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Abstract. In this paper, we focus on ideals and prime ideals of CKL-algebras and applications of prime ideals of CKL-algebras. Firstly, we prove that any self-distributive L-algebra is a CKL-algebra. Conversely, we give an example of CKL-algebras that is not a self-distributive L-algebra. Furthermore, we give a generation formula of ideals on CKL-algebras. Secondly, we give some equivalent descriptions of prime ideals and its properties on CKL-algebras. We mainly prove that maximal ideals are prime ideals on CKL-algebras. Next, we give a counterexample to show that commutative L-algebras may be not residuated lattices, much less MV-algebras. The results show that commutative L-algebras are a true promotion of MV-algebras. Therefore, we study some properties of commutative L-algebras.

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1. Introduction

The concept of L-algebras was introduced by Rump in [1], it is a quantum structure that is closely related to non-classical logical algebras and quantum Yang-Baxter equation solutions. In 2008, Rump [2] pointed that Hilbert algebras, (pseudo) MV-algebras, BCK-algebras, and ℓ-group negative cones are all special cases of L-algebras. Therefore, L-algebras are fundamental algebraic structures. As a result, the research on L-algebras has become a hot topic in the field of logic algebra in recent years, and it is attracting the attention of many domestic and foreign scholars [3, 4]. For example, in 2019, Yali Wu ect. [5] proved that every lattice-ordered effect algebras generates an L-algebra with the same orthogonal complement. In 2020, Yali Wu ect. [4] studied the relationship between L-algebras and Boolean algebras; Rump and Zhang X introduced the relationship between L-effect algebras [6] (as a class of L-algebras) and quantum sets; Jin Wang ect. studied the connection between L-effect algebras and basic algebras [7].

As an important concept in the field of non-classical algebraic logic research, filters theory plays a significant role in studying the completeness of various logical systems [3, 5, 7, 8, 9]. Filters are also called a deduction system. From the logical point of view, the filters correspond to some sets of formulas. Filters are also particularly interesting because they are closely related to congruence relations. In 2008, Rump proposed the definition of ideals on L-algebras, which ideals are called filters in the general logic algebras, and discussed the relationship between ideals and congruences [2]. In 2021, Ciungu study some classes of L-algebras and give characterizations of commutative L-algebras and CKL-algebras. He also present L-algebras arising from other structures, such as BCK-algebras, pseudo MV-algebras, pseudo BL-algebras, pseudo-hoops and Hilbert algebras [3]. In 2022, Rump proved that the ideal lattice of L-algebras is distributive [10]. He primarily studied the properties of prime ideals and quotient algebras induced by prime ideals, and gave the definition of topological space of L-algebras. Ciungu has proved that any commutative L-algebra is a BCK-algebra and any CKL-algebra is a BCK-algebra. We prove that any BCK-algebra is a CKL-algebra. Therefore any commutative L-algebra is a CKL-algebra. We know CKL-algebras may not commutative. Ciungu proved that CKL-algebras satisfying the self-similarity is commutative. Therefore, in this paper we consider the study of ideals and prime ideals on CKL-algebras and its applications. By studying ideals on CKL-algebras, we prove that maximal ideals are prime ideals on CKL-algebras. Next, we introduce special L-algebras (commutative L-algebras), which is a true extension of MV-algebras. We study of ideals and prime ideals on commutative L-algebras and its applications. Furthermore, the paper we study prime ideals and spectrum space on commutative L-algebras. We also study the divisibility and some properties of topological space on commutative L-algebras. By studying the properties of topological space on commutative L-algebras, we prove that Spec(S) is a T_1-space if and only if every prime ideal is maximal.

This paper is organized as follows. In Section 3, we first give the generation formula of ideals on CKL-algebras. Then we give a condition in which every L-algebra is a join semi-lattice. In Section 4, we mainly discuss some properties of prime ideals and maximal ideals on CKL-algebras. In addition, the equivalent condition in which CKL-algebras becomes a chain is given. Similarly, we discuss some properties of prime ideals and maximal ideals on commutative L-algebras. In Section 5, some topological properties of the space of all prime ideals are studied. Moreover, an equivalent condition is derived for every prime ideal of commutative L-algebras to become a maximal ideal.

2. Preliminaries

In this section, we give some definitions and results for L-algebras [2].

Definition 2.1 ([2])**.** Let S be a set with a binary operation $\to$. An element $1 \in S$ is said to be a *logical unit*, if for all $s \in S$,

$$s \to s = s \to 1 = 1, 1 \to s = s. \tag{2.1}$$

From Definition 2.1, logical unit 1 is unique.

If $(S, \to)$ with a logical unit 1 satisfies

$$(s \to t) \to (s \to w) = (t \to s) \to (t \to w) \tag{2.2}$$

and

$$s \to t = t \to s = 1 \Rightarrow s = t \tag{2.3}$$

for all $s, t, w \in S$, then $(S, \to, 1)$ is called an *L-algebra*.

If an L-algebra $(S, \to, 1)$ satisfies

$$s \to (t \to s) = 1, \tag{2.4}$$

then it is said to be a *KL-algebra*.

A *CKL-algebra* is an L-algebra $(S, \to, 1)$ such that

$$(s \to (t \to w)) \to (t \to (s \to w)) = 1. \tag{2.5}$$

It follows that in any L-algebra S satisfying condition (2.5), we have

$$s \to (t \to w) = t \to (s \to w) \tag{2.6}$$

for all $s, t, w \in S$.

Given an L-algebra $(S, \to, 1)$, a binary relation "$\leq$" is defined by: for any $s,\ t \in S$,

$$s \leq t \text{ iff } s \to t = 1.$$

The logical unit 1 is the greatest element of an L-algebra under partial order relation.

Example 2.2 ([11])**.** Let $S = \{0, e, f, g, h, 1\}$ be a lattice with Hasse diagram as Figure 1. Its implication operation $\to$ is defined as Table 1.

Table 1.

$\rightarrow$	0	e	f	g	h	1
0	1	1	1	1	1	1
e	h	1	1	h	1	1
f	g	h	1	g	h	1
g	f	f	f	1	1	1
h	e	f	f	h	1	1
1	0	e	f	g	h	1

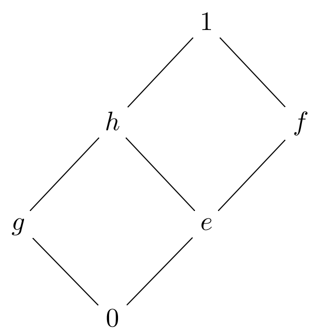


FIGURE 1. Hasse Diagram of S

Then $(S, \rightarrow, 1)$ is an L-algebra.

Proposition 2.3 ([2]). *Let S be an L-algebra. For all $s, t, w \in S$, we have*

(1) $s \leq t \Rightarrow w \rightarrow s \leq w \rightarrow t$,
(2) $s = t \Leftrightarrow s \rightarrow w = t \rightarrow w$,
(3) *if* $s, t \leq w$ *such that* $w \rightarrow s = w \rightarrow t$, *then* $s = t$,
(4) *if S satisfies condition* (2.5), *then*

$$s \rightarrow ((s \rightarrow t) \rightarrow t) = 1, \tag{2.7}$$

(5) *if S is a KL-algebra, then the following holds,*

$$s \leq t \Rightarrow t \rightarrow w \leq s \rightarrow w, \tag{2.8}$$

the converse is also true and condition (2.8) *implies* (2.4),
(6) $((s \rightarrow t) \rightarrow t) \rightarrow ((s \rightarrow t) \rightarrow w) = t \rightarrow w$,
(7) *if S is a KL-algebra satisfying condition* (2.7), *then*

$$((s \rightarrow t) \rightarrow t) \rightarrow t = s \rightarrow t.$$

Definition 2.4 ([12]). Let $(S, \rightarrow, 1)$ be an L-algebra. If for all $s, t, w \in S$, the following holds:

$$s \rightarrow (t \rightarrow w) = (s \rightarrow t) \rightarrow (s \rightarrow w),$$

then S is said to be *self-distributive*.

Obviously, any self-distributive L-algebra is a CKL-algebra. Conversely, it is not valid. In Example 2.2, we find that it is a CKL-algebra, but it is not a self-distributive L-algebra. Indeed, $h \rightarrow (e \rightarrow 0) = h \rightarrow h = 1 = e \rightarrow e = e \rightarrow (h \rightarrow 0)$, while $(h \rightarrow e) \rightarrow (h \rightarrow 0) = f \rightarrow e = h$.

