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Abstract. This paper deals with a new type of fuzzy separation axiom, viz., fuzzy α-b-regular space by introducing fuzzy α-b-open set as a basic tool. This newly defined class of sets is strictly larger than that of fuzzy open set as well as fuzzy preopen set, fuzzy semiopen set, fuzzy α-open set and fuzzy β-open set. Also, we introduce new type of fuzzy compact space and a strong form of fuzzy T_2-space. However, three different types of functions are introduced and studied. Also the mutual relationships of these functions are established. Lastly some applications of these functions on the spaces introduced here are established.

2020 AMS Classification: 54A40, 03E72

Keywords: Fuzzy α-b-open set, Fuzzy regular open set, Fuzzy α-b-r-continuous function, Fuzzy α-b-continuity, Fuzzy almost α-b-continuity, Fuzzy extremally disconnected space.

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1. Introduction

Fuzzy α-open set was introduced in [1]. Using this concept as a basic tool, here we introduce fuzzy α-b-open set. After introducing fuzzy continuous function in [2], different types of fuzzy continuous-like functions were introduced and studied. Using the concept of fuzzy regular closed set [3], here we introduce fuzzy α-b-r-continuous function, fuzzy α-b-continuity, and fuzzy almost α-b-continuity. Fuzzy regular space was introduced in [4]. Here we introduce fuzzy α-b-regular space, the class of which is strictly larger than that of fuzzy regular space. It is shown that in this space fuzzy open set and fuzzy α-b-open set coincide. Again fuzzy compact space was introduced by Chang [2]. Here we introduce fuzzy α-b-compactness which is weaker than fuzzy compactness. Also fuzzy α-b-T_2-space was introduced, the class of which is strictly larger than that of fuzzy T_2-space [4].

Recently, new types of fuzzy sets, viz., fuzzy soft set and fuzzy octahedron set are introduced and studied. A new branch of fuzzy topology is developed using these types of fuzzy sets. In this context we have to mention [5, 6, 7, 8, 9, 10].

2. Preliminaries

We recall the concepts related to fuzzy sets introduced by Zadeh [11]. A *fuzzy set* A in a nonempty set X is a mapping from X into the closed interval $I = [0, 1]$, i.e., $A \in I^X$. The *support* of a fuzzy set A, denoted by $suppA$, is defined by $suppA = \{x \in X : A(x) \neq 0\}$. The fuzzy set with the singleton support $\{x\} \subseteq X$ and the value t $(0 < t \leq 1)$ will be denoted by x_t. 0_X and 1_X are the constant fuzzy sets taking values 0 and 1 respectively in X. The *complement* of a fuzzy set A in X, denoted by $1_X \setminus A$, is defined by $(1_X \setminus A)(x) = 1 - A(x)$ for each $x \in X$. For any two fuzzy sets A, B in X, $A \leq B$ means $A(x) \leq B(x)$ for all $x \in X$, while AqB means there exists $x \in X$ such that $A(x) + B(x) > 1$ and A is said to be *quasi-coincident* (q-coincident, for short) *with* B [12]. The negation of these two statements will be denoted by $A \not\leq B$ and $A \not{q} B$ respectively.

Throughout the paper, (X, τ) or simply by X we shall mean a fuzzy topological space (fts, for short) in the sense of Chang [2]. For a fuzzy set A in a fts X, clA and $intA$ stand for the *fuzzy closure* and the *fuzzy interior* of A in X as follows [2]:

$$clA = \bigwedge\{F : A \leq F,\ F \text{ is a closed set in } X\},$$

$$intA = \bigvee\{G : G \leq A,\ G \text{ is an open set in } X\}.$$

For a fts X, $A \in I^X$ is said to be *fuzzy regular open* [3] (resp. *fuzzy semiopen* [3], *fuzzy preopen* [13], *fuzzy α-open* [1], *fuzzy β-open* [14]), if $A = int(clA)$ (resp. $A \leq cl(intA)$, $A \leq int(clA)$, $A \leq int(cl(intA))$, $A \leq cl(int(clA))$). The complement of a fuzzy regular open (resp. fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open) set is called a *fuzzy regular closed* (resp. *fuzzy semiclosed*, *fuzzy preclosed*, *fuzzy α-closed*, *fuzzy β-closed*) *set*. The smallest fuzzy semiclosed (resp., fuzzy preclosed, fuzzy α-closed, fuzzy β-closed) set containing a fuzzy set A in X is called the *fuzzy semiclosure* (resp *fuzzy preclosure*, *fuzzy α-closure*, *fuzzy β-closure*) of A, denoted by $sclA$ (resp. $pclA$, αclA, βclA). It is well-known that A is fuzzy semiclosed (resp. fuzzy preclosed, fuzzy α-closed, fuzzy β-closed) in a fts X if and only if $A = sclA$ (resp. $A = pclA$, $A = \alpha clA$, $A = \beta clA$). The collection of all fuzzy regular open (resp. fuzzy semiopen, fuzzy preopen, fuzzy α-open, fuzzy β-open) sets in X is denoted by $FRO(X)$ (rssp., $FSO(X)$, $FPO(X)$, $F\alpha O(X)$, $F\beta O(X)$) and the collection of all fuzzy regular closed (resp., fuzzy semiclosed, fuzzy preclosed, fuzzy α-closed, fuzzy β-closed) sets in X is denoted by $FRC(X)$ (rssp., $FSC(X)$, $FPC(X)$, $F\alpha C(X)$, $F\beta C(X)$). For a fuzzy open set A in X, $sclA = int(clA)$ [15].

3. Fuzzy α-b-open set : Some properties

In this section, we first introduce fuzzy α-b-open set and then establish the mutual relationships of this newly defined set with the sets defined in [1, 3, 13, 14].

First we recall some definitions from [16] for ready references.

Definition 3.1 ([16]). Let (X, τ) be an fts and $A \in I^X$. A fuzzy point x_α in X is said to be a *fuzzy θ-semicluster point* of A, if $clUqA$ for all $U \in FSO(X)$ with $x_\alpha qU$.

The union of all fuzzy θ-semicluster points of A is called the *fuzzy θ-semiclosure* of A and is denoted by θ-$sclA$. It is obvious that A is fuzzy θ-semiclosed in X if and only if $A = \theta$-$sclA$.

Definition 3.2 ([16])**.** Let (X, τ) be an fts and $A \in I^X$. Then the *r-kernel* of A, denoted by r-Ker$A$, is defined as follows :

$$r\text{-Ker}A = \bigwedge\{U : U \in FRO(X), A \leq U\}.$$

Let us now introduce the following concept.

