\diamond$ is obviously associative.

(iii) The law $\diamond$ is commutative because A is commutative.

(iv) Consider the map $\widehat{e}_A : \overline{G} \to A$ given by $\widehat{e}_A(x) = {}^{x}e_A$. Then $\widehat{e}_A$ is in $Z^1_r(G, A)$. Indeed, for $x,\ y \in \overline{G}$, we have

$$\widehat{e}_A(xy) = {}^{xy}e_A = {}^{x}({}^{y}e_A) = {}^{x}({}^{y}e_Ae_A) = {}^{x}({}^{y}e_A){}^{x}e_A = {}^{x}\widehat{e}_A(y)\widehat{e}_A(x).$$

The map $\widehat{e}_A$ is the neutral element of $Z^1_r(G, A)$ because for $f \in Z^1_r(G, A)$ and $x \in \overline{G}$, we have

$$\begin{aligned}(f \diamond \widehat{e}_A)(x) &= f(x)\widehat{e}_A(x)\\ &= {}^{e}f(x)\widehat{e}_A(x)\\ &= {}^{xx^{-1}}f(x){}^{x}e_A\\ &= {}^{x}({}^{x^{-1}}f(x)e_A)\\ &= {}^{x}({}^{x^{-1}}f(x))\\ &= {}^{e}f(x) = f(x).\end{aligned}$$

Thus $f \diamond \widehat{e}_A = f$.

(v) Let $f \in Z^1_r(G, A)$. Set $g(x) = (f(x))^{-1}$ for $x \in \overline{G}$. For $x,\ y \in \overline{G}$, we have

$$g(xy) = (f(xy))^{-1} = ({}^{x}f(y)f(x))^{-1} = ({}^{x}f(y))^{-1}(f(x))^{-1} = {}^{x}g(y)g(x).$$

Then $g \in Z^1_r(G, A)$. Moreover,

$$(f \diamond g)(x) = f(x)f(x)^{-1} = {}^{x}e_A = \widehat{e}_A(x).$$

Thus $f \diamond g = \widehat{e}_A$. So g is the inverse of f. Hence $(Z^1_r(G, A), \diamond)$ is an abelian group. □

Definition 3.3. Let $a \in A$. A map $f_a : \overline{G} \to A$ is called a *principal rough crossed homomorphism*, if

$$f_a(x) = {}^{x}aa^{-1}, \forall x \in \overline{G}.$$

Let us denote by $B_r^1(G,A)$ the set of principal rough crossed homomorphisms of G over A.

Theorem 3.4. *The set $B_r^1(G,A)$ is a normal subgroup of $(Z_r^1(G,A),\diamond)$.*

Proof. (i) Let us first show that $B_r^1(G,A)\subset Z_r^1(G,A)$. For $a\in A$, consider $f_a\in B_r^1(G,A)$. For $x,\ y\in\overline{G}$, we have

$$\begin{aligned}{}^{x}f_a(y)f_a(x) &= {}^{x}({}^{y}aa^{-1})({}^{x}aa^{-1})\\ &= ({}^{xy}a)({}^{x}a^{-1})({}^{x}aa^{-1})\\ &= ({}^{xy}a)({}^{x}a)^{-1}({}^{x}a)a^{-1}\\ &= {}^{xy}aa^{-1}\\ &= f_a(xy).\end{aligned}$$

Then $f_a\in Z_r^1(G,A)$. Thus $B_r^1(G,A)\subset Z_r^1(G,A)$.

