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Abstract. Rough set based on the L-fuzzy neighborhood system is a general rough set model. This paper studies the structure-preserving properties of its interior and closure operators. For an L-fuzzy neighborhood system on a single universe, we define the neighborhood system reduction and find the upper and lower bounds of the interior and closure operators. We also show that different L-fuzzy neighborhood systems can produce the same interior and closure operators under certain conditions. For L-fuzzy neighborhood systems on two universes, we use mapping and consistent function to study structure preservation of the interior and closure operators of the two systems.

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Keywords: L-fuzzy neighborhood system, Interior operator, Closure operator, Reduction, Consistent function

Corresponding Author: Yinshan Liu (yinshanliu@foxmail.com)

1. Introduction

Rough set theory was proposed by Pawlak [1] as an important mathematical tool for dealing with imprecise, fuzzy, or uncertain information. Rough set theory has been widely applied in fields such as medical diagnosis, machine learning, data mining, intelligent control, etc. [2, 3, 4, 5]. The core upper and lower approximation operators(also called closure and interior operators) of classical rough set theory are based on equivalence relations. Since the requirement of equivalence relations is too strict, equivalence relations are generalized to general binary relations [6], covering [7], neighborhoods, and neighborhood systems [8]. With the development of fuzzy mathematics, many concepts of rough sets are also generalized to fuzzy situations, such as fuzzy rough sets based on fuzzy relations [6], fuzzy rough sets based on fuzzy coverings [9], and fuzzy rough sets based on fuzzy neighborhood systems [10]. The rough set based on neighborhood system is an important rough set model that can unify various models, such as rough set models based on general relations, neighborhood, and covering into one framework. Therefore, the research on rough set based on neighborhood system has important significance. Lin first applied the concept of neighborhood systems to granular computing [8]. In Lin's model, each element of the universe is associated with a family of subsets of the universe, which are called neighborhoods. The neighborhoods of an element may or may not contain that element. To deal with more complex situations, neighborhood systems in fuzzy environments are introduced, including fuzzy neighborhood systems [11], fuzzifying neighborhood systems [12], and L-fuzzy neighborhood systems [10]. This article focuses on the interior and closure operators of neighborhood systems, and interior and closure are important concepts in topology. Many scholars pay attention to the topological structures of new uncertain mathematics such as fuzzy sets, rough sets, and soft sets. Lee et al. [13] studied neighborhood structures on cubic sets. Şenel gave a detailed discussion on topological structures on soft sets, such as Hausdorff space and topological subspace on soft sets [14, 15], soft closed sets on soft bitopological spaces [16], and soft topology generated by L- soft sets [17].

In rough set theory, knowledge is regarded as the ability to classify objects, and the uncertain concepts are approximated by precise concepts using interior and closure operators. The research on rough set models mainly includes constructive methods and axiomatic methods. The constructive method constructs interior and closure operators based on the relation between objects, the partition or covering of the universe, the neighborhood of objects, etc., and discusses the algebraic structure of interior and closure operators. The axiomatic method studies the conditions or axioms for a general set-valued mapping to become an interior or closure operator. Many scholars have conducted in-depth research on the axiomatization problem of rough sets. From the perspective of rough set axiomatization, when a set-valued mapping on the universe is an interior or closure operator, the induced neighborhood system is not unique. This is different from some traditional rough sets. Based on this, we can know that different neighborhood systems may generate the same interior and closure operators. This is the structure preservation of interior and closure operators of neighborhood systems on single universe, in addition to the structure preservation of interior and closure operators of neighborhood systems on two universe. Information systems, also known as information tables, are systems composed of domains and attribute sets. The rows of an information table represent objects, its columns represent attributes, and the contents of the table are corresponding attribute values. Since the early development of rough sets, people have studied the structural relationship and transformation between two information systems through OAD (object, attribute, attribute value domain)-homomorphism. Grzymala first proposed the concept of information system homomorphism and used it to study the relationship between two information systems [18]. Information system homomorphism is an effective tool for dealing with object and attribute fusion. In [19], Li studied the non-necessity of features and reduction of information systems under homomorphism. After that, Wang [20] discussed some invariant properties of relational information systems under homomorphism and proved that the reduction of two information systems under homomorphism is equivalent. Wang [21] first introduced the concept of consistent function into information systems and discussed some relevant properties and conclusions of consistent function in information systems. Zhu systematically discussed some relevant properties of consistent functions under binary relation information systems and simplified consistent functions [20] into two special types of consistent functions [22]. Consistent function is a special kind of mapping that can be used to study the structure preservation problem of neighborhood systems on two universes. Zhu et al. [22] first introduced the concept of consistent function into neighborhood systems and studied some preservation properties of neighborhood systems under consistent function. Subsequently, Liau [23] proposed concepts such as system consistent function and weak system consistent function based on neighborhood systems and obtained conditions for structure preservation of interior and closure operators by combining consistent function with interior and closure operators of neighborhood systems. Zhu et al. [22] generalized consistent function to fuzzy neighborhood systems, obtained relevant conclusions by combining fuzzy relation, fuzzy covering, etc., and used consistent function to study the relationship between multiple fuzzy neighborhood systems, but did not study relevant conclusions related to interior and closure operators and consistent functions in fuzzy neighborhood systems. In [23], Liau generalized the concept of consistent function on neighborhood systems to fuzzy neighborhood systems, but the definition of interior and closure operators on fuzzy neighborhood systems was defined by a special implication operator, which has limitations. He [11] generalized a special implication operator to a general implication and discussed structure preservation and property preservation of interior and closure operators in fuzzy neighborhood systems. Hou [24] used the consistent function to study the structure preservation of interior and closure operators in fuzzifying neighborhood systems.

This article explores how the interior and closure operators of L-fuzzy neighborhood systems preserve their structure. The second part introduces some basic knowledge and notation. The third part examines the structure preservation of the interior and closure operators of different neighborhood systems on single universe, and studies the fuzzy neighborhood systems, fuzzifying neighborhood systems, and L-fuzzy neighborhood systems respectively. The L-fuzzy neighborhood system can generate the same interior and closure operators in a special case. The fourth part discusses the structure preservation of the interior and closure operators of L-fuzzy neighborhood systems on two universes with the help of mappings and new consistent functions.

2. Preliminaries

Suppose L is a complete lattice, let 0 and 1 be the minimum element and maximum element of L. Then we have the following definition.

Definition 2.1. A *complete residuated lattice* refers to $(L, \otimes, \to, \bigwedge, \bigvee)$ satisfying

(i) $(L, \bigwedge, \bigvee, 1, 0)$ is a complete lattice with maximum element 1 and minimum element 0,

(ii) $(L, \otimes, 1)$ is a commutative monoid, i.e. $\otimes$ is commutative, associative and $x \otimes 1 = x$ holds for any $x \in L$,

(iii) $(\otimes, \to)$forms an adjoint pair,i.e., $x \otimes y \le z \Leftrightarrow x \le y \to z$ holds for any $x,\ y,\ z \in L$.

A complete residuated lattice L has the following properties.