Example 2.5. Let $S = \{1, e, f, g\}$ be an L-algebra with Hasse diagram as Figure 2. Its implication operatation $\to$ is defined as Table 2.

TABLE 2.

$\to$	1	e	f	g
1	1	e	f	g
e	1	1	f	g
f	1	e	1	g
g	1	e	f	1

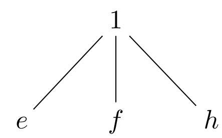


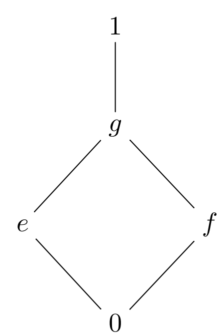
FIGURE 2. Hasse Diagram of S

It is easy to check that S is a commutative L-algebra and S is a join semi-lattice.

Example 2.6 ([5])**.** Let $S = \{0, e, f, g, 1\}$ be a lattice with Hasse diagram as Figure 3. Its implication operation $\to$ is defined as Table 3.

TABLE 3.

$\to$	0	e	f	g	1
0	1	1	1	1	1
e	f	1	f	1	1
f	e	e	1	1	1
g	0	e	f	1	1
1	0	e	f	g	1



We can check that S is a self-distributive L-algebra.

FIGURE 3. Hasse Diagram of S

Definition 2.7 ([10]). Let S be an L-algebra. If S satisfies the following conditions

$$t \leq s \rightarrow t \tag{2.9}$$

and

$$(s \rightarrow t) \rightarrow t = (t \rightarrow s) \rightarrow s, \tag{2.10}$$

then S is said to be *commutative*.

Definition 2.8 ([2]). Let $(S, \rightarrow, 1)$ be an L-algebra. We call $J \subseteq S$ an *ideal* of S, if it satisfies the following conditions for all $s, t \in S$,
(I0) $1 \in J$,
(I1) $s,\ s \rightarrow t \in J \Rightarrow t \in J$,
(I2) $s \in J \Rightarrow (s \rightarrow t) \rightarrow t \in J$,
(I3) $s \in J \Rightarrow t \rightarrow s,\ t \rightarrow (s \rightarrow t) \in J$.

Denote by $Id(S)$ the set of all ideals of S. $Id(S)^* := Id(S) - S$. Every morphism on L-algebras $h : S \rightarrow T$ is said to be a *homomorphism*, if $h(s \rightarrow t) = h(s) \rightarrow h(t)$ for all $s, t \in S$. Determines an ideal $kerh := \{s \in S | h(s) = 1\}$ of S, the kernel of h, and $S/kerh$ is isomorphic to the image $Imh := h(s)$ of h.

Example 2.9. In Example 2.2, $J_1 = \{1\}$, $J_2 = \{h, 1\}$, $J_3 = \{e, g, 1\}$ and $J_4 = S$ are ideals on S. In Example 2.6, $J_1 = \{1\}$, $J_2 = \{g, 1\}$, $J_3 = \{e, g, 1\}$, $J_4 = \{f, g, 1\}$ and $J_5 = S$ are ideals on S.

Remark 2.10. Let $(S, \rightarrow, 1)$ be an L-algebra and let $J \in Id(S)$. If S satisfies condition (2.6), then (I2) and (I3) can be omitted.

Theorem 2.11 ([12]). *The lattice of ideals of an L-algebra is distributive.*

In the following, we denote $J \rightarrow K := max\{H | J \cap H \subseteq K\}$.

Definition 2.12 ([12]). Let S be an L-algebra. A proper ideal B is said to be *prime* on S, if for every ideal J either $J \subseteq B$ or $J \rightarrow B \subseteq B$.

Denote by $PId(S)$ the set of all prime ideals on S.

Example 2.13. In Example 2.2, $J_1 = \{1\}$ is a prime ideal on S. In Example 2.6, $J_1 = \{1\}$, $J_3 = \{e, g, 1\}$ and $J_4 = \{f, g, 1\}$ are prime ideals on S.

Recall that a BCK-algebra is an algebra $(S, \rightarrow, 1)$ of type (2, 0) satisfying the following axioms for every $s, t, w \in S$,
(i) $(s \rightarrow t) \rightarrow ((t \rightarrow w) \rightarrow (s \rightarrow w)) = 1$,
(ii) $s \rightarrow ((s \rightarrow w) \rightarrow w) = 1$,
(iii) $s \rightarrow s = 1$,
(iv) $s \rightarrow t = 1$ and $t \rightarrow s = 1$ imply $s = t$.
Any BCK-algebra S satisfies condition (2.6), that is, $s \rightarrow (t \rightarrow w) = t \rightarrow (s \rightarrow w)$ for any $s, t, w \in S$.

Theorem 2.14 ([3]). *Let S be a set. If S is a CKL-algebra, then S is a BCK-algebra.*

However, the inverse of Theorem 2.14 generally does not hold, and the following example can illustrate it.

Example 2.15. Let $S = \{1, e, f, g\}$ be a join semi-lattice with Hasse diagram as Figure 4. Its implication operation $\to$ be defined as Table 4.

TABLE 4.

$\to$	1	e	f	g
1	1	e	f	g
e	1	1	e	1
f	1	f	1	1
g	1	f	e	1

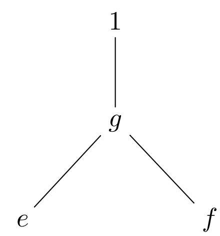


FIGURE 4. Hasse Diagram of S

It can be seen that $(S, \to, 1)$ is a BCK-algebra. But S is not a CKL-algebra, since $(c \to a) \to (c \to b) \neq (a \to c) \to (a \to b)$.

In the following, we study some properties of ideals and prime ideals on CKL-algebras.

3. IDEALS IN CKL-ALGEBRAS

The generation formula of ideals generally play a very important role in the study of prime ideals. Due to the complexity of the definitions for ideals in L-algebras, it is difficult to find the generated formula of ideals generated by a set. In this section, we first give the generation formula of ideals on CKL-algebras. Next, we introduce a binary operation $\vee$ on L-algebras, and the characterization such that L-algebras becomes a join semi-lattice with $\vee$ is given.

Let S be an L-algebra. An ideal J of S is said to be *proper*, if $J \neq S$. Next, we give generating formula of ideals on CKL-algebras.

Theorem 3.1. *Let S be a CKL-algebra and $\varnothing \neq D \subseteq S$. Then*

$$\langle D\rangle = \{s \in S | d_n \to (\cdots \to (d_1 \to s)\cdots) = 1 \text{ for some } d_1, d_2, \cdots, d_n \in D\}.$$

Proof. Assume that

$$E := \{s \in S | d_n \to (\cdots \to (d_1 \to s)\cdots) = 1 \text{ for some } d_1, \cdots, d_n \in D\}.$$

We first prove that $E \in Id(S)$. Obviously, $1 \in E$. Let $s \in E$ and $s \to t \in E$. Then there exist $d_1, \cdots, d_n \in D$ and $g_1, \cdots, g_m \in D$ such that

$$d_n \to (\cdots \to (d_1 \to s)...) = 1 \text{ and } g_m \to (\cdots \to (g_1 \to (s \to t))\cdots) = 1.$$

Thus we have

$$\begin{aligned} 1 &= g_m \to (\cdots \to (g_1 \to (s \to t))\cdots) \\ &= g_m \to (\cdots \to (s \to (g_1 \to t))\cdots) \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= s \to (g_m \to (\cdots \to (g_1 \to t)\cdots)). \end{aligned}$$

So $s \leq g_m \to (\cdots \to (g_1 \to t)\cdots)$. By Proposition 2.3 (1), we get

$$1 = d_n \to (\cdots \to (d_1 \to s)\cdots) \leq d_n \to (\cdots \to (d_1 \to (g_m \to (... \to (g_1 \to t)\cdots)))\cdots).$$

Since 1 is the greatest element of S, we have

$$d_n \to (\cdots \to (d_1 \to (g_m \to (\cdots \to (g_1 \to t)\cdots)))\cdots) = 1.$$

Hence $t \in E$. Therefore $E \in Id(S)$. Obviously, $D \subseteq E$.

We prove that E is the least ideal of S containing D. Let $D \subseteq J \in Id(S)$. Assume $s \in E$. Then there exist $d_1, \cdots, d_n \in D$ such that $d_n \to (\cdots \to (d_1 \to s)\cdots) = 1 \in J$. Since $D \subseteq J$ and $d_n \in D$, we have $d_n \in J$. Thus by (I1), we have

$$d_{n-1} \to (\cdots \to (d_1 \to s)\cdots) \in J.$$

Repeating the above argument, we conclude that $s \in J$. This shows that $E \subseteq J$. So E is the least ideal of S containing D. □

From the above theorem, we can get the following corollary.