(ii) Let $f_a,\ f_b$ be in $B_r^1(G,A)$ with $a,b\in A$. For $x\in\overline{G}$, we have

$$\begin{aligned}(f_a\diamond f_b)(x) &= f_a(x)f_b(x)\\ &= ({}^{x}aa^{-1})({}^{x}bb^{-1})\\ &= {}^{x}(ab)(ab)^{-1}\\ &= f_{ab}(x).\end{aligned}$$

Then $f_a\diamond f_b=f_{ab}$. Thus $f_a\diamond f_b\in B_r^1(G,A)$. Moreover, $f_a\diamond f_{a^{-1}}(x)=f_{e_A}(x)={}^{x}e_Ae_A^{-1}=\widehat{e}_A(x)$. So $f_a^{-1}=f_{a^{-1}}\in B_r^1(G,A)$. Furthermore, since $Z_r^1(G,A)$ is an abelian group, its subgroup $B_r^1(G,A)$ is a normal subgroup. □

Definition 3.5. We call the quotient-group $H_r^1(G,A):=Z_r^1(G,A)/B_r^1(G,A)$ the *first rough cohomology group* of G over A.

Example 3.6. Take $U=\mathbb{Z}_7=\{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6}\}$. The operation considered on $\mathbb{Z}_7$ is the multiplication of integers modulo 7. Let R be an equivalence relation on $\mathbb{Z}_7$ such that the equivalence classes are $E_1=\{\overline{0},\overline{2},\overline{4}\}$, $E_2=\{\overline{1},\overline{3},\overline{5}\}$ and $E_3=\{\overline{6}\}$.

Now, consider $G=\{\overline{2},\overline{3},\overline{4},\overline{5},\overline{6}\}$. Then, $\overline{G}=\mathbb{Z}_7$ and $\underline{G}=\{\overline{6}\}$. By the use of Definition 2.2, one may verify easily that G is a rough group. Take $A=(\mathbb{Z}_7,+)$. It is well known that $(\mathbb{Z}_7,+)$ is an abelian group. The map $\mathbb{Z}_7\times\mathbb{Z}_7\longrightarrow\mathbb{Z}_7$, $(x,a)\longmapsto xa$ is such that the equalities (3.1) are verified.

Here, $Z_r^1(G,\mathbb{Z}_7)$ is the set of functions $f:\mathbb{Z}_7\longrightarrow\mathbb{Z}_7$ such that

$$\forall x,y\in\mathbb{Z}_7,\ f(xy)=xf(y)+f(x).$$

Furthermore, $B_r^1(G,\mathbb{Z}_7)$ is the set of functions $f_a:\mathbb{Z}_7\longrightarrow\mathbb{Z}_7$, with a running through $\mathbb{Z}_7$, such that $\forall x\in\mathbb{Z}_7, f_a(x)=xa-a$.

Finally, $H_r^1(G,\mathbb{Z}_7)=\{f+B_r^1(G,\mathbb{Z}_7):f\in Z_r^1(G,\mathbb{Z}_7)\}$.

3.2. **The second rough cohomology group.** In this subsection, A is still an abelian group and G is a rough group.

Definition 3.7. A map $\alpha:\overline{G}\times\overline{G}\to A$ is called a *rough multiplier* or a *rough 2-cocycle* of G over A, if $\forall x,\ y,\ z\in\overline{G}$,

(i) $\alpha(x,y)\alpha(xy,z)=\alpha(x,yz)\alpha(y,z)$,

(ii) $\alpha(x,e)=\alpha(e,x)=e_A$.

The set of rough 2-cocycles is denoted by $Z_r^2(G, A)$.

For $\alpha,\ \beta \in Z_r^2(G, A)$, set

$$(\alpha \star \beta)(x, y) = \alpha(x, y)\beta(x, y),\ x,\ y \in \overline{G}.$$