Lemma 2.2 ([25]). *Let $(L, \otimes, \to, \bigwedge, \bigvee)$ be a complete residuated lattice. Then the followings hold: for any $x,\ y,\ z,\ x_i,\ y_i\ (i \in I) \in L$,*

(1) $x \otimes y = y \otimes x,\ x \to y = 1 \Leftrightarrow x \le y,\ 1 \to x = x$,
(2) $x \otimes \bigwedge_{i\in I} y_i \le \bigwedge_{i\in I} x \otimes y_i,\ x \otimes \bigvee_{i\in I} y_i = \bigvee_{i\in I} x \otimes y_i$,
(3) $\bigvee_{i\in I} x_i \to y = \bigwedge_{i\in I} (x_i \to y)$,
(4) $x \to \bigwedge_{i\in I} y_i = \bigwedge_{i\in I} (x \to y_i)$,
(5) $x \to \bigvee_{i\in I} y_i \ge \bigvee_{i\in I} (x \to y_i)$,
(6) $\bigwedge_{i\in I} x_i \to y \ge \bigvee_{i\in I} (x_i \to y)$,
(7) $(x \otimes y) \to z = x \to (y \to z) = y \to (x \to z)$.

Unless otherwise stated, L in this article is a complete residuated lattice.

If $(x \to 0) \to 0 = x$ for any $x \in L$, then L is said to *satisfy the law of double negation*. Generally speaking, $\neg x$ is used to denote $x \to 0$, then the residuated lattice that satisfies the law of double negation has the following properties.

Lemma 2.3 ([25]). *Let $(L, \otimes, \to, \bigwedge, \bigvee)$ satisfy the double negation law. Then the followings hold: for any $x,\ y,\ x_i\ (i \in I) \in L$,*

(1) $x \to y = \neg y \to \neg x$,
(2) $x \to y = \neg(x \otimes \neg y)$,
(3) $\neg(\bigwedge_{i\in I} x_i) = \bigvee_{i\in I} \neg x_i$.

For the classic set $A \subseteq U$, let 1_A be its characteristic function, we can know if $x \in A$, there is $1_A(x) = 1$, if $x \notin A$, there is $1_A(x) = 0$. The characteristic function 1_A of A can be regarded as an L fuzzy set on U, we can know when $L = \{0, 1\}$, L^U degenerates into 2^U.

For L fuzzy sets A, B in L^U, the subsethood degree and the intersection degree of A and B can be defined.

Definition 2.4 ([25, 26]). The *subsethood degree* of A and B is recorded as $S(A, B)$, which is specifically expressed as

$$S(A, B) = \bigwedge_{x\in U} (A(x) \to B(x))$$

The intersection degree of A and B is recorded as $N(A, B)$, which is specifically expressed as

$$N(A, B) = \bigvee_{x\in U} (A(x) \otimes B(x)).$$

Definition 2.5 ([11]). A mapping $N : U \to 2^{L^U}$ is called a *fuzzy neighborhood system operator* on U. For any $\mu \in L^U$, the *closure* and *interior operators* are defined as:

$$\overline{N}(\mu)(x)=\bigwedge_{\nu\in N(x)}\bigvee_{y\in U}(\nu(y)\otimes\mu(y)),\ \underline{N}(\mu)(x)=\bigvee_{\nu\in N(x)}\bigwedge_{y\in U}(\nu(y)\to\mu(y)).$$

In this paper, we generalize the definition of fuzzifying neighborhood systems and their interior and closure operators in [12].

Definition 2.6. A mapping $N:U\to L^{2^U}$ is called a *fuzzifying neighborhood system operator* on U, if $\forall x\in U$, $\bigvee\limits_{K\in L^U}N(x)(K)=1$, where $N(x)(K)$ represents the degree to which K is the neighborhood of x. for any $\mu\in L^U$, the *closure* and *interior operators* are defined as:

$$\overline{N}(\mu)(x)=\bigwedge_{H\in 2^U}(N(x)(H)\to(\bigvee_{z\in H}\mu(z))),\ \underline{N}(\mu)(x)=\bigvee_{H\in 2^U}(N(x)(H)\otimes(\bigwedge_{z\in H}\mu(z))).$$

In general, for any $x\in U, A,B\in L^U$, there is $N(x)(A\bigvee B)\geq N(x)(A)\bigwedge N(x)(B)$, it is the same for the L-fuzzy neighborhood system below.

Definition 2.7 ([10])**.** A mapping $N:U\to L^{L^U}$ is called a *L-fuzzy neighborhood system operator* on U if $\forall x\in U$, $\bigvee\limits_{K\in L^U}N(x)(K)=1$. For any $\mu\in L^U$, the *closure* and *interior operators* are defined as:

$$\overline{N}(\mu)(x)=\bigwedge_{K\in L^U}(N(x)(K)\to N(K,\mu)),\ \underline{N}(\mu)(x)=\bigvee_{K\in L^U}(N(x)(K)\otimes S(K,\mu)).$$

For the L-fuzzy neighborhood system, it degenerates into the general fuzzy neighborhood system and the fuzzifying neighborhood system. In the following discussion, we will discuss their structure preservation from two perspectives.

3. The single universe

First define the reduction of the fuzzy neighborhood system.

Definition 3.1. Let N be a fuzzy neighborhood system on U. For any $x\in U$, ν is called a *reducible element at* x, if $\nu\in N(x)$ and there exist $\nu_1,\ \nu_2,\ \cdots,\ \nu_n\in N(x)-\{\nu\}$ such that $\nu=\bigvee\limits_{i=1}^{n}\nu_i$.

Definition 3.2. For any $x\in U$, if $M(x)$ is all reducible elements at x, then let $RN(x)=N(x)-M(x)$, which is called RN is the *reduction* of N.

According to the above definition, we can have the following lemma.

Lemma 3.3. *For any $x\in U$, if ν is a reducible element at x, then*

$$\bigvee_{\nu'\in N(x)}\bigwedge_{y\in U}(\nu'(y)\to\mu(y))=\bigvee_{\nu''\in N(x)-\{\nu\}}\bigwedge_{y\in U}(\nu''(y)\to\mu(y)),$$

$$\bigwedge_{\nu'\in N(x)}\bigvee_{y\in U}(\nu'(y)\otimes\mu(y))=\bigwedge_{\nu''\in N(x)-\{\nu\}}\bigvee_{y\in U}(\nu''(y)\otimes\mu(y)).$$