Corollary 3.2. *Let S be a CKL-algebra and $J \in Id(S)$. Then $\langle J \cup \{x\}\rangle = \{s \in S | x^n \to s \in J$ for some $n \in \mathbb{N}\}$ and $\langle x\rangle = \{s \in S | x^n \to s = 1$ for some $n \in \mathbb{N}\}$.*

Proof. Let S be a CKL-algebra and $J \in Id(S)$. Then by Theorem 3.1, we have

$$\langle J \cup \{x\}\rangle = \{s \in S | j_n \to (\cdots \to (j_1 \to (x \to (... \to (x \to a)\cdots)))\cdots) = 1, j_i \in J \text{ and } i = 1, 2, \cdots, n \text{ for some } n \in \mathbb{N}\}.$$

Thus $j_n \leq j_{n-1} \to (\cdots \to (j_1 \to (x \to (\cdots \to (x \to a)\cdots)))\cdots)$. Since $j_n \in J \in Id(S)$. So we have $j_{n-1} \to (\cdots \to (j_1 \to (x \to (\cdots \to (x \to a)\cdots)))\cdots) \in J$. Repeating the above argument, we conclude that $x \to (\cdots \to (x \to a)\cdots) \in J$. Hence we can conclude that $\langle J \cup \{x\}\rangle = \{s \in S | x^n \to s \in J$ for some $n \in \mathbb{N}\}$. Obviously, by Theorem 3.1, we have

$$\langle x\rangle = \{s \in S | x \to (\cdots \to (x \to a)\cdots) = 1\} = \{s \in S | x^n \to s = 1 \text{ for some } n \in \mathbb{N}\}.$$

□

Let S be an L-algebra. We defined binary operation $\vee$ by $s \vee t = (s \to t) \to t$ for any $s, t \in S$. In order to inscribe prime ideal with $\vee$, we give some properties of the binary operation $\vee$ on L-algebras.

Proposition 3.3. *Let S be an L-algebra. Then the following hold:*

(1) *if S is a CKL-algebra, then $s \vee t$ is an upper bound of s and t for any $s, t \in S$,*

(2) *if S is a commutative L-algebra, then $s \vee t = t \vee s$ for any $s, t \in S$.*

Proof. (1) Suppose S is a CKL-algebra. Then for any $s, t, w \in S$, we get

$$s \to (t \to w) = t \to (s \to w).$$

Thus we have

$$\begin{aligned} s \to (s \vee t) &= s \to ((s \to t) \to t) \\ &= (s \to t) \to (s \to t) \\ &= 1, \end{aligned}$$

i.e., $s \leq s \vee t$.

$$\begin{aligned} t \to (s \vee t) &= t \to ((s \to t) \to t) \\ &= (s \to t) \to (t \to t) \\ &= (s \to t) \to 1 \\ &= 1, \end{aligned}$$

i.e., $t \leq s \vee t$.

(2) Suppose S is a commutative L-algebra. Then for any $s, t \in S$, we get $(s \to t) \to t = (t \to s) \to s$. Thus $s \vee t = t \vee s$. □

In what follows, an ordered set S is said to be a *join semi-lattice*, if every pair of elements in S has a supremum.

Example 3.4. Let $S = \{1, e, f, g\}$ be an L-algebra with Hasse diagram as Figure 5. Its implication operatation $\to$ be defined as Table 5.

Table 5.

$\to$	1	e	f	g
1	1	e	f	g
e	1	1	f	g
f	1	e	1	g
g	1	e	f	1

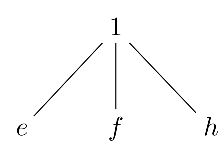


Figure 5. Hasse Diagram of S

It is easy to check that S is a commutative L-algebra and S is a join semi-lattice.

Remark 3.5. The Example 3.4 show that commutative L-algebras may not lattices. Then it is not residuated lattices and thus it is not a MV-algebra. It is meaningful to study commutative L-algebras.

The following proposition states the condition for L-algebras to be a join semi-lattice.

Proposition 3.6. *If $(S, \to, 1)$ is a commutative L-algebra, then it is a join semi-lattice.*

Proof. Suppose S is a commutative L-algebra and we denote binary operation $\vee$ by $s \vee t = (s \to t) \to t$ for any $s, t \in S$. Then by the hypothesis and Proposition 3.3, we have $s \vee t$ is an upper bound of s and t. Next, we shall show that $s \vee t$ is the least upper bound of s and t. Suppose that $s \leq w$ and $t \leq w$. Then $s \to w = t \to w = 1$. By the hypothesis, we get

$$w = 1 \to w = (s \to w) \to w = (w \to s) \to s \tag{3.1}$$

and

$$w = 1 \to w = (t \to w) \to w = (w \to t) \to t \tag{3.2}$$

Using (3.1) and (3.2), we have

$$w = (w \to s) \to s = (((w \to t) \to t) \to s) \to s. \tag{3.3}$$

Set $v := (w \to t) \to t$. Then from (3.3), $w = (v \to s) \to s$. Using (2.1) and (2.6), we have $1 = (w \to t) \to 1 = (w \to t) \to (t \to t) = t \to ((w \to t) \to t)$. Thus $t \leq (w \to t) \to t = v$. By Proposition 2.3 (5), $(t \to s) \to s \leq (v \to s) \to s = w$. So $t \vee s \leq w$. Since S is a commutative L-algebra, we get $s \vee t = t \vee s$. Hence we have $s \vee t \leq w$. Therefore we get $s \vee t$ is the least upper bound of s and t. □

Example 3.7. Let $S = \{1, e, f, g\}$ be an L-algebra with Hasse diagram as Figure 6. Its implication operation $\to$ be defined as Table 6. Define $s \vee t = (s \to t) \to t$ for all $s, t \in S$.

TABLE 6.

$\to$	1	e	f	g
1	1	e	f	g
e	1	1	e	e
f	1	1	1	e
g	1	1	e	1

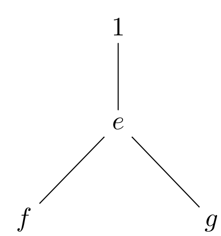


FIGURE 6. Hasse Diagram of S

Then S is a commutative L-algebra which is a join semi-lattice.

However, the inverse of Proposition 3.6 generally does not hold, and the following example can illustrate it.

Example 3.8. Let $S = \{1, e, f, g\}$ be a join semi-lattice with Hasse diagram as Figure 7. Its implication operation $\to$ be defined as Table 7. We define $s \vee t = (s \to t) \to t$ for any $s, t \in S$.

TABLE 7.

$\to$	1	e	f	g
1	1	e	f	g
e	1	1	f	g
f	1	1	1	g
g	1	1	f	1

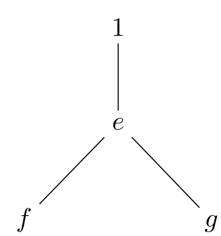


FIGURE 7. Hasse Diagram of S

It can be seen that $(S, \to, 1)$ is an L-algebra. From Figure 7, we can see $(S, \leq)$ is a join semi-lattice. But S is not commutative, since $e \vee f = 1 \neq e = f \vee e$.

4. PRIME IDEALS IN CKL-ALGEBRAS

In this section, some equivalent characterizations of prime ideals on CKL-algebras are given. Some properties of prime ideals and maximal ideals on CKL-algebras are studied. Moreover, we study some properties of prime ideals on commutative L-algebras.

Proposition 4.1. *Let S be an L-algebra. For any $J, K \in Id(S)$, we denote $M = \{H \in Id(S) | J \cap H \subseteq K\}$. Then the set M has the greatest element $m = \bigvee\{H \in Id(S) | J \cap H \subseteq K\}$.*

Proof. Let $J, K \in Id(S)$. Denote $M = \{H \in Id(S) | J \cap H \subseteq K\}$ and $m = \bigvee\{H \in Id(S) | J \cap H \subseteq K\}$. Clearly, for any $H \in \{H \in Id(S) | J \cap H \subseteq K\}$, we have $H \subseteq m$. In what following, we prove that $m \in M$, that is $m \cap J \subseteq K$. Note for any $H \in M$, we have $J \cap H \subseteq K$. Since lattice of ideals of an L-algebra S is distributive, then $m \cap J = \bigvee\{H \in Id(S) | J \cap H \subseteq K\} \cap J = \bigvee\{H \cap J \in Id(S) | J \cap H \subseteq K\} \subseteq K$. Thus $m \in M$. So $m = max\{H \in Id(S) | J \cap H \subseteq K\}$. □

Theorem 4.2. *Let S be an L-algebra and $B \in Id(S)^*$. Then the following assertions are equivalent:*
(1) $B \in PId(S)$,
(2) *for all $J, K \in Id(S)$, $J \cap K \subseteq B$ implies $J \subseteq B$ or $K \subseteq B$,*
(3) *for all $s, t \in B$, $\langle s\rangle \cap \langle t\rangle \subseteq B$ implies $s \in B$ or $t \in B$.*

Proof. (1)⇒ (2): For all $J, K \in Id(S)$ and $J \cap K \subseteq B$, the following two cases are discussed.