Definition 3.3. A fuzzy set A in an fts (X, τ) is said to be *fuzzy α-b-open* in X, if $A \leq cl(\alpha int(clA))$. The complement of a fuzzy α-b-open set is said to be *fuzzy α-b-closed* in X. The collection of all fuzzy α-b-open (resp. fuzzy α-b-closed) sets in an fts X is denoted by $F\alpha bO(X)$ (resp. $F\alpha bC(X)$).

Remark 3.4. The union of any two fuzzy α-b-open sets is also so. But intersection of any two fuzzy α-b-open sets may not be so, as it seen from the following example.

Example 3.5. Let $X = \{a, b\}$ and let $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.6$. Then (X, τ) is an fts. Consider two fuzzy sets $B,\ C$ in X defined by:

$$B(a) = 0.4,\ B(b) = 0.6,\ C(a) = 0.6,\ C(b) = 0.4.$$

Then clearly, $B,\ C \in F\alpha bO(X)$. Let $D = B \wedge C$. Then $D(a) = D(b) = 0.4$. Thus

$$cl(\alpha int(clD)) = cl(\alpha int(1_X \setminus A)) = cl0_X = 0_X \not\geq D.$$

So $D \notin F\alpha bO(X)$.

From Example 3.5, we can conclude that the set of all fuzzy α-b-open sets in an fts X does not form a fuzzy topology.

Remark 3.6. It is clear from definitions that fuzzy open set, fuzzy semiopen set, fuzzy preopen set, fuzzy α-open set, fuzzy β-open set implies fuzzy α-b-open set, but the reverse implications are not necessarily true follow from the following example.

Example 3.7. Let $X = \{a, b\}$ and let $\tau = \{0_X, 1_X, A\}$, where $A(a) = 0.5, A(b) = 0.4$. Then (X, τ) is an fts. Here $F\alpha O(X) = \{0_X, 1_X, U\}$, where $A \not\leq U \not\leq 1_X \setminus A$. Consider a fuzzy set B in X defined by $B(a) = B(b) = 0.5$. Then clearly, $B \notin \tau$, $B \notin FPO(X)$, $B \notin F\alpha O(X)$. But $cl(\alpha int(clB)) = 1_X \setminus A \geq B$. Thus $B \in F\alpha bO(X)$.

Next consider the fuzzy set C in X defined by $C(a) = C(b) = 0.4$. Then clearly, $C \notin FSO(X)$, but $C \in F\alpha bO(X)$.

As $\tau \subseteq F\alpha O(X)$, clearly fuzzy α-b-open set may not necessarily fuzzy β-open set.

Theorem 3.8. *Let (X, τ) be an fts. Then the union of any collection of fuzzy α-b-open sets in X is fuzzy α-b-open in X.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Lambda\}$ be any collection of fuzzy α-b-open sets in X and let $\alpha \in \Lambda$. Then clearly, $G_\alpha \leq cl(\alpha int(clG_\alpha))$. Also, $G_\alpha \leq \bigvee_{\alpha \in \Lambda} G_\alpha$. Thus $clG_\alpha \leq cl(\bigvee_{\alpha \in \Lambda} G_\alpha)$. So $G_\alpha \leq cl(\alpha int(clG_\alpha)) \leq cl(\alpha int(cl(\bigvee_{\alpha \in \Lambda} G_\alpha)))$. Taking union on both sides, $\bigvee_{\alpha \in \Lambda} G_\alpha \leq cl(\alpha int(cl(\bigvee_{\alpha \in \Lambda} G_\alpha)))$. Hence $\bigvee_{\alpha \in \Lambda} G_\alpha$ is fuzzy α-b-open in X. □

Definition 3.9. Let (X, τ) be an fts and $A \in I^X$. Then fuzzy α-b-closure of A, denoted by $\alpha bclA$, is defined by

$$\alpha bclA = \bigwedge\{U \in I^X : A \leq U, U \in F\alpha bC(X)\}$$

and fuzzy α-b-interior of A, denoted by $\alpha bintA$, is defined by

$$\alpha bintA = \bigvee\{G \in I^X : G \leq A, G \in F\alpha bO(X)\}.$$

Note 3.10. By Remark 3.4, we can conclude that for any fuzzy set A in an fts X, $\alpha bclA$ is fuzzy α-b-closed and $\alpha bintA$ is fuzzy α-b-open. Again, if $A \in F\alpha bC(X)$, then $A = \alpha bclA$ and if $A \in F\alpha bO(X)$, then $A = \alpha bintA$.

Result 3.11. *Let (X, τ) be an fts. Then the following statements are true:*

(1) *for any fuzzy point x_t in X and any $U \in I^X$, $x_t \in \alpha bclU$ implies that for any $V \in F\alpha bO(X)$ with $x_t qV$, VqU,*

(2) *for any two fuzzy sets U, V, where $V \in F\alpha bO(X)$, if $U \not{q} V$, then $\alpha bclU \not{q} V$.*

Proof. (1) Let $x_t \in \alpha bclU$ and $V \in F\alpha bO(X)$ with $x_t qV$. Then $x_t \notin 1_X \setminus V \in F\alpha bC(X)$. Thus $U \not\leq 1_X \setminus V$. So UqV.

(2) Assume that $\alpha bclUqV$ but $U \not{q} V$. Then there exists $x \in X$ such that $(\alpha bclU)(x) + V(x) > 1$. Thus $V(x) + t > 1$, where $t = (\alpha bclU)(x)$. So $x_t \in \alpha bclU$, where $x_t qV$, $V \in F\alpha bO(X)$. By (1), VqU. This is a contradiction. □

Result 3.12. *Let (X, τ) be an fts and $A \in I^X$. Then the following statements are true:*

(1) $\alpha bcl(1_X \setminus A) = 1_X \setminus \alpha bintA$,

(2) $1_X \setminus \alpha bclA = \alpha bint(1_X \setminus A)$.

Proof. (1) Let $x_t \in \alpha bcl(1_X \setminus A)$. Assume that $x_t \notin 1_X \setminus \alpha bintA$. Then $x_t q\alpha bintA$. Thus there exists $U \in F\alpha bO(X)$ with $U \leq A$ such that $x_t qU$. Since $x_t \in \alpha bcl(1_X \setminus A)$, by Result 3.11 (1), $Uq(1_X \setminus A)$. So $Aq(1_X \setminus A)$. This is a contradiction. Hence

$$\alpha bcl(1_X \setminus A) \leq 1_X \setminus \alpha bintA. \tag{3.1}$$

Conversely, let $x_t \in 1_X \setminus \alpha bintA$. Then we have

$$1 - \alpha bintA(x) \geq t \Rightarrow x_t \not{q} \alpha bintA \Rightarrow x_t \not{q} U,$$

$$\text{where } U \in F\alpha bO(X) \text{ with } U \leq A. \tag{3.2}$$

Let $V \in F\alpha bC(X)$ with $1_X \setminus A \leq V$. Then $1_X \setminus V \leq A$, where $1_X \setminus V \in F\alpha bO(X)$. Thus by (3.2), we get

$$x_t \not{q}(1_X \setminus V) \Rightarrow x_t \in V \Rightarrow x_t \in \alpha bcl(1_X \setminus A).$$

So we have

$$1_X \setminus \alpha bintA \leq \alpha bcl(1_X \setminus A). \tag{3.3}$$

Hence combining (3.1) and (3.3), we get the result.