Theorem 3.8. *$(Z_r^2(G, A), \star)$ is an abelian group.*

Proof. (i) The set $Z_r^2(G, A)$ is stable under the law $\star$. Indeed, let $\alpha,\ \beta$ be in $Z_r^2(G, A)$. For $x,\ y,\ z \in \overline{G}$, we have

$$\begin{aligned}(\alpha \star \beta)(x, y)(\alpha \star \beta)(xy, z) &= \alpha(x, y)\beta(x, y)\alpha(xy, z)\beta(xy, z)\\ &= \alpha(x, y)\alpha(xy, z)(\beta(x, y)\beta(xy, z)\\ &= \alpha(x, yz)\alpha(y, z)\beta(x, yz)\beta(y, z)\\ &= \alpha(x, yz)\beta(x, yz)\alpha(y, z)\beta(y, z)\\ &= (\alpha \star \beta)(x, yz)(\alpha \star \beta)(y, z).\end{aligned}$$

Moreover, $(\alpha\star\beta)(e, x) = \alpha(e, x)\beta(e, x) = e_A$ and $(\alpha\star\beta)(x, e) = \alpha(x, e)\beta(x, e) = e_A$. Then $\alpha \star \beta \in Z_r^2(G, A)$.

(ii) The law $\star$ is obviously associative and commutative.

(iii) The map $\widehat{e} : \overline{G} \times \overline{G} \to A$, defined by $\widehat{e}(x, y) = e_A$, is an element of $Z_r^2(G, A)$. It is the neutral element for the law $\star$.

(iv) Let $\alpha \in Z_r^2(G, A)$. Set $\alpha^{-1}(x, y) = \alpha(x, y)^{-1}$. A straightforward computation shows that $\alpha^{-1} \in Z_r^2(G, A)$ and $\alpha \star \alpha^{-1} = \widehat{e}$. Then $(Z_r^2(G, A), \star)$ is an abelian group. □

Definition 3.9. Let $f : \overline{G} \to A$ be a map such that $f(e) = e_A$. A map $\nu : \overline{G}\times\overline{G} \to A$ defined by

$$\nu(x, y) = f(x)f(y)f(xy)^{-1},\ \forall\, x, y \in \overline{G}.$$

is called a *rough coboundary.*

We denote by $B_r^2(G, A)$ the set of rough coboundaries.

Theorem 3.10. *The set $B_r^2(G, A)$ is a normal subgroup of $(Z_r^2(G, A), \star)$.*

Proof. (i) The neutral element $\widehat{e}$ of $Z_r^2(G, A)$ belongs to $B_r^2(G, A)$ since for $x,\ y \in \overline{G}$, $\widehat{e}(x, y) = f_e(x)f_e(y)(f_e(xy))^{-1}$, where $f_e : \overline{G} \to A$ is defined by $f_e(x) = e_A$.

(ii) Let $\nu \in B_r^2(G, A)$. Then there exists $f : \overline{G} \to A$ with $f(e) = e_A$ such that $\nu(x, y) = f(x)f(y)(f(xy))^{-1}$. Thus we have

$$\begin{aligned}\nu(x, y)\nu(xy, z) &= f(x)f(y)(f(xy))^{-1}f(xy)f(z)(f(xyz)^{-1})\\ &= f(x)f(y)f(z)(f(xyz))^{-1}\\ &= f(x)f(y)f(z)f(yz)(f(yz))^{-1}(f(xyz))^{-1}\\ &= f(y)f(z)(f(yz))^{-1}f(x)f(yz)(f(xyz))^{-1}\\ &= \nu(y, z)\nu(x, yz).\end{aligned}$$

Moreover, we get

$$\nu(e, x) = f(e)f(x)(f(ex))^{-1} = f(e)f(x)(f(x))^{-1}) = f(e) = e_A$$

and

$$\nu(x, e) = f(x)f(e)(f(xe))^{-1} = f(x)f(e)(f(x))^{-1}) = f(e) = e_A.$$

So $\nu \in Z_r^2(G, A)$. Hence $B_r^2(G, A) \subset Z_r^2(G, A)$.