Proof. Suppose ν is the reducible element at x. Then there exist $\nu_1,\ \nu_2,\ \cdots,\ \nu_n \in N(x)-\{\nu\}$ such that $\nu = \bigvee\limits_{i=1}^{n} \nu_i$. Thus

$$\begin{aligned}
&\bigvee_{\nu'\in N(x)} \bigwedge_{y\in U} (\nu'(y) \to \mu(y))\\
&= [\bigvee_{\nu'\in N(x)-\{\nu\}} \bigwedge_{y\in U} (\nu'(y) \to \mu(y))] \bigvee [\bigwedge_{y\in U} (\nu(y) \to \mu(y))]\\
&= [\bigvee_{\nu'\in N(x)-\{\nu\}} \bigwedge_{y\in U} (\nu'(y) \to \mu(y))] \bigvee [\bigwedge_{y\in U} ((\bigvee_{i=1}^{n} \nu_i(y)) \to \mu(y))]\\
&= [\bigvee_{\nu'\in N(x)-\{\nu\}} \bigwedge_{y\in U} (\nu'(y) \to \mu(y))] \bigvee [\bigwedge_{y\in U} \bigwedge_{i=1}^{n} (\nu_i(y) \to \mu(y))]\\
&= \bigvee_{\nu'\in N(x)-\{\nu\}} \bigwedge_{y\in U} (\nu'(y) \to \mu(y)),\\
&\bigwedge_{\nu'\in N(x)} \bigvee_{y\in U} (\nu'(y) \otimes \mu(y))\\
&= [\bigwedge_{\nu'\in N(x)-\{\nu\}} \bigvee_{y\in U} (\nu'(y) \otimes \mu(y))] \bigwedge [\bigvee_{y\in U} (\nu(y) \otimes \mu(y))]\\
&= [\bigwedge_{\nu'\in N(x)-\{\nu\}} \bigvee_{y\in U} (\nu'(y) \otimes \mu(y))] \bigwedge [\bigvee_{y\in U} ((\bigvee_{i=1}^{n} \nu_i(y)) \otimes \mu(y))]\\
&= [\bigwedge_{\nu'\in N(x)-\{\nu\}} \bigvee_{y\in U} (\nu'(y) \otimes \mu(y))] \bigwedge [\bigvee_{y\in U} \bigvee_{i=1}^{n} (\nu_i(y) \otimes \mu(y))]\\
&= \bigwedge_{\nu'\in N(x)-\{\nu\}} \bigvee_{y\in U} (\nu'(y) \otimes \mu(y)).
\end{aligned}$$

So we have

$$\bigvee_{\nu'\in N(x)} \bigwedge_{y\in U} (\nu'(y) \to \mu(y)) = \bigvee_{\nu''\in N(x)-\{\nu\}} \bigwedge_{y\in U} (\nu''(y) \to \mu(y)),$$

$$\bigwedge_{\nu'\in N(x)} \bigvee_{y\in U} (\nu'(y) \otimes \mu(y)) = \bigwedge_{\nu''\in N(x)-\{\nu\}} \bigvee_{y\in U} (\nu''(y) \otimes \mu(y)).$$

□

From the above lemma, it is obvious that the following proposition holds.

Proposition 3.4. *For any two neighborhood systems N_1 and N_2 on U, if $RN_1 = RN_2$, then $\overline{N_1} = \overline{N_2}, \underline{N_1} = \underline{N_2}$.*

It can be seen from the above proposition that for any two neighborhood systems N_1 and N_2 on U, the reduced neighborhood systems are the same, then N_1 and N_2 generate the same interior and closure operators, That is to say, their interior and closure operator structure is preserved.

Reductions can also be defined for fuzzifying neighborhood systems.

Definition 3.5. Let N be the fuzzifying neighborhood system on U. For any $x \in U$, H is called the reducible element at x, if $N(x)(H) > 0$ and there exist $H_1,\ H_2,\ \cdots,\ H_n \in 2^U - \{H\}(N(x)(H_i) > 0)$ such that $H = \bigcup\limits_{i=1}^{n} H_i$.

Definition 3.6. Let N be a fuzzifying neighborhood system. Then the *neighborhood system reduced by N*, denoted by RN, is defined as follows: for each $x \in U$ and each

$K \in 2^U$,

$$RN(x)(K) = \begin{cases} 0 & \text{if } K \text{ is the reducible element at x} \\ N(x)(K) & \text{otherwise.} \end{cases} \quad (3.1)$$

According to the above definition, we can have the following lemma.

Lemma 3.7. *For any $x \in U$, if H is a reducible element at x, then*

$$\bigvee_{H' \in 2^U} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z))) \geq \bigvee_{H'' \in 2^U - \{H\}} (N(x)(H'') \otimes (\bigwedge_{z \in H''} \mu(z))),$$

$$\bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z))) \leq \bigwedge_{H'' \in 2^U - \{H\}} (N(x)(H'') \to (\bigvee_{z \in H''} \mu(z))).$$

Proof. Suppose H is a reducible element at x. Then there exist $H_1, H_2, \cdots, H_n \in 2^U - \{H\}(N(x)(H_i) > 0)$ such that $H = \bigcup_{i=1}^{n} H_i$. Thus we have

$$\begin{aligned}
&\bigvee_{H' \in 2^U} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z))) \\
&= [\bigvee_{H' \in 2^U - \{H\}} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z)))] \bigvee [N(x)(H) \otimes (\bigwedge_{z \in H} \mu(z))] \\
&= [\bigvee_{H' \in 2^U - \{H\}} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z)))] \bigvee [N(x)(\bigcup_{i=1}^{n} H_i) \otimes (\bigwedge_{z \in H} \mu(z))] \\
&\geq [\bigvee_{H' \in 2^U - \{H\}} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z)))] \bigvee [(\bigwedge_{i=1}^{n} N(x)(H_i)) \otimes (\bigwedge_{z \in H} \mu(z))] \\
&= \bigvee_{H' \in 2^U - \{H\}} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z))),
\end{aligned}$$

$$\begin{aligned}
&\bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z))) \\
&= [\bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z)))] \bigwedge [N(x)(H) \to (\bigvee_{z \in H} \mu(z))] \\
&= [\bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z)))] \bigwedge [N(x)(\bigcup_{i=1}^{n} H_i) \to (\bigvee_{z \in H} \mu(z))] \\
&\leq [\bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z)))] \bigwedge [(\bigwedge_{i=1}^{n} N(x)(H_i)) \to (\bigvee_{z \in H} \mu(z))] \\
&= \bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z))).
\end{aligned}$$

So we get

$$\bigvee_{H' \in 2^U} (N(x)(H') \otimes (\bigwedge_{z \in H'} \mu(z))) \geq \bigvee_{H'' \in 2^U - \{H\}} (N(x)(H'') \otimes (\bigwedge_{z \in H''} \mu(z))),$$

$$\bigwedge_{H' \in 2^U} (N(x)(H') \to (\bigvee_{z \in H'} \mu(z))) \leq \bigwedge_{H'' \in 2^U - \{H\}} (N(x)(H'') \to (\bigvee_{z \in H''} \mu(z))).$$

□

Although the fuzzifying neighborhood system can not have the same interior and closure operators as the general fuzzy neighborhood system and its reduced neighborhood system, we can know the exact bounds of the interior and closure operators of the fuzzifying neighborhood system based on the above lemma.

Proposition 3.8. *Let $N_1, N_2, \ldots, N_m$ be the fuzzifying neighborhood systems on U. If the reduced neighborhood systems are all RN, then the followings hold:*

$$\overline{RN} = \underset{i=1,2,3,\ldots,m}{Sup} \overline{N_i},\ \underline{RN} = \underset{i=1,2,3,\ldots,m}{Inf} \underline{N_i}.$$

Define the reduction of the L-fuzzy neighborhood system.

Definition 3.9. Let N be the L-fuzzy neighborhood system on U. For any $x \in U$, H is called the *reducible element at* x, if $N(x)(K) > 0$ and there exist $K_1, K_2, \cdots, K_n \in L^U - \{K\}(N(x)(K_i) > 0)$ such that $K = \bigvee_{i=1}^{n} K_i$.