(2) Writing $1_X \setminus A$ for A in (1), we get the proof. □

Let us now recall the following Lemma from [16] for ready references.

Lemma 3.13 ([16]). *Let (X, τ) be an fts and $A \in I^X$. Then the following statements hold:*
(1) *for any $A \in FRO(X)$, θ-$sclA = A$,*
(2) *for any $A \in F\beta O(X), clA = \alpha clA$,*
(3) *for any $A \in FSO(X)$, $clA = pclA$,*
(4) *for any $A \in \tau, sclA = \theta$-$sclA$.*

4. Fuzzy α-b-r-continuous function : Some characterizations

In this section, we first introduce fuzzy α-b-r-continuous function and characterize it in several ways. Afterwards, two new types of functions, viz., fuzzy α-b-continuous function and fuzzy almost α-b-continuous function are introduced. The mutual relationships of these three functions are established here.

Definition 4.1. Let (X, τ) and (Y, τ_1) be two fts's. Then $f : X \to Y$ is called a *fuzzy α-b-r-continuous function*, if $f^{-1}(A) \in F\alpha bC(X)$ for all $A \in FRO(Y)$.

Theorem 4.2. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then the following statements are equivalent:*
(1) *f is fuzzy α-b-r-continuous,*
(2) *$f^{-1}(A) \in F\alpha bO(X)$ for all $A \in FRC(Y)$,*
(3) *$f(\alpha bcl_\tau U) \leq r$-$ker(f(U))$ for all $U \in I^X$,*
(4) *$\alpha bcl_\tau(f^{-1}(A)) \leq f^{-1}(r$-$ker(A))$ for all $A \in I^Y$,*
(5) *$\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta$-$scl_{\tau_1} R))$ for all $R \in \tau_1$,*
(6) *$\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(scl_{\tau_1} R))$ for all $R \in \tau_1$,*
(7) *$\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1} R))$ for all $R \in \tau_1$,*
(8) *$f^{-1}(int_{\tau_1}(cl_{\tau_1} A)) \in F\alpha bC(X)$ for all $A \in \tau_1$,*
(9) *$f^{-1}(cl_{\tau_1}(int_{\tau_1} F)) \in F\alpha bO(X)$ for all $F \in \tau_1^c$,*
(10) *$f^{-1}(cl_{\tau_1} U) \in F\alpha bO(X)$ for all $U \in F\beta O(Y)$,*
(11) *$f^{-1}(cl_{\tau_1} U) \in F\alpha bO(X)$ for all $U \in FSO(Y)$,*
(12) *$f^{-1}(int_{\tau_1}(cl_{\tau_1} U)) \in F\alpha bC(X)$ for all $U \in FPO(Y)$,*
(13) *$f^{-1}(\alpha cl_{\tau_1} U) \in F\alpha bO(X)$ for all $U \in F\beta O(Y)$,*
(14) *$f^{-1}(pcl_{\tau_1} U) \in F\alpha bO(X)$ for all $U \in FSO(Y)$.*

Proof. (1) $\Leftrightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Let $U \in I^X$ and suppose y_t is a fuzzy point in Y with $y_t \notin r$-ker$(f(U))$. Then there exists $V \in FRO(Y)$ such that $f(U) \leq V$ and $y_t \notin V$, i.e., $V(y) < t$. Thus $y_t q(1_Y \setminus V) \in FRC(Y)$ and $1_Y \setminus f(U) \geq 1_Y \setminus V$. So $f(U) \not{q}(1_Y \setminus V)$, i.e., $U \not{q} f^{-1}(1_Y \setminus V)$. By (2), $f^{-1}(1_Y \setminus V) = 1_X \setminus f^{-1}(V) \in F\alpha bO(X)$. By Result 3.11 (2), $\alpha bcl_\tau U \not{q}(1_X \setminus f^{-1}(V))$. This implies that $\alpha bcl_\tau U \leq f^{-1}(V)$, i.e., $f(\alpha bcl_\tau U) \leq V$ implies that $1_Y \setminus f(\alpha bcl_\tau U) \geq 1_Y \setminus V$. Hence we have

$$1 - f(\alpha bcl_\tau U)(y) \geq 1 - V(y) > 1 - t, \text{ i.e., } t > f(\alpha bcl_\tau U)(y), \text{ i.e., } y_t \notin f(\alpha bcl_\tau U).$$

Therefore $f(\alpha bcl_\tau U) \leq r$-ker$(f(U))$.

(3) $\Rightarrow$ (4): Let $A \in I^Y$. Then $f^{-1}(A) \in I^X$. Thus by (3), we get

$$f(\alpha bcl_\tau f^{-1}(A)) \leq \text{r-ker}(f(f^{-1}(A))) \leq \text{r-ker}(A).$$

So $\alpha bcl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A))$.

(4) ⇒ (1): Let $A \in FRO(Y)$. By (4), $\alpha bcl_\tau(f^{-1}(A)) \leq f^{-1}(r\text{-ker}(A)) = f^{-1}(A)$. But $f^{-1}(A) \leq \alpha bcl_\tau(f^{-1}(A))$. Thus $f^{-1}(A) = \alpha bcl_\tau(f^{-1}(A))$. So $f^{-1}(A) \in F\alpha bC(X)$. Hence f is a fuzzy α-b-r-continuous function.

(5) ⇔ (6): Follows from Lemma 3.13 (4).

(6) ⇔ (7): Obvious

(7) ⇒ (1): Let $A \in FRO(Y)$. Then by (7), we get

$$\alpha bcl_\tau(f^{-1}(A)) \leq f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(A).$$

Thus $f^{-1}(A) \in F\alpha bC(X)$. So f is a fuzzy α-b-r-continuous function.

(1) ⇒ (7): Let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1}A) \in FRO(Y)$. Then by (1), $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\alpha bC(X)$. Thus we have

$$\alpha bcl_\tau(f^{-1}(A)) \leq \alpha bcl_\tau(f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) = f^{-1}(int_{\tau_1}(cl_{\tau_1}A)).$$

(1)⇒ (8): Let $A \in \tau_1$. Then $int_{\tau_1}(cl_{\tau_1}A) \in FRO(Y)$. Thus by (1), $f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\alpha bC(X)$.