Case 1. Suppose $J \subseteq B$. Then (2) is true.

Case 2. Suppose $J \nsubseteq B$. Then $J \to B \subseteq B$. Since $J \to B = max\{H|J \cap H \subseteq B\}$ and $J \cap K \subseteq B$, $K \in max\{H|J \cap H \subseteq B\}$. Thus $K \subset J \to B \subseteq B$. So $K \subseteq B$.

(2)⇒ (1): For all $J \in Id(S)$, the following two cases are discussed.

Case 1. Suppose $J \subseteq B$. Then (1) is true.

Case 2. Suppose $J \nsubseteq B$. Then we need to prove that $J \to B \subseteq B$. Let $K = J \to B = max\{H|J \cap H \subseteq B\}$. Then $J \cap K \subseteq B$. Thus by (2), we have $J \subseteq B$ or $K \subseteq B$. Since $J \nsubseteq B$, $K = J \to B \subseteq B$. So (1) is true.

(2)⇒ (3): Suppose (2) holds. Let $s, t \in S$ be such that $\langle s\rangle \cap \langle t\rangle \subseteq B$. Then $\langle s\rangle \subseteq B$ or $\langle t\rangle \subseteq B$. Since $s \in \langle s\rangle$ and $t \in \langle t\rangle$, $s \in B$ or $t \in B$.

(3)⇒ (2): Let for all $J, K \in Id(S)$, $J \cap K \subseteq B$. We need to prove $J \subseteq B$ or $K \subseteq B$. Assume that $J \subseteq B$. Then (2) holds. Assume that $J \nsubseteq B$. Then there exists $s \in J \setminus B$. Thus $\langle s\rangle \subseteq J$. For any $t \in K$, $\langle t\rangle \subseteq K$. So $\langle s\rangle \cap \langle t\rangle \subseteq J \cap K \subseteq B$. By (3), $s \in B$ or $t \in B$. But $s \notin B$. Hence $t \in B$. Hence $K \subseteq B$. Therefore (2) is established. □

In the following we discussion of maximal ideals are prime ideals on CKL-algebras.

Theorem 4.3. *Let S be a CKL-algebra and $J \in Id(S)$. Then for any $e, f \in S$, we have*

$$\langle e\rangle \cap \langle f\rangle \subseteq J \Leftrightarrow \langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle = J.$$

Proof. (⇐): Suppose $\langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle = J$ for any $e, f \in S$. Since $e \in \langle J \cup \{e\}\rangle$ and $f \in \langle J \cup \{f\}\rangle$, we have $\langle e\rangle \subseteq \langle J \cup \{e\}\rangle$ and $\langle f\rangle \subseteq \langle J \cup \{f\}\rangle$. Then, it yields

$$\langle e\rangle \cap \langle f\rangle \subseteq \langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle = J.$$

Thus $\langle e\rangle \cap \langle f\rangle \subseteq J$.

(⇒): Suppose $\langle e\rangle \cap \langle f\rangle \subseteq J$. Obviously, $J \subseteq \langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle$. Let $s \in \langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle$. Then $s \in \langle J \cup \{e\}\rangle$ and $s \in \langle J \cup \{f\}\rangle$. Since $J \in Id(S)$, there exist $p, q \in \mathbb{N}$ such that $e^p \to s \in J$ and $f^q \to s \in J$ by Corollary 3.2. Thus there exist $p_1, p_2 \in J$ such that $e^p \to s = p_1$ and $f^q \to s = p_2$. So we have

$$e^p \to (p_1 \to s) = p_1 \to (e^p \to s) = p_1 \to p_1 = 1,$$

i.e., $p_1 \to s \in \langle e\rangle$. Similarly, $p_2 \to s \in \langle f\rangle$. Since

$$p_1 \to s \leq p_2 \to (p_1 \to s) = p_1 \to (p_2 \to s) \text{ and } p_2 \to s \leq p_1 \to (p_2 \to s),$$

$$p_1 \to (p_2 \to s) \in \langle e\rangle \text{ and } p_1 \to (p_2 \to s) \in \langle f\rangle.$$

Hence $p_1 \to (p_2 \to s) \in \langle e\rangle \cap \langle f\rangle \subseteq J$. Since $p_1, p_2 \in J$ and $J \in Id(S)$, we get $s \in J$. Hence $\langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle \subseteq J$. Therefore $\langle J \cup \{e\}\rangle \cap \langle J \cup \{f\}\rangle = J$. □

Definition 4.4. Let S be an L-algebra. $G \in Id(S)$ is said to be *maximal* on S, if $N \in Id(S)$ such that $G \subset N \subset S$, then $N = G$ or $N = S$.

Example 4.5. In Example 2.2, $J_3 = \{e, g, 1\}$ is a maximal ideal on S.

Denote by $MId(S)$ the set of all maximal ideals on S.

Theorem 4.6. *Let S be a CKL-algebra. Then every maximal ideal is prime ideal.*

Proof. Let S be a CKL-algebra and $G \in MId(S)$. By Theorem 4.2, assume that $\langle s\rangle \cap \langle t\rangle \subseteq G$ for $s, t \in S$. Then by Theorem 4.3,

$$\langle G \cup \{s\}\rangle \cap \langle G \cup \{t\}\rangle = G.$$

Let $s \notin G$ and $t \notin G$. By Definition 4.4, we get $\langle G \cup \{s\}\rangle = S$ and $\langle G \cup \{t\}\rangle = S$. Thus $\langle G \cup \{s\}\rangle \cap \langle G \cup \{t\}\rangle = S$. Theorem 4.3 and $G \neq S$ yield that $\langle s\rangle \cap \langle t\rangle \nsubseteq G$, which is a contradiction. So $s \in G$ or $t \in G$. Hence we have $G \in PId(S)$. □

Proposition 4.7. *Let S and T be two CKL-algebras and $h : S \to T$ a homomorphism such that $h(S) \in Id(T)$. If $J \in PId(T)$ and $h^{-1}(J) \neq S$, then $h^{-1}(J) \in PId(S)$.*

Proof. We prove that $h^{-1}(J) \in Id(S)$. Since $h(1) = 1 \in J$, we get $1 \in h^{-1}(J)$. Let $s, s \to t \in h^{-1}(J)$. Then $h(s) \in J$ and $h(s \to t) = h(s) \to h(t) \in J$. Since $J \in Id(T)$, we get $h(t) \in J$. Thus $t \in h^{-1}(J)$. So $h^{-1}(J) \in Id(S)$.

Next we verify $h^{-1}(J) \in PId(S)$. Let $s, t \in S$ be such that $\langle s\rangle \cap \langle t\rangle \subseteq h^{-1}(J)$. Assume that $a \in \langle h(s)\rangle \cap \langle h(t)\rangle$. Then there exist $p, q \in \mathbb{N}$ such that $h(s)^q \to a = 1 \in J$ and $h(t)^p \to a = 1 \in J$. Since $h(s) \in h(S)$ and $h(S) \in Id(T)$, which implies that $a \in h(S)$. Thus $a = h(b)$ for some $b \in S$. Moreover, $h(s^q \to b) = h(t^p \to b) = 1 \in J$, since h is a homomorphism. So we have

$$s^q \to b \in h^{-1}(J) \text{ and } t^p \to b \in h^{-1}(J), \text{ i.e.,}$$

$$b \in \langle h^{-1}(J) \cup \{s\}\rangle \cap \langle h^{-1}(J) \cup \{t\}\rangle.$$

Since $\langle s\rangle \cap \langle t\rangle \subseteq h^{-1}(J)$, by Theorem 4.3, we get $b \in h^{-1}(J)$, i.e., $a = h(b) \in J$. Hence $\langle h(s)\rangle \cap \langle h(t)\rangle \subseteq J$. Since $J \in PId(T)$, we get that $\langle h(s)\rangle \subseteq J$ or $\langle h(t)\rangle \subseteq J$. Hence $h(s) \in J$ or $h(t) \in J$. Therefore $s \in h^{-1}(J)$ or $t \in h^{-1}(J)$. By Theorem 4.2, we can concludes that $h^{-1}(J)$ is a prime ideal of S □

Proposition 4.8. *Let S be a CKL-algebra. Then $Id(S)$ is a chain iff every proper ideal of S is prime.*

Proof. ($\Rightarrow$): Suppose $Id(S)$ is a chain. Let $J \in Id(S)^*$ and there exist $e, f \in S$ such that $\langle e\rangle \cap \langle f\rangle \subseteq J$. Since $\langle e\rangle$ and $\langle f\rangle$ are ideals of S, by the hypothesis, we have either $\langle e\rangle \subseteq \langle f\rangle$ or $\langle f\rangle \subseteq \langle e\rangle$. Then it concludes that $e \in J$ or $f \in J$. Thus $J \in PId(S)$.