Definition 3.10. Let N be an L-fuzzy neighborhood system. Then the *neighborhood system reduced by N*, denoted by RN, is defined as follows: for each $x \in U$ and each $K \in L^U$,

$$RN(x)(K) = \begin{cases} 0 & \text{if } K \text{ is the reducible element at x} \\ N(x)(K) & \text{otherwise.} \end{cases} \tag{3.2}$$

According to the above definition, we can have the following lemma.

Lemma 3.11. *If K is a reducible element at any $x \in U$, then*

$$\bigvee_{K' \in L^U} (N(x)(K') \otimes S(K', \mu)) \geq \bigvee_{K'' \in L^U - \{K\}} (N(x)(K'') \otimes S(K'', \mu)),$$

$$\bigwedge_{K' \in L^U} (N(x)(K') \to N(K', \mu)) \leq \bigwedge_{K'' \in L^U - \{K\}} (N(x)(K'') \to N(K'', \mu)).$$

Proof. Suppose K is a reducible element at any $x \in U$. Then there exist $K_1, K_2, \cdots, K_n \in L^U - \{K\}(N(x)(K_i) > 0)$ such that $K = \bigvee_{i=1}^{n} K_i$. Thus we get

$$\begin{aligned}
&\bigvee_{K' \in L^U} (N(x)(K') \otimes S(K', \mu)) \\
&= [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [N(x)(K) \otimes S(K, \mu)] \\
&= [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [N(x)(\bigvee_{i=1}^{n} K_i) \otimes S(\bigvee_{j=1}^{n} K_j, \mu)] \\
&\geq [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [(\bigwedge_{i=1}^{n} N(x)(K_i)) \otimes (\bigwedge_{j=1}^{n} S(K_j, \mu))] \\
&= \bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu)),
\end{aligned}$$

$$\begin{aligned}
&\bigwedge_{K' \in L^U} (N(x)(K') \to N(K', \mu)) \\
&= [\bigwedge_{K' \in L^U - \{K\}} (N(x)(K') \to N(K', \mu))] \bigwedge [N(x)(K) \to N(K, \mu)] \\
&= [\bigwedge_{K' \in L^U - \{K\}} (N(x)(K') \to N(K', \mu))] \bigwedge [N(x)(\bigvee_{i=1}^{n} K_i) \to N(\bigvee_{j=1}^{n} K_j, \mu)] \\
&\leq [\bigwedge_{K' \in L^U - \{K\}} (N(x)(K') \to N(K', \mu))] \bigwedge [(\bigwedge_{i=1}^{n} N(x)(K_i)) \to (\bigvee_{j=1}^{n} N(K_j, \mu))]
\end{aligned}$$

$$= \bigwedge_{K' \in L^U - \{K\}} (N(x)(K') \to N(K', \mu)). \qquad \square$$

Similar to the fuzzifying neighborhood system, the exact bound of the interior and closure operator of the L-fuzzy neighborhood system can be known based on the above lemma.

Proposition 3.12. *Let $N_1, N_2, \ldots, N_m$ be the L-fuzzy neighborhood systems on U. If the reduced neighborhood systems are all RN, then*

$$\overline{RN} = \underset{i=1,2,3,\ldots,m}{Sup} \overline{N_i}, \ \underline{RN} = \underset{i=1,2,3,\ldots,m}{Inf} \underline{N_i}.$$

In some special cases, if the L-fuzzy neighborhood system has the same reduced neighborhood system, it has the same interior and closure operators.

Definition 3.13. Let N be the L-fuzzy neighborhood system on U. Then N is said to be *union preserving*, if for any $x \in U, \ A, \ B \in L^U$, there is $N(x)(A \bigvee B) = N(x)(A) \bigvee N(x)(B)$.

Then there is the following lemma.

Lemma 3.14. *Let N be union preserving. If K is a reducible element at any $x \in U$, then*

$$\bigvee_{K' \in L^U} (N(x)(K') \otimes S(K', \mu)) \leq \bigvee_{K'' \in L^U - \{K\}} (N(x)(K'') \otimes S(K'', \mu)),$$

$$\bigwedge_{K' \in L^U} (N(x)(K') \to N(K', \mu)) \geq \bigwedge_{K'' \in L^U - \{K\}} (N(x)(K'') \to N(K'', \mu)).$$

Proof. Suppose K is a reducible element at any $x \in U$. Then there exist $K_1, \ K_2, \ \cdots, \ K_n \in L^U - \{K\}(N(x)(K_i) > 0)$ such that $K = \bigvee_{i=1}^{n} K_i$. Thus we have

$$\begin{aligned}
&\bigvee_{K' \in L^U} (N(x)(K') \otimes S(K', \mu)) \\
&= [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [N(x)(K) \otimes S(K, \mu)] \\
&= [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [N(x)(\bigvee_{i=1}^{n} K_i) \otimes S(\bigvee_{j=1}^{n} K_j, \mu)] \\
&= [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [(\bigvee_{i=1}^{n} N(x)(K_i)) \otimes (\bigwedge_{j=1}^{n} S(K_j, \mu))] \\
&= [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [\bigvee_{i=1}^{n} (N(x)(K_i) \otimes (\bigwedge_{i=1}^{n} S(K_j, \mu)))] \\
&\leq [\bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu))] \bigvee [\bigvee_{i=1}^{n} (N(x)(K_i) \otimes S(K_i, \mu))] \\
&= \bigvee_{K' \in L^U - \{K\}} (N(x)(K') \otimes S(K', \mu)),
\end{aligned}$$

$$\begin{aligned}
&\bigwedge_{K' \in L^U} (N(x)(K') \to N(K', \mu)) \\
&= [\bigwedge_{K' \in L^U - \{K\}} (N(x)(K') \to N(K', \mu))] \bigwedge [N(x)(K) \to N(K, \mu)]
\end{aligned}$$

$$= [\bigwedge_{K'\in L^U-\{K\}}(N(x)(K')\to N(K',\mu))]\bigwedge[N(x)(\bigvee_{i=1}^{n}K_i)\to N(\bigvee_{j=1}^{n}K_j,\mu)]$$
$$= [\bigwedge_{K'\in L^U-\{K\}}(N(x)(K')\to N(K',\mu))]\bigwedge[(\bigvee_{i=1}^{n}N(x)(K_i))\to (\bigvee_{j=1}^{n}N(K_j,\mu))]$$
$$\geq [\bigwedge_{K'\in L^U-\{K\}}(N(x)(K')\to N(K',\mu))]\bigwedge[(\bigvee_{i=1}^{n}N(x)(K_i))\to N(K_i,\mu)]$$
$$= [\bigwedge_{K'\in L^U-\{K\}}(N(x)(K')\to N(K',\mu))]\bigwedge[\bigwedge_{i=1}^{n}(N(x)(K_i)\to N(K_i,\mu))]$$
$$= \bigwedge_{K'\in L^U-\{K\}}(N(x)(K')\to N(K',\mu)).$$