(iii) Let $\nu,\ \mu \in B_r^2(G, A)$. Then there exists $f : \overline{G} \to A$ with $f(e) = e_A$ and $g : \overline{G} \to A$ with $g(e) = e_A$ such that $\nu(x, y) = f(x)f(y)(f(xy))^{-1}$ and $\mu(x, y) = g(x)g(y)(g(xy))^{-1}$. Thus $\forall x,\ y \in \overline{G}$, we have

$$\begin{aligned}(\nu \star \mu)(x, y) &= \nu(x, y)\mu(x, y)\\ &= (f(x)f(y)(f(xy))^{-1})(g(x)g(y)(g(xy))^{-1})\\ &= f(x)f(y)(f(xy))^{-1}g(x)g(y)(g(xy))^{-1}\\ &= f(x)g(x)f(y)g(y)(f(xy))^{-1}(g(xy))^{-1}\\ &= (fg)(x)(fg)(y)(fg(xy))^{-1}.\end{aligned}$$

Also, $(f \star g)(e) = f(e)g(e) = e_Ae_A = e_A$. So $\nu \star \mu \in B_r^2(G, A)$.

(iv) Let $\nu \in B_r^2(G, A)$. Then there exists $f : \overline{G} \to A$ with $f(e) = e_A$ such that $\nu(x, y) = f(x)f(y)(f(xy))^{-1}$. Set $g(x) = (f(x))^{-1}$ and $\nu_0(x, y) = g(x)g(y)(g(xy))^{-1}$. Thus $\nu_0 \in B_r^2(G, A)$. Moreover,

$$\nu \star \nu_0(x, y) = f(x)f(y)(f(xy))^{-1}g(x)g(y)(g(xy))^{-1} = e_A = \widehat{e}(x, y).$$

So $\nu^{-1} = \nu_0$.

Finally, since $Z_r^2(G, A)$ is an abelian group, its subgroup $B_r^2(G, A)$ is a normal subgroup. □

Definition 3.11. We call the quotient-group $H_r^2(G, A) = Z_r^2(G, A)/B_r^2(G, A)$ the *second rough cohomology group* of G over A.

Example 3.12. We consider again the rough group $G = \{\overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ and the abelian group $A = (\mathbb{Z}_7, +)$ as in Example 3.6.

The group $Z_r^2(G, \mathbb{Z}_7)$ is made up of functions $\alpha : \overline{G} \times \overline{G} \to \mathbb{Z}_7$ such that

$$\alpha(x, y) + \alpha(xy, z) = \alpha(x, yz) + \alpha(y, z) \text{ and } \alpha(x, \overline{1}) = \alpha(\overline{1}, x) = \overline{0}, \forall x,\ y,\ z \in \overline{G}.$$

The elements of the subgroup $B_r^2(G, \mathbb{Z}_7)$ of $Z_r^2(G, \mathbb{Z}_7)$ are the functions ν from $\overline{G} \times \overline{G}$ into $\mathbb{Z}_7$ such that $\forall x, y \in \overline{G},\ \nu(x, y) = f(x) + f(y) - f(xy)$ where f is a function from $\overline{G}$ into $\mathbb{Z}_7$ with $f(\overline{1}) = \overline{0}$. Finally, $H_r^2(G, \mathbb{Z}_7) = \{f + B_r^2(G, \mathbb{Z}_7) : f \in Z_r^2(G, \mathbb{Z}_7)\}$.

4. Conclusion

The first and the second rough cohomology groups are constructed. This paved the way for the study of projective representations of rough groups.

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Essoyomewè Kieou (kieouesso@gmail.com)
Department of Mathematics, University of Kara, 1 P.O.Box 43 Kara, Togo

Mawoussi Todjro (todjrom7@gmail.com)
Department of Mathematics, University of Kara, 1 P.O.Box 43 Kara, Togo

Yaogan Mensah (mensahyaogan2@gmail.com, ymensah@univ-lome.tg)
Department of Mathematics, University of Lomé, 1 P.O.Box 1515 Lomé 1, Togo and International Chair in Mathematical Physics and Applications (ICMPA)-Unesco Chair, University of Abomey-Calavi, Benin