($\Leftarrow$): Suppose every proper ideal of S is prime. Let J and K be any two proper ideals of S. We verify $J \subseteq K$ or $K \subseteq J$. Assume that $J \nsubseteq K$ and $K \nsubseteq J$. Then there exist $j \in J - K$ and $k \in K - J$. Thus $\langle j\rangle \cap \langle k\rangle \subseteq J \cap K$. Since $S \neq J \cap K \in PId(S)$, we have $\langle j\rangle \subseteq J \cap K$ or $\langle k\rangle \subseteq J \cap K$. So we have $j \subseteq J \cap K$ or $k \subseteq J \cap K$, which is a contradiction. Hence $J \subseteq K$ or $K \subseteq J$. Therefore $Id(S)$ is a chain. □

In the following we focus on some properties of commutative L-algebras.

Proposition 4.9. *Let $(S, \to, 1)$ be a commutative L-algebra. For any $s, t, w \in S$, we have*
(1) $s \to (t \vee w) = (t \to w) \to (s \to w)$,
(2) $s \leq t$ *implies* $s \vee t = t$,
(3) $w \leq s$ *and* $s \to w \leq t \to w$ *imply* $t \leq s$.

Proof. (1) Let $s, t, w \in S$. Then by Proposition 2.3(4), we have

$$s \to (t \vee w) = s \to ((t \to w) \to w) = (t \to w) \to (s \to w).$$

(2) Let $s, t \in S$ be such that $s \leq t$. Then $s \to t = 1$. Thus $t = 1 \to t = (s \to t) \to t = s \vee t$.

(3) Let $s, t, w \in S$ be such that $w \leq s$ and $s \to w \leq t \to w$. Then $w \to s = 1$ and $(s \to w) \to (t \to w) = 1$. Thus we get

$$\begin{aligned} t \to s &= t \to (1 \to s) \\ &= t \to ((w \to s) \to s) \\ &= t \to ((s \to w) \to w) \\ &= (s \to w) \to (t \to w) \\ &= 1. \end{aligned}$$

So we have $t \leq s$. □

Definition 4.10. Let S be an L-algebra. If there exists $r \in S$ such that $r \leq s$ and $r \leq t$ for all $s, t \in S$, then S is said to be *down-directed*.

The real unit interval $[0, 1] = \{s \in \mathbb{R} | 0 \leq s \leq 1\}$, and for all $s, t \in [0, 1]$. Let $s \oplus t := min(1, s + t)$ and $s' := 1 - s$. It is easy to see that $([0, 1], \oplus, ', 0)$ is an MV-algebra. Then for any $s, t, w \in (0, 1]$, we have:
(1) $s \oplus (t \oplus w) = (s \oplus t) \oplus w$,
(2) $s \oplus t = t \oplus s$,
(3)$(s' \oplus t)' \oplus t = (t' \oplus s)' \oplus s$.

Example 4.11. Let $S = (0, 1] \subseteq \mathbb{R}$. Its implication operation $\to$ is given by

$$s \to t := \begin{cases} s' \oplus t & \text{for any } s \in S - \{1\} \\ t & s = 1, \end{cases} \tag{4.1}$$

where $s \oplus t := min(1, s + t)$ and $s' := 1 - s$ denotes the complement of s, for any $t \in S$. It is easy to check that for any $s, t \in S$, we have:
(i) if $s \in S$, then $1 \to s = s$, $s \to 1 = 1$, $s \to s = 1$,
(ii) if $s = 1$, then we get

$$(s \to t) \to (s \to w) = (1 \to t) \to (1 \to w) = t \to w$$

and

$$(t \to s) \to (t \to w) = (t \to 1) \to (t \to w) = 1 \to (t \to w) = t \to w,$$

if $s \neq 1$, then we have

$$\begin{aligned} (s \to t) \to (s \to w) &= (s' \oplus t)' \oplus (s' \oplus w) \\ &= [(s' \oplus t)' \oplus s'] \oplus w \\ &= [(t \oplus s')' \oplus s'] \oplus w \\ &= [(s \oplus t')' \oplus t'] \oplus w \end{aligned}$$

$$= (s \oplus t')' \oplus (t' \oplus w)$$
$$= (t' \oplus s)' \oplus (t' \oplus w)$$
$$= (t \to s) \to (t \to w),$$

(iii) if $s = 1$, then we get

$$(s \to t) \to t = (1 \to t) \to t = t \to t = 1$$

and

$$(t \to s) \to s = (t \to 1) \to 1 = 1 \to 1 = 1,$$

if $s \neq 1$, then $(s \to t) \to t = (s' \oplus t)' \oplus t = (t' \oplus s)' \oplus s = (t \to s) \to s$.
Then S is a commutative L-algebra.

Remark 4.12. The Example 4.11 shows that a commutative L-algebra can be a lattice and thus be down-directed, but it is not an MV-algebra because it is unbounded.

In what follows, we consider how this $\wedge$-operation gives on commutative L-algebras. Next, we go through definition of down-directed on L-algebras and give $\wedge$-operation on commutative L-algebras.

Proposition 4.13. *If S is a down-directed commutative L-algebra, then for any $s, t \in S$, $s \wedge t := [(s \to r) \vee (t \to r)] \to r$ is the greatest lower bound of s and t where $r \leq s, t$.*

Proof. Suppose S is a down-directed commutative L-algebra and let $r \leq s, t$. We first prove that $s \wedge t$ is a lower bound of s and t. Obviously, $r \leq s \vee t$. By the hypothesis, we have $s \to r \leq (s \to r) \vee (t \to r)$. Then Proposition 2.3 (5),

$$[(s \to r) \vee (t \to r)] \to r \leq (s \to r) \to r = s \vee r = s.$$

We concludes that $s \wedge t \leq s$. Similarly, we get $s \wedge t \leq t$. Thus $s \wedge t$ is a lower bound of s and t. Next we prove that $s \wedge t$ is the greatest lower bound of s and t. Let $z \leq s, t$. Then by Proposition 2.3 (5), we have $s \to r \leq z \to r$ and $t \to r \leq z \to r$. Thus $(s \to r) \vee (t \to r) \leq z \to r$. So we get

$$z \leq z \vee r = (z \to r) \to r \leq [(s \to r) \vee (t \to r)] \to r = s \wedge t.$$

Hence $s \wedge t$ is the greatest lower bound of s and t. □

Proposition 4.14. *Let S be a commutative L-algebra. For any $s, t, w \in S$, we have*

(1) $s \vee t \to w = inf(s \to w, t \to w)$,
(2) $(s \to t) \vee (t \to s) = 1$,

if S is a down-directed, then we have

(3) $s \to t \wedge w = (s \to t) \wedge (s \to w)$,
(4) $s \to s \wedge t = s \to t$,
(5) $s \wedge t \to w = (s \to w) \vee (t \to w)$.

Proof. (1) We first prove that $s \vee t \to w$ is a lower bound of $s \to w$ and $t \to w$. Since S is a commutative L-algebra, we have $s, t \leq s \vee t$. Then by Proposition 2.3 (5), we have

$$s \vee t \to w \leq s \to w \text{ and } s \vee t \to w \leq t \to w.$$

Thus $s \vee t \to w$ is a lower bound for $s \to w$ and $t \to w$. Next we prove that $s \vee t \to w$ is the greatest lower bound of $s \to w$ and $t \to w$. Let there exist $r \in S$ be such that $r \leq s \to w$ and $t \to w$. Then by (2.6), $s \leq r \to w$ and $t \leq r \to w$. Thus $s \vee t \leq r \to w$. So $r \leq s \vee t \to w$. Hence $s \vee t \to w$ is the greatest lower bound of $s \to w$ and $t \to w$.