(8) ⇒ (1): Let $A \in FRO(Y)$. Then $A \in \tau_1$. Thus by (8), $f^{-1}(A) = f^{-1}(int_{\tau_1}(cl_{\tau_1}A)) \in F\alpha bC(X)$.

(2) ⇒ (9): Let $F \in \tau_1^c$. Then $cl_{\tau_1}int_{\tau_1}F \in FRC(Y)$. Thus by (2), $f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\alpha bO(X)$.

(9) ⇒ (2): Let $F \in FRC(Y)$. Then by (9), $f^{-1}(F) = f^{-1}(cl_{\tau_1}(int_{\tau_1}F)) \in F\alpha bO(X)$.

(2) ⇒ (10): Let $U \in F\beta O(Y)$. Then $U \leq cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)) \leq cl_{\tau_1}U$. Thus

$$cl_{\tau_1}U \leq cl_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)) \leq cl_{\tau_1}(cl_{\tau_1}U) = cl_{\tau_1}U.$$

So $cl_{\tau_1}U = cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))$. Hence $cl_{\tau_1}U \in FRC(Y)$. Therefore by (2), $f^{-1}(cl_{\tau_1}U) \in F\alpha bO(X)$.

(10) ⇒ (6): Since $FSO(Y) \subseteq F\beta O(Y)$, by (10), $f^{-1}(cl_{\tau_1}U) \in F\alpha bO(X)$ for all $U \in FSO(Y)$.

(11) ⇒ (12): Let $U \in FPO(Y)$. Then $U \leq int_{\tau_1}(cl_{\tau_1}U)$. We claim that $int_{\tau_1}(cl_{\tau_1}U) \in FRO(Y)$. Note that the following inequalities hold:

$$int_{\tau_1}(cl_{\tau_1}U) \leq int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) \leq int_{\tau_1}(cl_{\tau_1}U).$$

Then $int_{\tau_1}(cl_{\tau_1}U) = int_{\tau_1}(cl_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U)))$. Thus $1_Y \setminus int_{\tau_1}(cl_{\tau_1}U) \in FRC(Y)$. So $1_Y \setminus int_{\tau_1}(cl_{\tau_1}U) \in FSO(Y)$. By (11), $f^{-1}(cl_{\tau_1}(1_Y \setminus int_{\tau_1}(cl_{\tau_1}U))) \in F\alpha bO(X)$. Hence we have

$$1_X \setminus f^{-1}(int_{\tau_1}(int_{\tau_1}(cl_{\tau_1}U))) = 1_X \setminus f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\alpha bO(X).$$

Therefore $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\alpha bC(X)$.

(12) ⇒ (1): Let $U \in FRO(Y)$. Then $U \in FPO(Y)$. By (12), $f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\alpha bC(X)$. Thus $f^{-1}(U) = f^{-1}(int_{\tau_1}(cl_{\tau_1}U)) \in F\alpha bC(X)$. So (1) holds.

(10) ⇔ (13): The proof follows from Lemma 3.13 (2).

(11) ⇔ (14): The proof follow from Lemma 3.13 (3). □

Theorem 4.3. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Consider the following statements:*

(a) for each fuzzy point x_t in X and each $A \in FSO(Y)$ with $f(x_t)qA$, there exists $U \in F\alpha bO(X)$ with x_tqU, $f(U) \leq cl_{\tau_1}A$,

(b) $f(\alpha bcl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$ for all $P \in I^X$,

(c) *for each fuzzy point* x_t *in* X *and each* $A \in FSO(Y)$ *with* $f(x_t) \in A$*, there exists* $U \in F\alpha bO(X)$ *such that* $x_t \in U$ *and* $f(U) \leq cl_{\tau_1}A$*,*
(d) $f^{-1}(A) \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$ *for all* $A \in FSO(Y)$*,*
(e) $\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R))$ *for all* $R \in I^Y$*,*
(f) f *is a fuzzy* α*-b-r-continuous function.*

Then we have:
(1) (a), (b), (c), (d) *and* (e) *are equivalent,*
(2) (e) *implies* (f).

Proof. (1) (a) $\Rightarrow$ (b): Let $P \in I^X$ and x_t be any fuzzy point in X such that $x_t \in \alpha bcl_\tau P$ and let $G \in FSO(Y)$ with $f(x_t)qG$. By (a), there exists $U \in F\alpha bO(X)$ with x_tqU, $f(U) \leq cl_{\tau_1}G$. As $x_t \in \alpha bcl_\tau P$, by Result 3.11 (1), UqP. Then $f(U)qf(P)$. Thus $f(P)qcl_{\tau_1}G$. So $f(x_t) \in \theta\text{-}scl_{\tau_1}(f(P))$. Hence $f(\alpha bcl_\tau P) \leq \theta\text{-}scl_{\tau_1}(f(P))$.

(b)$\Rightarrow$ (e): Let $R \in I^Y$. Then $f^{-1}(R) \in I^X$. Thus by (b), we have
$$f(\alpha bcl_\tau(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}(f(f^{-1}(R))) \leq \theta\text{-}scl_{\tau_1}R.$$
So $\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$.

(e) $\Rightarrow$ (a): Let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t)qA$. Since $cl_{\tau_1}A \not q\ (1_Y \setminus cl_{\tau_1}A)$, by definition, $f(x_t) \notin \theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1}A)$. Then $x_t \notin f^{-1}(\theta\text{-}scl_{\tau_1}(1_Y \setminus cl_{\tau_1}A))$. By (e), $x_t \notin \alpha bcl_\tau(f^{-1}(1_Y \setminus cl_{\tau_1}A))$. Thus there exists $U \in F\alpha bO(X)$ with x_tqU such that $U \not q f^{-1}(1_Y \setminus cl_{\tau_1}A)$. So $f(U) \not q(1_Y \setminus cl_{\tau_1}A)$. Hence $f(U) \leq cl_{\tau_1}A$.

(a) $\Rightarrow$ (d): Let $A \in FSO(Y)$ and x_t be any fuzzy point in X such that $x_tqf^{-1}(A)$. Then $f(x_t)qA$. By (a), there exists $U \in F\alpha bO(X)$ with x_tqU such that $f(U) \leq cl_{\tau_1}A$. Thus $x_tqU \leq f^{-1}(cl_{\tau_1}A)$. So $x_tqU = \alpha bint_\tau U \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$. Since $\alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$ is the union of all fuzzy α-b-open sets in X contained in $f^{-1}(cl_{\tau_1}A)$, $x_tq\alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$. Hence $f^{-1}(A) \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$.

(d) $\Rightarrow$ (a): Let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t)qA$. Then by (d), $x_tqf^{-1}(A) \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$. Thus there exists $U \in F\alpha bO(X)$ with x_tqU such that $U \leq f^{-1}(cl_{\tau_1}A)$. So $f(U) \leq cl_{\tau_1}A$.