So we get

$$\bigvee_{K'\in L^U}(N(x)(K')\otimes S(K',\mu))\leq \bigvee_{K''\in L^U-\{K\}}(N(x)(K'')\otimes S(K'',\mu),)$$

$$\bigwedge_{K'\in L^U}(N(x)(K')\to N(K',\mu))\geq \bigwedge_{K''\in L^U-\{K\}}(N(x)(K'')\to N(K'',\mu)).$$

□

From the above two lemmas, we have

Proposition 3.15. *For any two union-preserving L-fuzzy neighborhood systems* N_1 *and* N_2 *on* U*, if* $RN_1 = RN_2$*, then* $\overline{N_1} = \overline{N_2}$*,* $\underline{N_1} = \underline{N_2}$*.*

4. The two universe

Let f be the mapping from U to V, denote $f^{-1}(\{y\}) = \{x \in U|f(x) = y\}$. According to the Zadeh extension principle for fuzzy sets, the mapping f can be extended to $f : L^U \to L^V$ and $f^{-1} : L^V \to L^U$, specifically The form is defined as follows: for any $\mu \in L^U,\ \rho \in L^V$ and any $x \in U,\ y \in V$,

$$f(\mu)(y) = \begin{cases} \bigvee_{x\in f^{-1}(\{y\})}\mu(x) & \text{if } f^{-1}(\{y\}) \neq \varnothing \\ 0 & \text{otherwise,} \end{cases}$$

$$f^{-1}(\rho)(x) = \rho(f(x)).$$

Obviously, if f is surjective, then the extended f is also surjective.

Definition 4.1. Let f be the mapping from U to V and let N be the L-fuzzy neighborhood system on U. Then we define the *L-fuzzy neighborhood system* $f(N)$ on V $f(N)$ as follows: for any $y \in V,\ H' \in L^V$, there exists $H \in L^U$ such that $f(H) = H'$,

$$f(N)(y)(H') = \begin{cases} \bigvee_{f(x)=y}\bigvee_{f(H)=H'}N(x)(H) & \text{if } f^{-1}(\{y\}) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

If $f^{-1}(\{y\}) \neq \varnothing$ but there is no $H \in L^U$ such that $f(H) = H'$, then $f(N)(y)(H') = 0$.

Definition 4.2. f is said to be a *consistent function with respect to* N, if for any $x,\ y \in U$, $f(x) = f(y)$ implies for any $z \in U$ and any $K \in L^U$ and $N(z)(K) > 0$ have $K(x) = K(y)$.

Remark 4.3. If f is consistent function with respect to N and there exists $H \in L^U$ and $N(z)(H) > 0$ such that $f(H) = H'$, then H is unique.

Remark 4.4. If f is consistent function with respect to N and $N(z)(K) > 0$, then $K = f^{-1}(f(K))$.

Similarly, there can be system consistent functions.

Definition 4.5. f is said to be a *system consistent function with respect to* N, if for any $x,\ y \in U$, $f(x) = f(y)$ implies $N(x) = N(y)$.

Lemma 4.6. *For any* $\mu,\ \nu \in L^U$ *and any* $\rho,\ \varrho \in L^V$*, The followings hold:*
(1) $\mu \le \nu \to f(\mu) \le f(\nu)$,
(2) $\rho \le \varrho \to f^{-1}(\rho) \le f^{-1}(\varrho)$,
(3) $\mu \le f^{-1}(f(\mu))$,
(4) $\rho \le f(f^{-1}(\rho))$.

The above definitions and lemma have the following consequences.

Lemma 4.7. *If* f *is a consistent function with respect to* N*, then for any* $z \in U,\ A \in L^U\,(N(z)(A) > 0)$ *and* $B \in L^U$*,* $S(A, B) \le S(f(A), f(B))$*.*

Proof. Suppose f is a consistent function with respect to N and let $z \in U,\ A \in L^U(N(z)(A) > 0)$ and $B \in L^U$. Then we have

$$\begin{aligned} S(f(A), f(B)) &= \bigwedge_{y\in V} (f(A)(y) \to f(B)(y)) \\ &= \bigwedge_{y\in V} \Big(\bigvee_{x'\in f^{-1}(\{y\})} A(x') \to \bigvee_{x''\in f^{-1}(\{y\})} B(x'')\Big) \\ &= \bigwedge_{y\in V} \Big(A(x'') \to \bigvee_{x''\in f^{-1}(\{y\})} B(x'')\Big) \\ &\ge \bigwedge_{y\in V} \bigvee_{x''\in f^{-1}(\{y\})} (A(x'') \to B(x'')) \\ &\ge \bigwedge_{x''\in f^{-1}(\{y\})} (A(x'') \to B(x'')) \\ &\ge \bigwedge_{x''\in U} (A(x'') \to B(x'')) \\ &= S(A, B). \end{aligned}$$

□

Lemma 4.8. *If* f *is a consistent function with respect to* N*, then for any* $z \in U,\ A \in L^U\,(N(z)(A) > 0)$ *and* $B \in L^U$*,* $N(A, B) = N(f(A), f(B))$*.*

Proof. Suppose f is a consistent function with respect to N and let $z \in U,\ A \in L^U(N(z)(A) > 0)$ and $B \in L^U$. Then we get

$$\begin{aligned} N(f(A), f(B)) &= \bigvee_{y\in V} (f(A)(y) \otimes f(B)(y)) \\ &= \bigvee_{y\in V} \Big(\bigvee_{x'\in f^{-1}(\{y\})} A(x') \otimes \bigvee_{x''\in f^{-1}(\{y\})} B(x'')\Big) \\ &= \bigvee_{y\in V} \Big(A(x'') \otimes \bigvee_{x''\in f^{-1}(\{y\})} B(x'')\Big) \\ &= \bigvee_{y\in V} \bigvee_{x''\in f^{-1}(\{y\})} (A(x'') \otimes B(x'')) \end{aligned}$$

$$= \bigvee_{x''\in U} (A(x'') \otimes B(x''))$$
$$= N(A, B).$$ □

Lemma 4.9. *For any $H,\ K \in L^V$, $S(H, K) \leq S(f^{-1}(H), f^{-1}(K))$. In particular, when f is surjective, we have $S(H, K) = S(f^{-1}(H), f^{-1}(K))$.*

Proof. Let $H,\ K \in L^V$. Then we have

$$\begin{aligned} S(f^{-1}(H), f^{-1}(K)) &= \bigwedge_{x\in U} (f^{-1}(H)(x) \to f^{-1}(K)(x) \\ &= \bigwedge_{x\in U} (H(f(x)) \to K(f(x))) \\ &= \bigwedge_{x\in U} (f^{-1}(H)(x) \to f^{-1}(K)(x) \\ &= \bigwedge_{y\in f(U)} (H(y) \to K(y)) \\ &\geq \bigwedge_{y\in V} (H(y) \to K(y)) \\ &= S(H, K). \end{aligned}$$

Obviously, the equal sign holds when f is surjective. □

Lemma 4.10. *For any $H,\ K \in L^V$, $N(H, K) \geq N(f^{-1}(H), f^{-1}(K))$. In particular, when f is surjective, we have $N(H, K) = N(f^{-1}(H), f^{-1}(K))$.*

Proof. Let $H,\ K \in L^V$. Then we get

$$\begin{aligned} N(f^{-1}(H), f^{-1}(K)) &= \bigvee_{x\in U} (f^{-1}(H)(x) \otimes f^{-1}(K)(x) \\ &= \bigvee_{x\in U} (H(f(x)) \otimes K(f(x))) \\ &= \bigvee_{x\in U} (f^{-1}(H)(x) \otimes f^{-1}(K)(x) \\ &= \bigvee_{y\in f(U)} (H(y) \otimes K(y)) \\ &\leq \bigvee_{y\in V} (H(y) \otimes K(y)) \\ &= N(H, K). \end{aligned}$$

Obviously, the equal sign holds when f is surjective. □

The above lemma has the following results.