(2) Let $s, t, w \in S$. Since S is a commutative L-algebra, we have

$$\begin{aligned}(s \to t) \vee (t \to s) &= ((s \to t) \to (t \to s)) \to (t \to s)\\ &= ((s \to (s \wedge t)) \to (t \to (s \wedge t))) \to (t \to s)\\ &= (t \to (s \to (s \wedge t)) \to (s \wedge t)) \to (t \to s)\\ &= (t \to ((s \wedge t) \to s) \to s) \to (t \to s)\\ &= (t \to (1 \to s)) \to (t \to s)\\ &= (t \to s) \to (t \to s)\\ &= 1.\end{aligned}$$

Now suppose S is a down-directed.

(3) Let $s, t, w \in S$ such that $r \leq t, w$. Then by the hypothesis, we get

$$t \wedge w = ((t \to r) \vee (w \to r)) \to r.$$

Since $r \leq t$, we have $(t \to r) \to r = (r \to t) \to t = 1 \to t = t$. Similarly, we get $(w \to r) \to r = w$. Thus we have

$$\begin{aligned}s \to (t \wedge w) &= s \to [((t \to r) \vee (w \to r)) \to r]\\ &= ((t \to r) \vee (w \to r)) \to (s \to r)\\ &= ((t \to r) \vee (s \to r)) \wedge ((w \to r) \vee (s \to r))\\ &= (s \to ((t \to r) \to r)) \wedge (s \to ((w \to r) \to r))\\ &= (s \to t) \wedge (s \to w).\end{aligned}$$

(4) It is immediately we set t by s and w by t in (3), then

$$s \to s \wedge t = (s \to s) \wedge (s \to t) = 1 \wedge (s \to t) = s \to t.$$

(5) We first prove that $s \wedge t \to w$ is a upper bound of $s \to w$ and $t \to w$. Since S is a down-directed, we get $s \wedge t \leq s, t$. Then $s \wedge t \to w \geq s \to w$ and $s \wedge t \to w \geq t \to w$, where $w \leq s, t$. Thus $(s \wedge t) \to w$ is a upper bound of $s \to w$ and $t \to w$. Next we prove that $s \wedge t \to w$ is the least upper bound of $s \to w$ and $t \to w$. Let $v \in S$ be such that $w \leq v$ and $s \to w, t \to w \leq v$. Then $(s \to w) \vee (t \to w) \leq v$, i.e., $v \to w \leq ((s \to w) \vee (t \to w)) \to w$. Thus $(s \wedge t) \to w \leq (v \to w) \to w$. Since $w \leq v$, we get $w \to v = 1$. So $v = 1 \to v = (w \to v) \to v = (v \to w) \to w$. Hence $s \wedge t \to w \leq v$. Therefore $s \wedge t \to w$ is the least upper bound of $s \to w$ and $t \to w$. □

Lemma 4.15. *Let S be a commutative L-algebra. For any $e, f \in S$, we have*

(1) *$e \leq f$ implies $\langle f \rangle \subseteq \langle e \rangle$,*

(2) *$\langle e \vee f \rangle = \langle e \rangle \cap \langle f \rangle$.*

Proof. (1) Suppose $e \leq f$ and let $s \in \langle f \rangle$. We need prove that $s \in \langle e \rangle$. Since $e \leq f$, by (2.4), $f \to s \leq e \to s$. Then $f \to (f \to s) \leq f \to (e \to s) \leq e \to (e \to s)$, i.e., $f^2 \to s \leq e^2 \to s$. Repeating the above argument, we conclude that $f^n \to s \leq$

$e^n \to s$. Since $f^n \to s = 1$ and 1 is the greatest element of S, $e^n \to s = 1$. Thus by Corollary 3.2, $s \in \langle e\rangle$. So $\langle f\rangle \subseteq \langle e\rangle$.

(2) Since $e, f \leq e \vee f$, $\langle e \vee f\rangle \subseteq \langle e\rangle$ and $\langle e \vee f\rangle \subseteq \langle f\rangle$ by (1). Then $\langle e \vee f\rangle \subseteq \langle e\rangle \cap \langle f\rangle$. Conversely, let $s \in \langle e\rangle \cap \langle f\rangle$. Then $s \in \langle e\rangle$ and $s \in \langle f\rangle$. Thus $e^n \to s = f^n \to s = 1$. Since S is a commutative L-algebra, by Proposition 4.14, we have

$$(e \vee f)^n \to s = (e^n \to s) \wedge (f^n \to s) = 1 \wedge 1 = 1.$$

So $s \in \langle e \vee f\rangle$ by Corollary 3.2. Hence $\langle e\rangle \cap \langle f\rangle \subseteq \langle e \vee f\rangle$. Therefore $\langle e \vee f\rangle = \langle e\rangle \cap \langle f\rangle$. □

Next we study the equivalence inscription of prime ideals and some properties on commutative L-algebras.

Theorem 4.16. *Let S be a commutative L-algebra and $B \in Id(S)^*$. Then the following assertions are equivalent:*
(1) *$B \in PId(S)$,*
(2) *$s \vee t \in B$ implies $s \in B$ or $t \in B$ for all $s, t \in S$.*

Proof. (1)⇒(2): Suppose the condition (1) holds. By Theorem 4.2, we have $J, K \in Id(S)$, $J \cap K \subseteq B$ implies $J \subseteq B$ or $K \subseteq B$. Let $s, t \in S$ such that $s \vee t \in B$. Then $\langle s\rangle \cap \langle t\rangle \subseteq B$ by Lemma 4.15 (2). Then $\langle s\rangle \subseteq B$ or $\langle t\rangle \subseteq B$. Thus $s \in B$ or $t \in B$ for all $s, t \in S$.

(2) ⇒(1): Suppose the condition (2) holds. Let $J, K \in Id(S)$ such that $J \cap K \subseteq B$. Assume that $J \nsubseteq B$. Then there exists $e \in S$ such that $e \in J$ and $e \notin B$. Let f be an arbitrary element on K. Obviously, $\langle e\rangle \cap \langle f\rangle \subseteq J \cap K \subseteq B$. Thus $\langle e \vee f\rangle \subseteq J \cap K \subseteq B$ by Lemma 4.15 (2). So $e \vee f \in B$. Since $e \notin B$, we have $f \in B$. Hence $K \subseteq B$. Therefore by Theorem 4.2, we can get $B \in PId(S)$. □

Proposition 4.17. *Let S be a commutative L-algebra and $J \in Id(S)^*$. Then $J \in PId(S)$ iff $s \to t \in J$ or $t \to s \in J$ for all $s, t \in S$.*

Proof. (⇒): Suppose $J \in PId(S)$. Since S is a commutative L-algebra. Then by Proposition 4.14 (2), we have $(s \to t) \vee (t \to s) = 1 \in J$ for all $s, t \in S$. Thus we get either $s \to t \in J$ or $t \to s \in J$ by Theorem 4.16.

(⇐): Suppose $s \to t \in J$ or $t \to s \in J$ for all $s, t \in S$. Let $s \vee t \in J$. We need to prove that $t \in J$ or $s \in J$. Assume that $s \to t \in J$. Since $J \in Id(S)$, $s \to t \in J$ and $(s \to t) \to t \in J$. Then $t \in J$. Similarly, assume that $t \to s \in J$. Then we get $s \in J$. Thus $J \in PId(S)$ by Theorem 4.16. □

From Proposition 4.17, we get the following.

Corollary 4.18. *Let S be a commutative L-algebra and $J \in PId(S)$. If $K \in Id(S)^*$ such that $J \subseteq K$, then $K \in PId(S)$.*

Theorem 4.19. *Let S be a commutative L-algebra with $|S| \geq 2$. Then the following conditions are equivalent:*
(1) *$Id(S)^* \subseteq PId(S)$,*
(2) *$\{1\} \in PId(S)$,*
(3) *S is a totally ordered set with respect to L-ordering.*

Proof. (1)⇒(2): It is obvious.

(2)⇒ (3): Suppose $\{1\} \in PId(S)$ and $\{1\}$ is prime. Then by Proposition 4.17, we get that either $s \to t \in \{1\}$ or $t \to s \in \{1\}$ for any $s, t \in S$. Thus $s \to t = 1$ or $t \to s = 1$. So $s \leq t$ or $t \leq s$. Hence S is totally ordered.