(c) $\Rightarrow$ (d): Let $A \in FSO(Y)$ and x_t be any fuzzy point in X such that $x_t \in f^{-1}(A)$. Then $f(x_t) \in A$. By (c), there exists $U \in F\alpha bO(X)$ with $x_t \in U$ such that $f(U) \leq cl_{\tau_1}A$. Thus $U \leq f^{-1}(cl_{\tau_1}A)$. So $x_t \in U = \alpha bint_\tau U \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$. Hence $f^{-1}(A) \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$.

(d) $\Rightarrow$ (c): Let x_t be any fuzzy point in X and $A \in FSO(Y)$ with $f(x_t) \in A$. Then by (d), $x_t \in f^{-1}(A) \leq \alpha bint_\tau(f^{-1}(cl_{\tau_1}A))$. Thus there exists $U \in F\alpha bO(X)$ with $x_t \in U$ such that $U \leq f^{-1}(cl_{\tau_1}A)$. So $f(U) \leq cl_{\tau_1}A$.

(2) Suppose (e) holds and let $A \in FRO(Y)$. Then by (e), we get
$$\alpha bcl_\tau(f^{-1}(A)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}A) = f^{-1}(A).$$
Thus by Lemma 3.13 (1), $f^{-1}(A) \in F\alpha bC(X)$. So f is a fuzzy α-b-r-continuous function. □

Theorem 4.4. *Let* (X,τ) *and* (Y,τ_1) *be two fts's and* $f : X \to Y$ *be a function satisfying* $\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$*, for all* $R \in I^Y$*. Then the following statements hold:*
(1) $\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$ *for all* $R \in FSO(Y)$
(2) $\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$ *for all* $R \in FPO(Y)$*,*
(3) $\alpha bcl_\tau(f^{-1}(R)) \leq f^{-1}(\theta\text{-}scl_{\tau_1}R)$ *for all* $R \in F\beta O(Y)$*.*

Proof. Obvious. □

Definition 4.5. Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be fuzzy

(i) *α-b-continuous*, if $f^{-1}(A) \in F\alpha bO(X)$ for all $A \in \tau_1$,

(ii) *almost α-b-continuous*, if $f^{-1}(A) \in F\alpha bO(X)$ for all $A \in FRO(Y)$.

Let us now recall the following definition from [2] for ready references.

Definition 4.6 ([2])**.** Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. Then f is said to be a *fuzzy continuous function*, if $f^{-1}(U) \in \tau$ for all $U \in \tau_1$.

Remark 4.7. It is clear from definitions that:

(1) fuzzy continuity $\Rightarrow$ fuzzy α-b-continuity $\Rightarrow$ fuzzy almost α-b-continuity, but reverse implications are not necessarily true, in general, follow from the next examples,

(2) fuzzy α-b-r-continuity is an independent concept of fuzzy continuity, fuzzy α-b-continuity and fuzzy almost α-b-continuity, follow from the next examples.

Example 4.8. Fuzzy continuity, fuzzy α-b-continuity and fuzzy almost α-b-continuity $\not\Rightarrow$ fuzzy α-b-r-continuity.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, B\}$, where $A(a) = A(b) = 0.5, B(a) = B(b) = 0.4$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Then clearly, i is fuzzy continuous and thus fuzzy α-b-continuous as well as fuzzy almost α-b-continuous function. Now $1_X \setminus B \in FRC(X, \tau_2)$. $i^{-1}(1_X \setminus B) = 1_X \setminus B$. Then we get

$$cl_{\tau_1}(\alpha int_{\tau_1}(cl_{\tau_1}(1_X \setminus B))) = A \not\geq 1_X \setminus B \Rightarrow 1_X \setminus B \notin F\alpha bO(X, \tau_1).$$

Thus i is not a fuzzy α-b-r-continuous function.

Example 4.9. Fuzzy α-b-r-continuity, fuzzy α-b-continuity and fuzzy almost α-b-continuity $\not\Rightarrow$ fuzzy continuity.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X\}$, $\tau_2 = \{0_X, 1_X, A\}$, where $A(a) = A(b) = 0.5$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Since every fuzzy set in (X, τ_1) is fuzzy α-b-open in (X, τ_1), i is fuzzy α-b-r-continuous, fuzzy α-b-continuous and fuzzy almost α-b-continuous. Since $A \in \tau_2$, $i^{-1}(A) = A \notin \tau_1$. Then i is not a fuzzy continuous function.

Example 4.10. Fuzzy α-b-r-continuity, fuzzy almost α-b-continuity $\not\Rightarrow$ fuzzy α-b-continuity.

Lt $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X, C\}$, where $A(a) = A(b) = 0.4, B(a) = B(b) = 0.5, C(a) = 0.5, C(b) = 0.6$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $C \in \tau_2$, $i^{-1}(C) = C$. Then $cl_{\tau_1}(\alpha int_{\tau_1}(cl_{\tau_1} C)) = B \not\geq C$. Thus $C \notin F\alpha bO(X, \tau_1)$. So i is not fuzzy α-b-continuous. Since $0_X, 1_X \in FRO(X, \tau_2)$ only, i is a fuzzy α-b-r-continuous function and a fuzzy almost α-b-continuous function.

Example 4.11. Fuzzy α-b-r-continuity $\not\Rightarrow$ fuzzy almost α-b-continuity.

Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X, B\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1) and (X, τ_2) are fts's. Consider the identity function $i : (X, \tau_1) \to (X, \tau_2)$. Now $B \in FRO(X, \tau_2)$, $i^{-1}(B) = B$. Then $int_{\tau_1}(\alpha cl_{\tau_1}(int_{\tau_1} B)) = 0_X \leq B$. Thus $B \in F\alpha bC(X, \tau_1)$. So i is a fuzzy α-b-r-continuous function. But $cl_{\tau_1}(\alpha int_{\tau_1}(cl_{\tau_1} B)) = 0_X \not\geq B$, i.e., $B \notin F\alpha bO(X, \tau_1)$. Hence i is not a fuzzy almost α-b-continuous function.

Definition 4.12 ([17])**.** An fts (X, τ) is said to be *fuzzy extremally disconnected*, if the closure of every fuzzy open set in X is fuzzy open in X.