Proposition 4.11. *If f is a consistent function with respect to N, then we have the following results: for any $\mu \in L^U$,*

(1) $f(\underline{N}(\mu)) \leq \underline{f(N)}(f(\mu))$,

(2) $\underline{N}(\mu) \leq f^{-1}(\underline{f(N)}(f(\mu)))$,

(3) $f(\overline{N}(\mu)) \geq \overline{f(N)}(f(\mu))$.

(4) $\overline{N}(\mu) \geq f^{-1}(\overline{f(N)}(f(\mu)))$.

Proof. (1) Let $\mu \in L^U$ and let $y \in V$. Then we have

$$\begin{aligned} f(\underline{N}(\mu))(y) &= \bigvee_{x\in f^{-1}(\{y\})} \underline{N}(\mu)(x) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigvee_{K\in L^U} (N(x)(K) \otimes S(K, \mu)). \end{aligned}$$

Thus according to Lemma 4.7, we get: for any $f(x) = y$, $K \in L^U$ and $f(K) = K' \in L^V$,

$$\begin{aligned} N(x)(K)\otimes S(K,\mu) &\leq N(x)(K)\otimes S(f(K),f(\mu)) \\ &\leq \bigvee_{x\in f^{-1}(\{y\})} N(x)(K)\otimes S(f(K),f(\mu)) \\ &\leq \bigvee_{K'\in L^{f(U)}} (\bigvee_{x\in f^{-1}(\{y\})} N(x)(K)\otimes S(K',f(\mu))) \\ &\leq \bigvee_{K'\in L^{V}} (\bigvee_{x\in f^{-1}(\{y\})} N(x)(K)\otimes S(K',f(\mu))) \\ &= \underline{f(N)}(f(\mu))(y). \end{aligned}$$

So $f(\underline{N}(\mu))(y) \leq \underline{f(N)}(f(\mu))(y)$.

(2) It can be obtained from (1) and Lemma 4.6.

(3) Let $\mu \in L^U$ and let $y \in V$. Then we have

$$\begin{aligned} f(\overline{N}(\mu))(y) &= \bigvee_{x\in f^{-1}(\{y\})} \overline{N}(\mu)(x) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigwedge_{K\in L^U, N(x)(K)>0} (N(x)(K)\to N(K,\mu)) \\ &\geq \bigwedge_{K\in L^U, N(x)(K)>0} (N(x)(K)\to N(K,\mu)). \end{aligned}$$

On the other hand, by Lemma 4.8, we get: for any $K \in L^U$ and $f(K) = K' \in L^V$,

$$\begin{aligned} N(x)(K)\to N(K,\mu) &= N(x)(K)\to N(f(K),f(\mu)) \\ &= N(x)(K)\to N(K',f(\mu)) \\ &\geq \bigwedge_{x\in f^{-1}(\{y\})} (N(x)(K)\to N(K',f(\mu))) \\ &\geq \bigwedge_{K'\in F(f(U))} (\bigwedge_{x\in f^{-1}(\{y\})} (N(x)(K)\to N(K',f(\mu)))) \\ &= \bigwedge_{K'\in F(f(U))} ((\bigvee_{x\in f^{-1}(\{y\})} N(x)(K))\to N(K',f(\mu))) \\ &= \bigwedge_{K'\in F(f(U))} (f(N)(y)(K')\to N(K',f(\mu))) \\ &\geq \bigwedge_{K'\in L^V} (f(N)(y)(K')\to N(K',f(\mu))) \\ &= \overline{f(N)}(f(\mu))(y). \end{aligned}$$

Thus $f(\overline{N}(\mu))(y) \geq \overline{f(N)}(f(\mu))(y)$.

(4) Let $\mu \in L^U$ and let $x \in U$. Then we have

$$\begin{aligned} &f^{-1}(\overline{f(N)}(f(\mu)))(x) \\ &= \overline{f(N)}(f(\mu))(f(x)) \\ &= \bigwedge_{K'\in L^V} (f(N)(f(x))(K')\to N(K',f(\mu))) \\ &= \bigwedge_{K'\in L^V, f(K)=K'} ((\bigvee_{x'\in f^{-1}(\{f(x)\})} N(x')(K))\to N(f(K),f(\mu))) \\ &= \bigwedge_{K'\in L^V, f(K)=K'} \bigwedge_{x'\in f^{-1}(\{f(x)\})} (N(x')(K)\to N(f(K),f(\mu))), \end{aligned}$$

$\overline{N}(\mu)(x) = \bigwedge_{K\in L^U, N(x)(K)>0} (N(x)(K)\to N(K,\mu))$. Thus according to Lemma 4.8, we get: for any $K \in L^U$ and $f(K) = K'$,

$$\begin{aligned} &N(x)(K)\to N(K,\mu) \\ &= N(x)(K)\to N(f(K),f(\mu)) \\ &\geq \bigwedge_{x'\in f^{-1}(\{f(x)\})} (N(x')(K)\to N(f(K),f(\mu))) \\ &\geq \bigwedge_{K'\in L^V, f(K)=K'} \bigwedge_{x'\in f^{-1}(\{f(x)\})} (N(x')(K)\to N(f(K),f(\mu))) \end{aligned}$$

$$= f^{-1}(\overline{f(N)}(f(\mu)))(x).$$

So $\overline{N}(\mu)(x) \geq f^{-1}(\overline{f(N)}(f(\mu)))(x)$. □

Proposition 4.12. *If f is a consistent function and a system consistent function with respect to N, then the followings hold: for any $\mu \in L^U$,*

(1) $f(\overline{N}(\mu)) = \overline{f(N)}(f(\mu))$,

(2) $\overline{N}(\mu) = f^{-1}(\overline{f(N)}(f(\mu)))$.

Proof. (1) If $y \notin f(U)$, The conclusion is obvious. If $y \in f(U)$, then we get

$$\begin{aligned} f(\overline{N}(\mu))(y) &= \bigvee_{x\in f^{-1}(\{y\})} \overline{N}(\mu)(x) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigwedge_{K\in L^U, N(x)(K)>0} (N(x)(K) \to N(K,\mu)) \\ &= \bigwedge_{K\in L^U, N(x)(K)>0} (N(x)(K) \to N(K,\mu)), \end{aligned}$$

$$\begin{aligned} \overline{f(N)}(f(\mu))(y) &= \bigwedge_{K'\in L^V} (f(N)(y)(K') \to N(K', f(\mu))) \\ &= \bigwedge_{K'\in L^V, f(K)=K'} ((\bigvee_{x\in f^{-1}(\{y\})} N(x)(K)) \to N(f(K), f(\mu))) \\ &= \bigwedge_{K'\in L^V, f(K)=K'} (N(x)(K) \to N(K,\mu)). \end{aligned}$$

Thus $f(\overline{N}(\mu))(y) = \overline{f(N)}(f(\mu))(y)$.