(3)⇒(1): Suppose S is a totally ordered set with respect to L-ordering " $\leq$ " and Suppose $J \in Id(S)^*$. Let there exist $s, t \in S$ be such that $s \to t \notin J$. Since $J \in Id(S)^*$. Then we have $s \to t \neq 1$. Thus $s \nleq t$. So $t \leq s$. Hence for any $s, t \in S$, we have $s \leq t$ or $t \leq w$. Therefore $s \to t = 1 \in J$ or $t \to s = 1 \in J$. By Proposition 4.17, we concludes that $J \in PId(S)$. □

5. The space of prime ideals of commutative L-algebras

In this section, some topological properties about the space of all prime ideals on commutative L-algebras are studied. A necessary and sufficient condition is derived for a prime ideal of a commutative L-algebra to become maximal.

Theorem 5.1. *S be a commutative L-algebra. Then every maximal ideal is prime ideal.*

Proof. Let $G \in MId(S)$. We prove that $G \in PId(S)$. Let $s, t \in G$ be such that $s \vee t \in G$. Then $\langle s \vee t \rangle \subseteq G$. By Lemma 4.15 (2), we have $\langle s \vee t \rangle = \langle s \rangle \cap \langle t \rangle$. Thus $\langle s \rangle \cap \langle t \rangle \subseteq G$. So by Theorem 4.3, we have

$$\langle G \cup \{s\} \rangle \cap \langle G \cup \{t\} \rangle = G.$$

Since $e \notin G$, we have $e \notin \langle G \cup \{s\} \rangle$ or $e \notin \langle G \cup \{t\} \rangle$. It follows that $\langle G \cup \{s\} \rangle = G$ or $\langle G \cup \{t\} \rangle = G$ since G is maximal. Hence $s \in G$ or $t \in G$. Therefore $G \in PId(S)$. □

Theorem 5.2. *Let S be a commutative L-algebra and $e \in S$. If $J \in Id(S)$ such that $e \notin J$, then there exists $B \in PId(S)$ such that $e \notin B$ and $J \subseteq B$.*

Proof. Let $e \in S$ and suppose $J \in Id(S)$ such that $e \notin J$. Denote $\mathscr{K} = \{K \in Id(S) | e \notin K \text{ and } J \subseteq K\}$. Then $\mathscr{K} \neq \varnothing$ and $\mathscr{K}$ is a partially order set under inclusion relation $\subseteq$. Assume that $\{K_i | i \in I\}$ is a chain in $\mathscr{K}$. We can easily check that $e \notin \bigcup\{K_i | i \in I\}$ and $J \subseteq \bigcup\{K_i | i \in I\}$. In what following, we prove that $\bigcup\{K_i | i \in I\}$ is a ideal of S. Denote $K = \bigcup\{K_i | i \in I\}$. Obviously, $1 \in K$. For any $s, t \in S$, $s \in K$ and $s \to t \in K$, there exists $i \in I$ such that $s \in K_i$ and there exists $j \in I$ such that $s \to t \in K_j$. Since $\{K_i | i \in I\}$ is a chain in $\mathscr{K}$, we have $K_i \subseteq K_j$ or $K_j \subseteq K_i$. Assume that $K_i \subseteq K_j$. Then $x \in K_j$. Since $K_j \in Id(S)$, we have $y \in K_j$. Thus $y \in K$. For any $s \in S$, $s \in K$, there exists $i \in I$ such that $s \in K_i$. Since $K_i \in Id(S)$, we have $(s \to t) \to t \in K_i$, $t \to s \in K_i$ and $s \to (t \to s) \in K_i$. So $(s \to t) \to t \in K$, $t \to s \in K$ and $s \to (t \to s) \in K$. Hence $K \in Id(S)$. Therefore $\bigcup\{K_i | i \in I\} \in \mathscr{K}$ and thus it is the upper bound for this chain. By Zorn's Lemma, there exists a maximum element M in $\mathscr{K}$. By Theorem 5.1, we get $M \in PId(S)$. □

From Theorem 5.2, we can directly obtain the following corollary.

Corollary 5.3. *Let S be a commutative L-algebra and $1 \neq e \in S$. Then there exists $B \in PId(S)$ such that $e \notin B$.*

Let S be a commutative L-algebra and $Spec(S)$ denote the set of all prime ideals of S. For any $D \subseteq S$, let $R(D) = \{B \in Spec(S)|D \nsubseteq B\}$ and for any $s \in S$, $R(s) = R(\{s\})$. Then we have the following observations:

Lemma 5.4. *Let S be a commutative L-algebra. For any $s, t \in S$, the followings hold:*
(1) $R(s) \cap R(t) = R(s \vee t)$,
(2) $R(s) = \varnothing \Leftrightarrow s = 1$,
(3) *if S is a down-directed, then* $R(s) \cup R(t) = R(s \wedge t)$.

Proof. (1) Let $B \in Spec(S)$ be such that $B \in R(s) \cap R(t)$. Then we have $B \in R(s)$ and $B \in R(t)$. Thus $s \notin B$ and $t \notin B$. Since B is prime, we get $s \vee t \notin B$. So $B \in R(s \vee t)$. Hence $R(s) \cap R(t) \subseteq R(s \vee t)$.

Conversely, let $B \in Spec(S)$ and let $B \in R(s \vee t)$. Then $s \vee t \notin B$. If $s \in B$, then $s \vee t \in B$ because of $s \leq s \vee t$. Thus it yields that $s \notin B$. So $B \in R(s)$. Similarly, we get $B \in R(t)$. Hence $B \in R(s) \cap R(t)$. Therefore $R(s \vee t) \subseteq R(s) \cap R(t)$. We concludes that $R(s) \cap R(t) = R(s \vee t)$.

(2) It is clear, since $\{1\} \subseteq B$ for all $B \in Spec(S)$.

(3) Suppose S is a down-directed and let $B \in Spec(S)$ be such that $B \in R(s) \cup R(t)$. Then $B \in R(s)$ or $B \in R(t)$. Thus $s \notin B$ or $t \notin B$. If $s \wedge t \in B$, then $s \in B$ and $t \in B$. So $s \wedge t \notin B$. Hence $B \in R(s \wedge t)$. Therefore $R(s) \cup R(t) \subseteq R(s \wedge t)$.

Conversely, let $B \in Spec(S)$ be such that $B \in R(s \wedge t)$. Then $s \wedge t \notin B$. Since S is a down-directed, we have $s \wedge t = inf\{s, t\}$ by Proposition 4.13. Thus $s \notin B$ and $t \notin B$. So $B \in R(s) \cup R(t)$. Hence $R(s \wedge t) \subseteq R(s) \cup R(t)$. Therefore $R(s) \cup R(t) = R(s \wedge t)$. □

Proposition 5.5. *Let S be a commutative L-algebra. Then* $\bigcup_{s \in S} R(s) = Spec(S)$.

Proof. Let $B \in Spec(S)$. Since $B \in Id(S)^*$, there exists $e \in S$ such that $e \notin B$. Then $B \in R(e) \subseteq \bigcup_{s \in S} R(s)$. Thus $Spec(S) \subseteq \bigcup_{s \in S} R(s)$. Clearly, $\bigcup_{s \in S} R(s) \subseteq Spec(S)$. So $\bigcup_{s \in S} R(s) = Spec(S)$. □

By Proposition 5.5, it can be seen that $\{R(s)|s \in S\}$ forms a covering of $Spec(S)$. Then $\{R(s)|s \in S\}$ is a topology consisting of open bases on $Spec(S)$, named *hull-kernel technology*. In what follows, we will discuss some properties about this topology.

Example 5.6. In Example 2.6, $Spec(S) = \{\{1\}, \{e, g, 1\}, \{f, g, 1\}\}$. $R(0) = \{B \in Spec(S)|0 \notin B\}$ is a hull-kernel technology on S.

Lemma 5.7. *Let S be a commutative L-algebra. Then the followings hold:*
(1) *for any $s \in S$,* $R(\langle s \rangle) = R(s)$,
(2) *for any $J, K \in Id(S)$,* $R(J) \cap R(K) = R(J \cap K)$.

Proof. (1) Let $B \in Spec(S)$ be such that $B \in R(\langle s \rangle)$. Then $\langle s \rangle \nsubseteq B$. Thus $s \notin B$. So $B \in R(s)$. Hence $R(\langle s \rangle) \subseteq R(s)$.

Conversely, let $B \in R(s)$. Then $s \notin B$. Thus $\langle s \rangle \nsubseteq B$. So $B \in R(\langle s \rangle)$. Hence $R(s) \subseteq R(\langle s \rangle)$. Therefore $R(\langle s \rangle) = R(s)$.