Theorem 4.13. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a function. If (Y, τ_1) is a fuzzy extremally disconnected space, then f is a fuzzy α-b-r-continuous function if and only if f is a fuzzy almost α-b-continuous function.*

Proof. First suppose that f is fuzzy α-b-r-continuous function and let $U \in FRO(Y)$. Then $U = int_{\tau_1}(cl_{\tau_1} U)$. As Y is fuzzy extremally disconnected, $cl_{\tau_1} U \in \tau_1$. Thus $U = int_{\tau_1} cl_{\tau_1} U = cl_{\tau_1} U = cl_{\tau_1} int_{\tau_1} U$. So $U \in FRC(Y)$. By the hypothesis, $f^{-1}(U) \in F\alpha bO(X)$. Hence f is a fuzzy almost α-b-continuous function.

Conversely, suppose f is a fuzzy almost α-b-continuous function and let $U \in FRC(Y)$. As Y is a fuzzy extremally disconnected space, $U \in FRO(Y)$. Then By the hypothesis, $f^{-1}(U) \in F\alpha bO(X)$. Thus f is a fuzzy α-b-r-continuous function. □

Remark 4.14. Composition of two fuzzy α-b-r-continuous (resp. fuzzy α-b-continuous and fuzzy almost α-b-continuous) functions need not be so, as it seen from the following examples.

Example 4.15. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A, B\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$, where $A(a) = A(b) = 0.5, B(a) = B(b) = 0.4$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$, $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly, i_1 and i_2 are fuzzy α-b-r-continuous functions. Let $i_3 = i_2 \circ i_1$. Now $1_X \setminus B \in FRC(X, \tau_3)$, $i_3^{-1}(1_X \setminus B) = 1_X \setminus B$. Then we have

$$cl_{\tau_1}(\alpha int_{\tau_1}(cl_{\tau_1}(1_X \setminus B))) = A \not\geq 1_X \setminus B.$$

Thus $1_X \setminus B \notin F\alpha bO(X, \tau_1)$. So i_3 is not a fuzzy α-b-r-continuous function.

Example 4.16. Let $X = \{a, b\}$, $\tau_1 = \{0_X, 1_X, A\}$, $\tau_2 = \{0_X, 1_X\}$, $\tau_3 = \{0_X, 1_X, B\}$, where $A(a) = 0.5, A(b) = 0.6, B(a) = 0.5, B(b) = 0.3$. Then (X, τ_1), (X, τ_2) and (X, τ_3) are fts's. Consider two identity functions $i_1 : (X, \tau_1) \to (X, \tau_2)$ and $i_2 : (X, \tau_2) \to (X, \tau_3)$. Clearly, i_1 and i_2 are fuzzy α-b-continuous and thus fuzzy almost α-b-continuous functions. Let $i_3 = i_2 \circ i_1$. Bow $B \in \tau_3$ as well as $B \in FRO(X, \tau_3)$. $i_3^{-1}(B) = B$. Now $cl_{\tau_1}(\alpha int_{\tau_1}(cl_{\tau_1} B)) = 0_X \not\geq B \Rightarrow B \notin F\alpha bO(X, \tau_1) \Rightarrow i_3$ is not a fuzzy α-b-continuous function and also a fuzzy almost α-b-continuous function.

5. Fuzzy α-b-regular, α-b-compact and α-b-T_2-spaces

In this section, new types of separation axioms and compactness are introduced and studied. Then the mutual relationships of these spaces with the spaces defined in [2, 4] are established.

Definition 5.1. An fts (X,τ) is called an *α-b-regular space*, if for each fuzzy point x_t in X and each fuzzy α-b-closed set F with $x_t \notin F$, there exist a fuzzy open set U and a fuzzy α-b-open set V in X such that $x_t qU$, $F \leq V$ and $U \not{q} V$.

Theorem 5.2. *For an fts (X,τ), the following statements are equivalent:*
(1) *X is fuzzy α-b-regular,*
(2) *for each fuzzy point x_t in X and each fuzzy α-b-open set U in X with $x_t qU$, there exists a fuzzy open set V in X such that $x_t qV \leq \alpha bclV \leq U$,*
(3) *for each fuzzy α-b-closed set F in X, $\bigwedge\{clV : F \leq V, V \in F\alpha bO(X)\} = F$,*
(4) *for each fuzzy set G in X and each fuzzy α-b-open set U in X such that GqU, there exists a fuzzy open set V in X such that GqV and $\alpha bclV \leq U$.*

Proof. (1)⇒(2): Let x_t be a fuzzy point in X and let U be a fuzzy α-b-open set in X with $x_t qU$. Then $x_t \notin 1_X \setminus U \in F\alpha bC(X)$. By (1), there exist a fuzzy open set V and a fuzzy α-b-open set W in X such that $x_t qV$, $1_X \setminus U \leq W$ and $V \not{q} W$. Thus $x_t qV \leq 1_X \setminus W \leq U$. So $x_t qV \leq \alpha bclV \leq \alpha bcl(1_X \setminus W) = 1_X \setminus W \leq U$.

(2)⇒(1): Let F be a fuzzy α-b-closed set in X and let x_t be a fuzzy point in X with $x_t \notin F$. Then $x_t q(1_X \setminus F) \in F\alpha bO(X)$. By (2), there exists a fuzzy open set V in X such that $x_t qV \leq \alpha bclV \leq 1_X \setminus F$. Put $U = 1_X \setminus \alpha bclV$. Then $U \in F\alpha bO(X)$ and $x_t qV$, $F \leq U$ and $U \not{q} V$.

(2)⇒(3): Let F be fuzzy α-b-closed set in X. Then we get

$$F \leq \bigwedge\{clV : F \leq V, V \in F\alpha bO(X)\}.$$

Conversely, let $x_t \notin F \in F\alpha bC(X)$. Then $F(x) < t$. Thus $x_t q(1_X \setminus F)$, where $1_X \setminus F \in F\alpha bO(X)$. By (2), there exists a fuzzy open set U in X such that $x_t qU \leq \alpha bclU \leq 1_X \setminus F$. Put $V = 1_X \setminus \alpha bclU$. Then $F \leq V$ and $U \not{q} V$. Thus $x_t \notin clV$. So $\bigwedge\{clV : F \leq V, V \in F\alpha bO(X)\} \leq F$. Hence $\bigwedge\{clV : F \leq V, V \in F\alpha bO(X)\} = F$.

(3)⇒(2): Let V be any fuzzy α-b-open set in X and let x_t be any fuzzy point in X with $x_t qV$. Then $V(x) + t > 1$, i.e., $x_t \notin (1_X \setminus V)$, where $1_X \setminus V \in F\alpha bC(X)$. By (3), there exists $G \in F\alpha bO(X)$ such that $1_X \setminus V \leq G$ and $x_t \notin clG$. Thus there exists a fuzzy open set U in X with $x_t qU$ such that $U \not{q} G$. So $U \leq 1_X \setminus G \leq V$. Hence $x_t qU \leq \alpha bclU \leq \alpha bcl(1_X \setminus G) = 1_X \setminus G \leq V$.