(2) The proof is similar to (1). □

Proposition 4.13. *If f is a consistent function with respect to N and surjective, then we have: for any $\rho \in L^V$,*

(1) $f(\underline{N}(f^{-1}(\rho))) = \underline{f(N)}(\rho)$,

(2) $\underline{N}(f^{-1}(\rho)) = f^{-1}(\underline{f(N)}(\rho))$,

(3) $f(\overline{N}(f^{-1}(\rho))) \geq \overline{f(N)}(\rho)$,

(4) $\overline{N}(f^{-1}(\rho)) \geq f^{-1}(\overline{f(N)}(\rho))$.

Proof. (1) Let $\rho \in L^V$ and let $y \in V$. Then we have

$$\begin{aligned} f(\underline{N}(f^{-1}(\rho)))(y) &= \bigvee_{x\in f^{-1}(\{y\})} \underline{N}(f^{-1}(\rho))(x) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigvee_{K\in L^U} (N(x)(K) \otimes S(K, f^{-1}(\rho)))\ \underline{f(N)}(\rho)(y) \\ &= \bigvee_{K'\in L^V} (f(N)(y)(K') \otimes S(K',\rho)) \\ &= \bigvee_{K'\in L^V, f(K)=K'} ((\bigvee_{x\in f^{-1}(\{y\})} (N(x)(K)) \otimes S(K',\rho))) \\ &= \bigvee_{K'\in L^V, f(K)=K'} \bigvee_{x\in f^{-1}(\{y\})} (N(x)(K) \otimes S(K',\rho)). \end{aligned}$$

On the other hand, by Lemma 4.9, we get: for any $f(x) = y$, $K \in L^U$ and $f(K) = K'$,

$$\begin{aligned} N(x)(K) \otimes S(K, f^{-1}(\rho)) &= N(x)(K) \otimes S(f^{-1}(K'), f^{-1}(\rho))) \\ &= N(x)(K) \otimes S(K',\rho)). \end{aligned}$$

Thus $f(\underline{N}(f^{-1}(\rho)))(y) = \underline{f(N)}(\rho)(y)$.

(2) It can be obtained from (1) and Lemma 4.6.

(3) It can be obtained from (4) and Lemma 4.6.

(4) Let $\rho \in L^V$ and let $y \in V$. Then we have

$$\begin{aligned} f^{-1}(\overline{f(N)}(\rho))(x) &= \overline{f(N)}(\rho)(f(x)) \\ &= \bigwedge_{K'\in P(V)} (f(N)(f(x))(K') \to N(K',\rho)) \\ &= \bigwedge_{K'\in P(V), f(K)=K'} ((\bigvee_{x'\in f^{-1}(\{f(x)\})} N(x')(K)) \to N(K',\rho)) \\ &= \bigwedge_{K'\in P(V), f(K)=K'} \bigwedge_{x'\in f^{-1}(\{f(x)\})} (N(x')(K) \to N(K',\rho)), \end{aligned}$$

$$\overline{N}(f^{-1}(\rho))(x) = \bigwedge_{K\in P(U)} (N(x)(K) \to N(K, f^{-1}(\rho))).$$

On the other hand, by Lemma 4.10, we get: for any $f(x) = y$, $K \in L^U$ and $f(K) = K'$,

$$\begin{aligned} & N(x)(K) \to N(K, f^{-1}(\rho)) \\ &= N(x)(K) \to N(f^{-1}(K'), f^{-1}(\rho)) \\ &= N(x)(K) \to N(K',\rho) \\ &\geq \bigwedge_{x'\in f^{-1}(\{f(x)\})} (N(x')(K) \to N(K',\rho)) \\ &\geq \bigwedge_{K'\in P(V), f(K)=K'} \bigwedge_{x'\in f^{-1}(\{f(x)\})} (N(x')(K) \to N(K',\rho)) \\ &= f^{-1}(\overline{f(N)}(\rho))(x). \end{aligned}$$

Thus $\overline{N}(f^{-1}(\rho))(x) \geq f^{-1}(\overline{f(N)}(\rho))(x)$. □

Let f be the mapping from U to V and let N be the L-fuzzy neighborhood system on V. Then we can define the L-neighborhood system $f^{-1}(N)$ on U for any $x \in U,\ H \in L^U$,

$$f^{-1}(N)(x)(H) = N(f(x))(f(H)).$$

Proposition 4.14. *Let N be an L-fuzzy neighborhood system on V and let f be a mapping from U to V. If f is consistent with $f^{-1}(N)$ and f is surjective, then the followings hold: for any $\rho \in L^V$,*

(1) $f(\underline{f^{-1}(N)}(f^{-1}(\rho))) = \underline{N}(\rho)$,
(2) $\underline{f^{-1}(N)}(f^{-1}(\rho)) = f^{-1}(\underline{N}(\rho))$,
(3) $f^{-1}(\overline{N}(\rho)) = \overline{f^{-1}(N)}(f^{-1}(\rho))$,
(4) $\overline{N}(\rho) = f(\overline{f^{-1}(N)}(f^{-1}(\rho)))$.

Proof. Suppose f is consistent with $f^{-1}(N)$ and f is surjective, and let $\rho \in L^V$.

(1) Let $y \in V$. Then by Lemma 4.9, we have

$$\underline{N}(\rho)(y) = \bigvee_{K'\in L^V} (N(y)(K') \otimes S(K',\rho)),$$

$$\begin{aligned} f(\underline{f^{-1}(N)}(f^{-1}(\rho)))(y) &= \bigvee_{x\in f^{-1}(\{y\})} \underline{f^{-1}(N)}(f^{-1}(\rho))(x) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigvee_{K\in L^U} (f^{-1}(N)(x)(K) \otimes S(K, f^{-1}(\rho))) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigvee_{K\in L^U} (N(f(x))(f(K)) \otimes S(K, f^{-1}(\rho))) \\ &= \bigvee_{K\in L^U} (N(y)(f(K)) \otimes S(K, f^{-1}(\rho))) \\ &= \bigvee_{f(K)=K'\in L^V} (N(y)(K') \otimes S(f^{-1}(K'), f^{-1}(\rho))) \\ &= \bigvee_{f(K)=K'\in L^V} (N(y)(K') \otimes S(K',\rho)) \end{aligned}$$

$$= \bigvee_{K'\in L^V} (N(y)(K') \otimes S(K',\rho))$$
$$= \underline{N}(\rho)(y).$$

Thus $f(\underline{f^{-1}(N)}(f^{-1}(\rho)))(y) = \underline{N}(\rho)(y)$.

(2) It can be obtained from (1) and Lemma 4.6.