(2) Let $B \in Spec(S)$ be an arbitrary prime ideal and let $B \in R(J) \cap R(K)$. Then we get $B \in R(J)$ and $B \in R(K)$. Thus $J \nsubseteq B$ and $K \nsubseteq B$. So there exist $s \in J$ and $t \in J$ such that $s \notin B$ and $t \notin B$. So $s \vee t \notin B$. Since $s \in J$ and $t \in J$, $s \vee t \in J \cap K$. Hence $J \cap K \nsubseteq B$ which means $B \in R(J \cap K)$. Therefore $R(J) \cap R(K) \subseteq R(J \cap K)$.

Conversely, we need to prove that $R(J \cap K) \subseteq R(J) \cap R(K)$. Let $B \in R(J \cap K)$. Then $J \cap K \notin B$. Thus we get $J \notin B$ and $K \notin B$, i.e., $B \in R(J)$ and $B \in R(K)$. So we get $B \in R(J) \cap R(K)$. Hence $R(J \cap K) \subseteq R(J) \cap R(K)$. Therefore $R(J) \cap R(K) = R(J \cap K)$. □

Lemma 5.8. *Let S be a commutative L-algebra, $s \in S$ and $J \in Id(S)$. Then $s \in J$ iff $R(s) \subseteq R(J)$.*

Proof. ($\Rightarrow$): Suppose $s \in J$ and and let $B \in Spec(S)$ such that $B \in R(s)$. Then $s \notin B$. Thus $J \nsubseteq B$. So $B \in R(J)$.

($\Leftarrow$): Suppose $R(s) \subseteq R(J)$. Assume that $s \notin J$. Then there exists $B \in Spec(S)$ such that $s \notin B$ and $J \subseteq B$ by Theorem 5.2. Thus $B \in R(s)$ and $B \nsubseteq R(J)$. So $R(s) \nsubseteq R(J)$, which is a contradiction. Hence $s \in J$. □

Proposition 5.9. *Let S be a commutative L-algebra. Then for any $s \in S$, $R(S)$ is compact in $Spec(S)$.*

Proof. Let $s \in S$ and let $D \subseteq S$ be such that $R(s) \subseteq \bigcup_{t \in D} R(t)$. Assume that $J = \langle D \rangle$ and $s \notin J$. Then there exists $B \in PId(S)$ such that $J \subseteq B$ and $s \notin J$ by Theorem 5.2. Thus $B \in R(s) \subseteq \bigcup_{t \in D} R(t)$. So $t \notin B$ for some $t \in D$, which is a contradiction, because of $t \in D \subseteq J \subseteq B$. Hence $s \in J$. Therefore there exist $d_1, d_2, ..., d_n \in D$ such that

$$d_n \to (\cdots \to (d_1 \to s) \cdots) = 1.$$

Let $B \in R(s)$. Then $s \notin B$. Assume that $d_j \in B$ for all $j = 1, 2, \cdots, n$. Since $d_n \to (\cdots \to (d_1 \to s) \cdots) = 1 \in B$ and $B \in Id(S)$, we get $s \in B$, which is a contradiction. Then $d_j \notin B$ for some $j = 1, 2, \cdots, n$, i.e., $B \in R(d_j)$ for some d_j. Thus $B \in \bigcup_{j=1} R(d_j)$. So $R(s) \subseteq \bigcup_{j=1}^{n} R(d_j)$, which is a finite subcover of $R(s)$. Hence $R(s)$ is compact in $Spec(S)$. Therefore for each $s \in S$, $R(s)$ is a compact open subset of $Spec(S)$. □

Theorem 5.10. *Let S be a down-directed commutative L-algebra and X be a compact open subset of $Spec(S)$. Then $X = R(s)$ for some $s \in S$.*

Proof. Let X be a compact open subset of $Spec(S)$. Since X is open, we get $X = \bigcup_{d \in D} R(d)$ for some $D \subseteq S$. Since X is compact, there exist $d_1, d_2, \cdots, d_n \in D$ such that

$$X = \bigcup_{j=1}^{n} R(d_j) = R(\bigwedge_{j=1}^{n} d_j).$$

Then $X = R(s)$ for some $s \in S$. □

Corollary 5.11. *For any down-directed commutative L-algebra S, the set $\{R(s)|s \in S\}$ is an open base for the prime space $Spec(S)$.*

Theorem 5.12. *Let S be a down-directed commutative L-algebra. Then $Spec(S)$ is a T_0-space.*

Proof. Let $B, C \in PId(S)$. Without loss of generality assume that $B \nsubseteq C$. Choose $s \in S$ such that $s \in B$ and $s \notin C$. Then $B \notin R(s)$ and $C \in R(s)$. Thus $Spec(S)$ is a T_0-space. □

Based on the Theorem 5.12, we can directly obtain the following corollary.

Corollary 5.13. *The map $s \mapsto R_0(s)$ is an anti-homomorphism from S onto the lattice of all compact open subsets on $Spec(S)$.*

For any $D \subseteq S$, denote $F(D) = \{B \in Spec(S)|D \subseteq B\}$. Then clearly $F(D) = Spec(S) - R(D)$. Thus $F(D)$ is a closed set in $Spec(S)$. Also every closed set in $Spec(S)$ is of the form $F(D)$ for some $D \subseteq S$. So we have the following.

Proposition 5.14. *The closure of any $T \subseteq Spec(S)$ is given by $\bar{T} = F(\bigcap\limits_{B \in T} B)$.*

Proof. Let $T \subseteq Spec(S)$. Now we prove that $F(\bigcap\limits_{B \in T} B)$ is a closed set containing T. Let $C \in T$. Then $\bigcap\limits_{B \in T} B \subseteq C$. Thus $C \in F(\bigcap\limits_{B \in T} B)$. So $F(\bigcap\limits_{B \in T} B)$ is a closed set containing T. Next we prove that $F(\bigcap\limits_{B \in T} B)$ is the smallest closed set containing T. Let X be any closed set in $Spec(S)$. Then $X = F(D)$ for some $D \subseteq S$. Since $T \subseteq X = F(D)$, we have $D \subseteq B$ for all $B \in T$. Thus $D \subseteq \bigcap\limits_{B \in T} B$. So $F(\bigcap\limits_{B \in T} B) \subseteq F(D) = X$. Hence $F(\bigcap\limits_{B \in T} B)$ is the smallest closed set containing T. Therefore it concludes that $\bar{T} = F(\bigcap\limits_{B \in T} B)$. □

Theorem 5.15. *For any down-directed commutative L-algebra S, $Spec(S)$ is a T_1-space iff every prime ideal is maximal.*

Proof. ($\Rightarrow$): Suppose $Spec(S)$ is a T_1-space and let $B \in PId(S)$. Assume that there exists $S \neq C \in Id(S)$ such that $B \subseteq C$. Since $Spec(S)$ is a T_1-space, there exist two basic open sets $R(s)$ and $R(t)$ such that $B \in R(s) - R(t)$ and $C \in R(t) - R(s)$. Since $B \notin R(t)$, we have $t \in B \subset C$, which is a contradiction to that $C \in R(t)$. Then $B \in MId(S)$.

($\Leftarrow$): Suppose every prime ideal is maximal. Let B_1 and B_2 be two distinct elements of $Spec(S)$. Then by the hypothesis, $B_1, B_2 \in MId(S)$. Thus $B_1 \nsubseteq B_2$ and $B_2 \nsubseteq B_1$. So there exist $e, f \in S$ such that $e \in B_1 - B_2$ and $f \in B_2 - B_1$. Hence $B_1 \in R(f) - R(e)$ and $B_2 \in R(e) - R(f)$. Therefore $Spec(S)$ is a T_1-space. □

6. Conclusion

In order to study some properties of prime ideals, we gave the generated formula of ideals on CKL-algebras and we defined the $\vee$-operation on L-algebras. Next, we studied the relationship between prime ideals and maximal ideals on CKL-algebras.

Such as, we proved that maximal ideals is prime ideals on CKL-algebras. We gave a counterexample to show that commutative L-algebras can be not residuated lattices, much less MV-algebras. In this paper, we introduced commutative L-algebras, which is a true extension of MV-algebras. Furthermore, we studied prime ideals and gave two equivalent characterizations of prime ideals on commutative L-algebras. Finally, we studied some topological spaces composed by prime ideals on commutative L-algebras. For example, we proved that Space(S) is a T_0-space. When Space(S) is a T_1-space, prime ideals and maximal ideals coincide. In our future work, we generalize the important conclusions of this paper, which studies prime and maximal ideals on L-algebras, to pseudo L-algebras. In addition to extending those to EL-algebra, we will discuss applications of prime ideals, such as the continuity of topological spaces, and if the topological space is continuous, then we can further discuss its analytic structure.

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