(3)⇒(4): Let G be any fuzzy set in X and let U be any fuzzy α-b-open set in X with GqU. Then there exists $x \in X$ such that $G(x)+U(x) > 1$. Let $G(x) = t$. Then $x_t qU \Rightarrow x_t \notin 1_X \setminus U$, where $1_X \setminus U \in F\alpha bC(X)$. By (3), there exists $W \in F\alpha bO(X)$ such that $1_X \setminus U \leq W$ and $x_t \notin clW$. Thus $(clW)(x) < t$. So $x_t q(1_X \setminus clW)$. Let $V = 1_X \setminus clW$. Then V is fuzzy open set in X and $V(x)+t > 1$. Thus $V(x)+G(x) > 1$. So VqG and $\alpha bclV = \alpha bcl(1_X \setminus clW) \leq \alpha bcl(1_X \setminus W) = 1_X \setminus W \leq U$.

(4)⇒(2): Obvious. □

Note 5.3. It is clear from Theorem 5.2 that in a fuzzy α-b-regular space, every fuzzy α-b-closed set is fuzzy closed and hence every fuzzy α-b-open set is fuzzy open. As a result, in a fuzzy α-b-regular space, the collection of all fuzzy closed (resp., fuzzy open) sets and fuzzy α-b-closed (resp., fuzzy α-b-open) sets coincide.

Definition 5.4. Let A be a fuzzy set in X. A collection $\mathcal{U}$ of fuzzy sets in X is called a *fuzzy cover* of A, if $sup\{U(x) : U \in \mathcal{U}\} = 1$ for each $x \in suppA$ [18]. In particular, if $A = 1_X$, we get the definition of fuzzy cover of X [2].

Definition 5.5. A fuzzy cover $\mathcal{U}$ of a fuzzy set A in X is said to *have a finite subcover* $\mathcal{U}_0$, if $\mathcal{U}_0$ is a finite subcollection of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 \geq A$, i.e., $\mathcal{U}_0$ is also a fuzzy cover of A [18]. In particular, if $A = 1_X$, we get $\bigcup \mathcal{U}_0 = 1_X$ [2].

Definition 5.6. A fuzzy set A in an fts (X, τ) is said to be *fuzzy compact* [18], if every fuzzy covering $\mathcal{U}$ of A by fuzzy open sets in X has a finite subcovering $\mathcal{U}_0$ of $\mathcal{U}$. In particular, if $A = 1_X$, we get the definition of fuzzy compact space [2].

Definition 5.7. An fts (X, τ) is said to be *fuzzy s-closed* [19] (resp. *fuzzy nearly compact* [17]), if every fuzzy covering of X by fuzzy regular closed (resp. fuzzy regular open) sets of X contains a finite subcovering.

Let us now introduce the following concept.

Definition 5.8. A fuzzy set A in an fts (X, τ) is called *fuzzy α-b-compact*, if every fuzzy covering of A by fuzzy α-b-open sets of X has a finite subcovering. In particular, if $A = 1_X$, we get the definition of fuzzy α-b-compact space.

Remark 5.9. It is clear from above discussion that fuzzy α-b-compact space is fuzzy compact. But the converse is not necessarily true follows from the next example.

Example 5.10. Let $X = \{a\}$, $\tau = \{0_X, 1_X\}$. The clearly (X, τ) is a fuzzy compact space. Here every fuzzy set is fuzzy α-b-open set in X. Consider the fuzzy cover $\mathcal{U} = \{U_n(a) : n \in \mathbf{N}\}$, where $U_n(a) = \{\frac{n}{n+1} : n \in \mathbf{N}\}$. Then $\mathcal{U}$ is a fuzzy α-b-open cover of X. But it does not have any subcovering of X. Thus X is not fuzzy α-b-compact space.

Theorem 5.11. *Every fuzzy α-b-closed set A in a fuzzy α-b-compact space X is fuzzy α-b-compact.*

Proof. Let A be a fuzzy α-b-closed set in a fuzzy α-b-compact space X and let $\mathcal{U}$ be a fuzzy covering of A by fuzzy α-b-open sets in X. Then $\mathcal{V} = \mathcal{U} \bigcup (1_X \setminus A)$ is a fuzzy α-b-open covering of X. By the hypothesis, there exists a finite subcollection $\mathcal{V}_0$ of $\mathcal{V}$ which also covers X. If $\mathcal{V}_0$ contains $1_X \setminus A$, we omit it and get a finite subcovering of A. Consequently, A is fuzzy α-b-compact. $\square$

Let us now recall the following definition from [4] for ready references.

Definition 5.12 ([4])**.** Let (X, τ) be an fts. Then X is said to be a *fuzzy T_2-space*, if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy open sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Now we introduce the following concept.

Definition 5.13. Let (X, τ) be an fts. Then X is said to be a *fuzzy α-b-T_2-space*, if for each pair of distinct fuzzy points x_α, y_β : when $x \neq y$, there exist fuzzy α-b-open sets U_1, U_2, V_1, V_2 such that $x_\alpha \in U_1, y_\beta q V_1$ and $U_1 \not{q} V_1$ and $x_\alpha q U_2$, $y_\beta \in V_2$ and $U_2 \not{q} V_2$; when $x = y$, $\alpha < \beta$ (say), there exist fuzzy α-b-open sets U, V in X such that $x_\alpha \in U, y_\beta q V$ and $U \not{q} V$.

Let us now recall the following definition from [4] for ready references.

Definition 5.14 ([4]). An fts (X, τ) is said to be a *fuzzy regular space*, if for any fuzzy point x_t in X and any fuzzy closed set F in X with $x_t \notin F$, there exist fuzzy open sets U, V in X such that $x_t qU, F \leq V$ and $U \not{q} V$.

Remark 5.15. It is clear from Note 5.3 that fuzzy α-b-regular space is fuzzy regular and fuzzy T_2-space is fuzzy α-b-T_2-space. But the reverse implications are not necessarily true, follow from the next example.

Example 5.16. Consider Example 5.10. It is clear that (X, τ) is fuzzy regular and fuzzy α-b-T_2-space (as every fuzzy set is fuzzy α-b-open set as well as fuzzy α-b-closed set). Now consider the fuzzy point $a_{0.4}$ and a fuzzy set A defined by $A(a) = 0.3$. Then $a_{0.4} \notin A \in F\alpha bC(X)$. But there do not exist any fuzzy open set U and a fuzzy α-b-open set V in X such that $a_{0.4}qU, A \leq V$ and $U \not{q} V$ (because 1_X is the only fuzzy open set in X with $a_{0.4}q1_X$ and $1_X qV$ for all fuzzy set $V(\neq 0_X)$ in X). Thus X is not fuzzy α-b-regular space.