(3) Let $x \in U$. Then by Lemma 4.10, we have

$$\begin{aligned} f^{-1}(\overline{N}(\rho))(x) &= \overline{N}(\rho)(f(x)) \\ &= \bigwedge_{K'\in L^V} (N(f(x))(K') \to N(K',\rho)), \end{aligned}$$

$$\begin{aligned} \overline{f^{-1}(N)}(f^{-1}(\rho))(x) &= \bigwedge_{K\in L^U} (f^{-1}(x)(K) \to N(K, f^{-1}(\rho))) \\ &= \bigwedge_{K\in L^U} (N(f(x))(f(K))) \to N(K, f^{-1}(\rho))) \\ &= \bigwedge_{f(K)=K'\in L^V} (N(f(x))(f(K))) \to N(f^{-1}(K'), f^{-1}(\rho))) \\ &= \bigwedge_{f(K)=K'\in L^V} (N(f(x))(K') \to N(K',\rho))\ ] \\ &= \bigwedge_{K'\in L^V} (N(f(x))(K') \to N(K',\rho)) \\ &= f^{-1}(\overline{N}(\rho))(x). \end{aligned}$$

Thus $\overline{f^{-1}(N)}(f^{-1}(\rho))(x) = f^{-1}(\overline{N}(\rho))(x)$.

(4) The proof is similar to (1). □

Proposition 4.15. *Let N be L-fuzzy neighborhood system on V and let f be a mapping from U to V. If f is consistent function with respect to $f^{-1}(N)$, then the followings hold: for any $\mu \in L^U$,*

(1) $f(\underline{f^{-1}(N)}(\mu)) \leq \underline{N}(f(\mu))$,

(2) $\underline{f^{-1}(N)}(\mu) \leq f^{-1}(\underline{N}(f(\mu)))$,

(3) $f(\overline{f^{-1}(N)}(\mu)) \geq \overline{N}(f(\mu))$,

(4) $\overline{f^{-1}(N)}(\mu) \leq f^{-1}(\overline{N}(f(\mu)))$.

Proof. Suppose f is consistent function with respect to $f^{-1}(N)$.

(1) Let $y \in V$ and let $\mu \in L^U$. Then by Lemma 4.7, we get

$$\underline{N}(f(\mu))(y) = \bigvee_{K'\in L^V} (N(y)(K') \otimes S(K', f(\mu))),$$

$$\begin{aligned} f(\underline{f^{-1}(N)}(\mu))(y) &= \bigvee_{x\in f^{-1}(\{y\})} \underline{f^{-1}(N)}(\mu)(x) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigvee_{K\in L^U} (f^{-1}(N)(x)(K) \otimes S(K,\mu)) \\ &= \bigvee_{x\in f^{-1}(\{y\})} \bigvee_{K\in L^U} (N(f(x))(f(K)) \otimes S(K,\mu)) \\ &\leq \bigvee_{f(K)=K'\in L^V} (N(y)(K') \otimes S(K', f(\mu))) \\ &\leq \bigvee_{K'\in L^V} (N(y)(K') \otimes S(K', f(\mu))) \\ &= \underline{N}(f(\mu))(y). \end{aligned}$$

Thus $f(\underline{f^{-1}(N)}(\mu))(y) \leq \underline{N}(f(\mu))(y)$.

(2) It can be obtained from (1) and Lemma 4.6.

(3) Let $y \in V$ and let $\mu \in L^U$. Then by Lemma 4.8, we get

$$\overline{N}(f(\mu))(y) = \bigwedge_{K' \in L^V} (N(y)(K') \to N(K', f(\mu))),$$

$$\begin{aligned} f(\overline{f^{-1}(N)}(\mu))(y) &= \bigvee_{x \in f^{-1}(\{y\})} \overline{f^{-1}(N)}(\mu)(x) \\ &= \bigvee_{x \in f^{-1}(\{y\})} \bigwedge_{K \in L^U} (f^{-1}(N)(x)(K) \to N(K, \mu)) \\ &= \bigvee_{x \in f^{-1}(\{y\})} \bigwedge_{K \in L^U} (N(f(x))(f(K)) \to N(K, \mu)) \\ &= \bigwedge_{K \in L^U} (N(y)(f(K)) \to N(K, \mu)) \\ &= \bigwedge_{f(K) = K' \in F(f(U))} (N(y)(K') \to N(K', f(\mu))) \\ &\geq \bigwedge_{K' \in L^V} (N(y)(K') \to N(K', f(\mu))) \\ &= \overline{N}(f(\mu))(y). \end{aligned}$$

Thus $f(\overline{f^{-1}(N)}(\mu))(y) \geq \overline{N}(f(\mu))(y)$.

(4) Let $x \in U$ and let $\mu \in L^U$. Then by Lemma 4.8, we get

$$\begin{aligned} f^{-1}(\overline{N}(f(\mu)))(x) &= \overline{N}(f(\mu))(f(x)) \\ &= \bigvee_{K' \in L^V} (N(f(x))(K') \to N(K', f(\mu))) \\ &\geq \bigvee_{f(K) = K', K \in L^U} (N(f(x))(K') \to N(K', f(\mu))) \\ &= \bigvee_{K \in L^U} (N(f(x))(f(K)) \to N(f(K), f(\mu))) \\ &= \bigvee_{K \in L^U} (N(f(x))(f(K)) \to N(K, \mu)) \\ &= \bigvee_{K \in L^U} (f^{-1}(N)(x)(K) \to N(K, \mu)) \\ &= \overline{f^{-1}(N)}(\mu). \end{aligned}$$

□

From the above proposition, we have

Proposition 4.16. *Let N be L-fuzzy neighborhood system on V and let f be a mapping from U to V. If f is consistent function with respect to $f^{-1}(N)$ and surjective, then the followings hold: for any $\mu \in L^U$,*

(1′) $f(\overline{f^{-1}(N)}(\mu)) = \overline{N}(f(\mu))$,

(4′) $\overline{f^{-1}(N)}(\mu) = f^{-1}(\overline{N}(f(\mu)))$.

5. Conclusion

This paper investigates how the interior and closure operators of L-fuzzy neighborhood systems preserve their structure from two perspectives: different L-fuzzy neighborhood systems on the single universe and L-fuzzy neighborhood systems on two universes. First, for neighborhood systems on the single universe, the paper defines neighborhood system reduction and gives several conditions for different neighborhood systems to generate the same interior and closure operators or their exact bounds. This is different from previous rough set models. Second, for L-fuzzy neighborhood systems on two universes, the paper extends the consistent function to the L-fuzzy neighborhood system and the preservation of the structure of the interior and closure operators of the L-fuzzy neighborhood system on two universes is achieved with the help of consistent functions and general mappings. The conclusion of this article is meaningful for the study of topology structures in neighborhood systems.

This article does not give the reduction algorithm of the neighborhood system or the property preservation of the neighborhood system itself due to space limitations. In the future, we plan to do the following: First, we will present specific neighborhood system reduction algorithms and their applications. Second, we will study how properties such as seriality, reflexivity, transitivity, and symmetry of neighborhood systems are preserved in single and two universes.Third, more topological properties on neighborhood systems will be studied.

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Yinshan Liu (yinshanliu@foxmail.com)
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China

Hao Wang (wanghao@my.swjtu.edu.cn)
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China

Keyun Qin (Keyunqin@263.net)
School of Mathematics, Southwest Jiaotong University, Chengdu 611756, China