Consider two fuzzy points $a_{0.4}$ and $a_{0.5}$ in X. But there do not exist fuzzy open sets U, V in X such that $a_{0.4} \in U, a_{0.5}qV$ and $U \not{q} V$. So X is not fuzzy T_2-space.

6. Applications of fuzzy α-b-r-continuous, α-b-continuous and almost α-b-continuous functions

In this section, the applications of the functions introduced in this paper are established. First we recall the following definition from [20] for ready references.

Definition 6.1 ([20]). A function $f : X \to Y$ is said to be a *fuzzy open function*, if $f(U)$ is a fuzzy open set in Y for every fuzzy open set U in X.

Theorem 6.2. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a surjective, fuzzy α-b-r-continuous function. If X is a fuzzy α-b-compact space, then Y is a fuzzy s-closed space.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of Y by fuzzy regular closed sets of Y. As f is a fuzzy α-b-r-continuous function, $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ covers X by fuzzy α-b-open sets of X. As X is a fuzzy α-b-compact space, there exists a finite subset Λ_0 of Λ such that $1_X = \bigvee\limits_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)$. Then we have

$$1_Y = f(\bigvee_{\alpha \in \Lambda_0} f^{-1}(U_\alpha)) = \bigvee_{\alpha \in \Lambda_0} f(f^{-1}(U_\alpha)) \leq \bigvee_{\alpha \in \Lambda_0} U_\alpha.$$

Thus Y is a fuzzy s-closed space. □

Theorem 6.3. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a fuzzy α-b-continuous function. If A is fuzzy α-b-compact set relative to X, then the image $f(A)$ is fuzzy compact relative to Y.*

Proof. Let A be fuzzy α-b-compact relative to X and $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a fuzzy covering of $f(A)$ by fuzzy open sets of Y, i.e, $f(A) \leq \bigvee\limits_{\alpha \in \Lambda} U_\alpha$. Then $A \leq f^{-1}(\bigvee\limits_{\alpha \in \Lambda} U_\alpha) = \bigvee\limits_{\alpha \in \Lambda} f^{-1}(U_\alpha)$. Thus $\mathcal{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a fuzzy covering of A by fuzzy α-b-open sets in X. As A is fuzzy α-b-compact set relative to X, there exists a finite subcollection $\mathcal{V}_0 = \{f^{-1}(U_{\alpha_i}) : 1 \leq i \leq n\}$ of $\mathcal{V}$ such that $A \leq \bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})$. So $f(A) \leq f(\bigvee_{i=1}^{n} f^{-1}(U_{\alpha_i})) = \bigvee_{i=1}^{n} f(f^{-1}(U_{\alpha_i})) \leq \bigvee_{i=1}^{n} U_{\alpha_i}$. Hence $\mathcal{U}_0 = \{U_{\alpha_i} : 1 \leq i \leq n\}$ is a finite subcovering of $f(A)$. Therefore $f(A)$ is fuzzy compact relative to Y. □

Theorem 6.4. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a fuzzy almost α-b-continuous function. If A is fuzzy α-b-compact relative to X, then the image $f(A)$ is fuzzy nearly compact relative to Y.*

Proof. The proof is similar to that of Theorem 6.3. □

Theorem 6.5. *Let (X, τ) and (Y, τ_1) be two fts's and $f : X \to Y$ be a injective, fuzzy α-b-continuous function and Y is a fuzzy T_2-space. Then X is a fuzzy α-b-T_2-space.*

Proof. Let x_α and y_β be two distinct fuzzy points in X, where $x \neq y$. As f is injective, $f(x_\alpha) \neq f(y_\beta)$. As Y is a fuzzy T_2-space, there exist fuzzy open sets U_1, U_2, V_1, V_2 in Y such that $f(x_\alpha) \in U_1, f(y_\beta)qV_1$ and $U_1 \not{q} V_1$ and $f(x_\alpha)qU_2$, $f(y_\beta) \in V_2$ and $U_2 \not{q} V_2$. Then $x_\alpha \in f^{-1}(U_1), y_\beta q f^{-1}(V_1)$ and $f^{-1}(U_1) \not{q} f^{-1}(V_1)$. Assume that $f^{-1}(U_1) q f^{-1}(V_1)$. Then there exists $z \in X$ such that

$$f^{-1}(U_1)(z) + f^{-1}(V_1)(z) > 1.$$

Thus $U_1(f(z)) + V_1(f(z)) > 1$. So $U_1 q V_1$. This is a contradiction. Also, we have

$$x_\alpha q f^{-1}(U_2), y_\beta \in f^{-1}(V_2) \text{ and } f^{-1}(U_2) \not{q} f^{-1}(V_2),$$

where $f^{-1}(U_1), f^{-1}(V_1), f^{-1}(U_2), f^{-1}(V_2) \in F\alpha bO(X, \tau_1)$.

Similarly, when $x = y$, $\alpha < \beta$ (say), there exist $U_1, U_2 \in \tau_1$ such that

$$f(x_\alpha) \in U_1, f(y_\beta)qU_2 \text{ and } U_1 \not{q} U_2.$$

Then $x_\alpha \in f^{-1}(U_1), y_\beta q f^{-1}(U_2)$ and $f^{-1}(U_1) \not{q} f^{-1}(U_2)$, where $f^{-1}(U_1), f^{-1}(U_2) \in F\alpha bO(X, \tau_1)$. Thus X is a fuzzy α-b-T_2-space. □

Theorem 6.6. *If a bijective function $h : X \to Y$ is a fuzzy α-b-continuous, fuzzy open function from a fuzzy α-b-regular space X onto an fts Y, then Y is fuzzy regular space.*

Proof. Let y_α be a fuzzy point in Y and F, a fuzzy closed set in Y with $y_\alpha \notin F$. As h is bijective, there exists unique $x \in X$ such that $h(x) = y$. Then $h(x_\alpha) \notin F$. As h is fuzzy α-b-continuous function, $x_\alpha \notin h^{-1}(F) \in F\alpha bC(X)$. As X is fuzzy α-b-regular space, there exist a fuzzy open set U and a fuzzy α-b-open set V in X such that

$$x_\alpha q U, h^{-1}(F) \leq V \text{ and } U \not{q} V.$$

Since X is fuzzy α-b-regular, by Note 5.3, V is also fuzzy open set in X. As h is fuzzy open function, we have $h(x_\alpha)qh(U), F \leq h(V)$ and $h(U) \not{q} h(V)$, where $h(U), h(V)$ are fuzzy open sets in Y. Thus Y is a fuzzy regular space. □

7. Conclusions

In this paper we have introduced and characterized only fuzzy α-b-regularity, fuzzy α-b-compactness and fuzzy α-b-T_2 property. A thorough discussion on these three spaces are done in a separate article which has already been communicated.

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